The Steinhaus property and Haar null sets

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Introduction

The purpose is to find a satifactory extension in Polish Groups of the following theorem :

Theorem (Steinhaus theorem)**.** *Let μ be a translation-invariant regular measure defined on the Borel sets of $\mathbb{R}$, and A is a Borel measurable set with $\mu(A) > 0$, then $0 \in Int(A - A)$.*

We just must find on which set we would apply the result. In fact it will be on non generically left Haar-null sets. But what is a left Haar-null set ? Let's start with (a lot of) definitions.

1 Preliminaries

1.1 Polish group

Let's start with some definition on Polish groups.

Definition (Polish space)**.** *A topological space X is said to be a* Polish space *if it is separable and completely metrizable.*

Definition (Polish group)**.** *A group G is said to be a* Polish group *if it is a topological group which is also a Polish space.*

And let's continue with definitions on meager.

Definition. *A subset A of a topological space X is said to be* meager *if A is covered by a countable union of closed nowhere dense sets.*
A subset A of a topological space X is said to be co-meager *if the complement of A is meager.*

1.2 Topological spaces

We will need to work with compacts. For any Polish space we denote by $\mathcal{K}(X)$ the space of all compacts of X with the Vietoris topology. We consider the Hausdorff metric d_H on $\mathcal{K}(X)$ associated to d, defined by :

$$d_H(K, C) = \inf\{\varepsilon > 0 \ / \ K \subset C_\varepsilon \