Automatic continuity of some group homomorphisms

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February 1, 2010

Abstract

This short text was written for the Séminaire bleu of the ENS Lyon which is a part of the Master of Mathematics; I had to pick some sections of Rosendal's article *Automatic continuity of group homomorphisms* [3] and present them to an audience of professors and students. Here is what I intended to talk about.

Introduction

It is a classical exercise to find all the continuous homomorphisms from $(\mathbb{R}, +)$ onto itself: these are the $\mathbb{R}$-linear applications. Are there any other homomorphisms? The axiom of choice gives us an example through the existence of a Hamel basis of $\mathbb{R}$ seen as a $\mathbb{Q}$ vector space. So if we want to have automatic continuity, we must ask stronger conditions on the homomorphism: for instance (see section 1) measurability implies continuity. But we can also change the domain and the range of the homomorphisms. One way to do this is to consider $\mathbb{R}$ as a Polish group and thus extend the question to this class of groups:

Question. Let $\phi : G \to H$ be a group homomorphism, where G and H are actually Polish groups. What conditions could make ϕ continuous?

The first section is dedicated to regularity conditions for the homomorphism and the second to counterexamples, while the two last sections respectively deal with the range and the domain of the homomorphism.

1 Baire measurability and continuity

Let us first recall some definitions:

Definition 1.1. A **Polish space** is a separable topological space (X, τ) which has a compatible complete metric. A **Polish group** is a topological group whose topology is Polish.

Definition 1.2. Let G be a subset of a topological space (X, τ). We say that G is **meager** if it is included in a countable union of nowhere dense subsets of X (i.e. G is a subset of a countable union of closed sets, each of them having an empty interior). Taking the complementary, we say that G is **comeager** if it contains a countable union of dense open sets.

We also recall the two following fundamental theorems:

Theorem 1.3. *If (X,τ) is a Polish space and $(U_\alpha)_{\alpha\in A}$ is a family of open subsets of X, then there exists countably many $\alpha_i \in A$ so that*

$$\bigcup_{\alpha\in A} U_\alpha = \bigcup_{i\in\mathbb{N}} U_{\alpha_i}$$

Theorem 1.4 (Baire)**.** *If (X,τ) is a Polish space, then every comeager subset of X is dense in X (equivalently, every meager subset is of empty interior).*

In particular, this implies that in a Polish space (X,τ), a subset $A \subseteq X$ cannot be meager and comeager at the same time, for its complementary would also be comeager and hence $A \cap X \setminus A = \emptyset$ would be comeager too, a contradiction.

So, in a Polish space we can see meager sets as "small" sets, hence the following definition, which will encompass every set "not too far from being nice":

Definition 1.5. Let (X,τ) be a Polish space. A subset $A \subseteq X$ is called **Baire-measurable** if there exists an open set U so that $U\Delta A$ is meager.

Note that A is then comeager in U. Using theorem 1.4, one can prove that the set of Baire-measurable subsets of X is a σ-algebra, and thus that every Borel set is Baire-measurable. Before considering the question of the existence of non Baire-measurable sets, let us jump to the main result of this section, which uses the following definition:

Definition 1.6. Let (X,τ) and (Y,τ') be Polish spaces. An application $f : X \to Y$ is **Baire-measurable** if for every open set $U \subseteq Y$, $f^{-1}(U)$ is Baire-measurable in X.

Theorem 1.7. *Let (G,τ) and (H,τ') be Polish groups. Then every Baire-measurable homomorphism between them is continuous.*

The proof uses the following definition and lemmas :

Definition 1.8. If A is any subset of a Polish space (X,τ), we note $U(A)$ the greatest open set in which A is comeager (it exists by taking the union of open sets in which A is comeager, and using theorems 1.3 and 1.4).

Lemma 1.9 (Pettis [2])**.** *Let (G,τ) be a Polish group, A and B be Baire-measurable subsets of G, then:*

$$U(A)\cdot U(B) \subseteq A\cdot B$$

Proof. Let us first notice that, as the multiplication by a fixed element and the inversion are homeomorphisms, we have $g\cdot U(A) = U(g\cdot A)$ and $U(A^{-1}) = U(A)^{-1}$. Now, if A and B are subsets of X, let $x \in U(A)\cdot U(B)$. Then $x\cdot U(B)^{-1}\cap U(B) = U(xB^{-1})\cap U(B)$ is a nonempty open subset of G in which $xA^{-1}\cap B$ is comeager, thus nonempty by theorem 1.4. But this means exactly that $x \in A\cdot B$. □

Lemma 1.10. *Let (G,τ) and (H,τ') be Polish groups, and $\phi : G \to H$ a homomorphism. Then the inverse image of any open subset is either non-meager or empty.*

Proof. Suppose it were not the case: let $V \subseteq H$ be an open subset so that $\phi^{-1}(V)$ is meager and not empty. We have $G = \bigcup_{g\in G} g\cdot\phi^{-1}(V)$ so that

$$\phi(G) = \bigcup_{g\in G}\phi(g)\cdot\phi(\phi^{-1}(V)) \subseteq \bigcup_{g\in G}\phi(g)\cdot V$$

Using theorem 1.3 in H, we find $(g_n)_{n\in\mathbb{N}}$ so that $\phi(G) \subseteq \bigcup_{n\in\mathbb{N}} \phi(g_n) \cdot V$, and then

$$\begin{aligned} G &= \phi^{-1}(\phi(G) \\ &= \bigcup_{n\in\mathbb{N}} \phi^{-1}(\phi(g_n) \cdot V) \\ G &= \bigcup_{n\in\mathbb{N}} g_n \cdot \phi^{-1}(V) \end{aligned}$$

But as $\phi^{-1}(V)$ is meager, G would be meager, a contradiction to theorem 1.4. □

We are now ready to prove theorem 1.7:

Proof. Let (G, τ) and (H, τ') be Polish groups, and $\phi : G \to H$ a Baire-measurable homomorphism. We wish to prove that the inverse image of any open set is open; by homogeneity it suffices to prove that the inverse image of any open neighborhood of 1_H is a neighborhood of 1_G. So let V be an open set containing 1_H; by the continuity of group operations we find an open neighborhood W of 1_H so that $W \cdot W^{-1} \subseteq V$. By lemma 1.10, $\phi^{-1}(W)$ cannot be meager; as it is Baire-measurable, $U(\phi^{-1}(W)) \neq \emptyset$. But then

$$1_H \in U(\phi^{-1}(W)) \cdot U(\phi^{-1}(W))^{-1} \;=\; U(\phi^{-1}(W)) \cdot U(\phi^{-1}(W)^{-1})$$

But by lemma 1.9, $U(\phi^{-1}(W)) \cdot U(\phi^{-1}(W)^{-1}) \subseteq \phi^{-1}(W) \cdot \phi^{-1}(W)^{-1} \subseteq \phi^{-1}(W \cdot W^{-1}) \subseteq \phi^{-1}(V)$, which is thus a neighborhood of 1_G. □

This result strongly restricts the class of possible noncontinuous homomorphisms between Polish groups: it excludes in particular many definable homomorphisms, e.g. the Borel ones. Furthermore, extending Solovay's ideas in [5], Shelah has proven in [4] that it is consistent with ZF to suppose that every subset of the reals is Baire-measurable. As this implies that every subset of a Polish space is Baire-measurable, it is thus consistent with ZF that every homomorphism between Polish groups is continuous. So in order to find non-continuous homomorphisms, we will have to use the full ZFC. Let us now see some of those counterexamples.

2 Non-continuous homomorphisms

Counterexample 2.1. *There exists discontinuous functionals ϕ on any separable infinite-dimensional Banach space X.*

Proof. Let us first find a basis $(x_i)_{i\in I}$ of X as an $\mathbb{R}$-vector space so that $\{x_i/i \in I\}$ is dense in X. We start by enumerating a countable open ball basis $(B_n)_{n\in\mathbb{N}}$ of X; and then we build by induction on $\mathbb{N}$ a free family $(x_n)_{n\in\mathbb{N}}$ such that $x_n \in B_n$. Such a construction is possible as the linear span of finitely many vectors is of empty interior, thus cannot cover entirely an open subset of X. Then we just have to complete $(x_n)_{n\in\mathbb{N}}$ in order to obtain an $\mathbb{R}$-basis $(x_i)_{i\in I}$ of X. Now if $j \in I$, ϕ_j is defined as the unique functional so that $\phi_j(x_i) = \delta_{j,i}$. It is discontinuous as by density we can find a sequence (x_{i_n}) with $i_n \neq j$ so that $x_{i_n} \to x_j$. □

Counterexample 2.2. *There exists a discontinuous group isomorphism between $(\mathbb{R}, +)$ and $(\mathbb{R}^2, +)$.*

Proof. Seeing both $\mathbb{R}$ and $\mathbb{R}^2$ as vector spaces over $\mathbb{Q}$, each of them has a basis whose cardinality is the continuum. So there is a bijection between those basis, which can be extended to a $\mathbb{Q}$-isomorphism of $\mathbb{R}$ and $\mathbb{R}^2$. This isomorphism is in particular a group isomorphism, and it cannot be a homeomorphism by a standard connectedness argument. □

Let now G be a Polish group, and H a subgroup of countable index of G. If H is open, then it is closed as its complementary is the union of open cosets. Conversely, suppose H is closed. By Baire's theorem (cf. theorem 1.4) it cannot be meager, but it still is Baire-measurable so $U(H)$ is not empty, and hence $1_G \in U(H) \cdot U(H)^{-1} \subseteq H$ by Pettis theorem (cf. lemma 1.9). So $1_G \in \mathrm{Int}(H)$, and by homogeneity H must be open.

Proposition 2.3. *If G is a Polish group with a non open/closed subgroup H of countable index, then there exists a non-continuous homomorphism of domain G.*

Proof. Let $\phi : G \to \mathcal{S}(G/H)$ be the action of G on G/H by left translation. If H is of finite index, we equip $\mathcal{S}(G/H)$ with the discrete topology, else $\mathcal{S}(G/H)$ is group isomorphic to $\mathcal{S}_\infty$ which has a natural Polish topology. Either way, $\{\sigma \in \mathcal{S}(G/H)/\sigma(1H) = 1H\}$ is an open subset of $\mathcal{S}(G/H)$, and its inverse image by ϕ is H which is not open: ϕ is thus non-continuous. □

Using this, we have the :

Counterexample 2.4. *If F is any finite group equipped with the discrete topology, there exists discontinuous homomorphisms whose domain is $F^{\mathbb{N}}$.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and let $H = \{(f_n)_{n\in\mathbb{N}}, \{n/f_n = 1_F\} \in \mathcal{U}\}$. As $\mathcal{U}$ is a filter, one can easily see that H is a subgroup of $F^{\mathbb{N}}$; its index is $|F|$ because if $(g_n)_{n\in\mathbb{N}} \in F^{\mathbb{N}}$, there exists $f \in F$ so that $\{n \in \mathbb{N}/f = g_n\} \in \mathcal{U}$. In fact, if it were not the case, for all $f \in F$, $A_f = \{n \in \mathbb{N}/f \neq g_n\}$ would be in $\mathcal{U}$, and thus so would $\cap A_f = \emptyset$, a contradiction.

Also, H is dense in $F^{\mathbb{N}}$ because $\mathcal{U}$ is not principal, and so H cannot be closed. By the preceding proposition 2.3, we are done. □

Now that we are convinced that the theory of automatic continuity for Polish groups is rich, let us see some of its content.

3 Conditions on the range

Definition 3.1. Let G be a group. We say it is **normed** if there exists a norm on it, i.e. an application $||\cdot|| : G \to \mathbb{N}$ satisfying the following properties for every $g, h \in G$:

(1) $||g \cdot h|| \leqslant ||g|| + ||h||$

(2) $||1_G|| = 0$

(3) $||g|| = ||g^{-1}||$

(4) $\forall n \in \mathbb{N}$, $g \neq 1_G \Rightarrow ||g^n|| \geqslant \max\{n, ||g||\}$

Examples: $\mathbb{Z}$, any free group, any free abelian group. (To show this, equip $\mathbb{Z}$ with the usual norm, and then show that the class of normed groups is stable by free product and direct sum)

Theorem 3.2 (Dudley [1])**.** *Let G be a Polish group, d a compatible complete metric and H a normed group equipped with the discrete topology. Then if $\phi : G \to H$ is a group morphism, ϕ is continuous.*

Proof. Let us suppose the contrary: then ϕ is not continuous at 1_G. We define by induction sequences $(x_{m,n})_{n,m\in\mathbb{N}^*}$, $(g_n)_{n\in\mathbb{N}^*}$ of elements of G, and $(k_n)_{n\in\mathbb{N}^*}$ of natural integers such that :

- For all $n \in \mathbb{N}^*$ and $0 < m < n$, $x_{n,m} = g_m x_{n,m+1}^{k_m}$
- For all $n \in \mathbb{N}^*$, $k_n = n + \sum_{i=1}^n \|\phi(g_i)\|$, $x_{n,n} = 1$ and $\phi(g_n) \neq 1_H$
- For all $m, n \in \mathbb{N}^*$, $d(x_{n+1,m}, x_{n,m}) < 2^{-n}$

We start with any g_1 such that $\phi(g_1) \neq 1_H$, then if we have $g_1, ..., g_n$ we let $k_n = n + \sum_{i=1}^n \|\phi(g_i)\|$. Note that we have, for $1 \leq m \leq n$,

$$x_{n,m} = g_m \left(g_{m+1} \left(g_{m+2} \cdots (g_{n-1}(g_n)^{k_{n-1}})^{k_{n-2}} \cdots \right)^{k_{m-1}}\right)^{k_m}$$

Now, to build every $x_{n+1,m}$, we wish to replace g_n by $g_n \cdot g_{n+1}^{k_n}$ so that the new expression is not to far away from the old one. But as there are finitely many m, and as $g \mapsto g_m \left(g_{m+1} \left(g_{m+2} \cdots (g_{n-1}(g_n \cdot g^{k_n})^{k_{n-1}})^{k_{n-2}}\right.\right.$ is continuous, it is always possible to do so, and to have $\phi(g_{n+1}) \neq 1_H$ by non-continuity of ϕ at 1_G.

Now, if m is fixed, the sequence $(x_{n,m})$ is Cauchy, let y_m be its limit. Then by continuity we have for all $m \in \mathbb{N}^*$, $y_m = g_m (y_{m+1})^{k_m}$. Applying this formula, we get

$$y_1 = g_1 \left(g_2 \left(\cdots (g_m y_{m+1}^{k_m}) \cdots \right)^{k_2}\right)^{k_1}$$

Let us remark that, if $\phi(y_{m+1}) = 1_H$, we have

$$\|\phi(y_m^{k_{m-1}})\| = \|\phi(g_m)^{k_{m-1}}\| \geqslant k_{m-1}$$

while if $\phi(y_{m+1}) \neq 1_H$,

$$\|\phi(y_m^{k_{m-1}})\| \geqslant \|\phi(y_m)\| = \|\phi(g_m)\phi(y_{m+1})^{k_m}\| \geqslant k_m - \|\phi(g_m)\| = k_{m-1} + 1$$

So either way, $\|\phi(y_m^{k_{m-1}})\| \geqslant k_{m-1}$. Now, we write

$$\begin{aligned}
\|\phi(y_1)\| &\geqslant \left\|\phi\left[\left(g_2 \left(g_3 \cdots\right)^{k_2}\right)^{k_1}\right]\right\| - \|\phi(g_1)\| \\
&\geqslant \left\|\phi\left[g_2 \left(g_3 \cdots\right)^{k_2}\right]\right\| - \|\phi(g_1)\| \\
&\geqslant \left\|\phi\left[\left(g_3 \cdots\right)^{k_2}\right]\right\| - \|\phi(g_2)\| - \|\phi(g_1)\| \\
&\geqslant \cdots \\
&\geqslant \left\|\phi\left[(g_m y_{m+1})^{k_{m-1}}\right]\right\| - \|\phi(g_{m-1})\| - \cdots - \|\phi(g_1)\| \\
\|\phi(y_1)\| &\geqslant k_{m-1} - \|\phi(g_{m-1})\| - \cdots - \|\phi(g_1)\| = m - 1
\end{aligned}$$

But as this holds for all m, we have a contradiction. □

Corollary 3.3. *There is no Polish topology on $\mathbb{F}_{2^\omega}$ (the free group on a continuum of generators).*

Proof. Suppose it where the case, then using the preceding theorem, the identity from $\mathbb{F}_{2^\omega}$ equipped with its Polish topology into $\mathbb{F}_{2^\omega}$ with the discrete topology is continuous, so the Polish topology we have is discrete, but as it is separable we have a contradiction. □

Corollary 3.4. *Any homomorphism from a Polish group into $(\mathbf{F}_2)^\omega$ is continuous (where $\mathbf{F}_2$ is the free group on two generators equipped with the discrete topology, and $(\mathbf{F}_2)^\omega$ has the product topology).*

Proof. Let ϕ be such a homomorphism. By definition of the product topology, we only need to show that $\phi \circ \pi$ is continuous for every projection $\pi : \mathbf{F}_2^\omega \to \mathbf{F}_2$, but then theorem 3.2 applies. □

4 Conditions on the domain

We begin with two definitions:

Definition 4.1. Let G be a Polish group. A subset A of X is said to be **σ-syndetic** if there exists $(g_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ so that $G = \cup g_n \cdot A$, i.e. G can be covered by countably many left translates of A.

Example: Open subsets of G.

Definition 4.2. Let G be a Polish group. We say that G is **Steinhaus** if there exists $k \in \mathbb{N}$ such that every symmetric, σ-syndetic subset $A \subseteq G$ containing 1_G verifies the following property:

$$1_G \in \mathrm{Int}(A^k)$$

In this case G is called Steinhaus with exponent k.

We now prove the main theorem of this section:

Theorem 4.3. *Let G be a Steinhaus Polish group; then if H is a Polish group, any homomorphism $\phi : G \to H$ is continuous.*

Proof. Suppose G is Steinhaus of exponent k. Let V be an open neighborhood of 1_H, by continuity of group operations we can find an open symmetric neighborhood W of 1_H so that $W^{2k} \subseteq V$. Now W is σ-syndetic; let $(h_n)_{n\in\mathbb{N}}$ be so that $(h_n \cdot W)_{n\in\mathbb{N}}$ covers H. Then for each n so that $h_n \cdot W$ intersects $\phi(G)$, take g_n satisfying $\phi(g_n) \in h_n \cdot W$. Then $h_n \in \phi(g_n) \cdot W^{-1}$, but by symmetry $W^{-1} = W$ so $h_n \cdot W \subseteq \phi(g_n) \cdot W^2$. But then G is covered by countably many left translates of $\phi^{-1}(W^2)$, which is thus σ-syndetic. As it is symmetric, we have $1_G \in \mathrm{Int}(\phi^{-1}(W^2))^k \subseteq \phi^{-1}(V)$. □

One example of such groups is the groups having ample generics, unfortunately we don't have the time to go deeper into this subject.

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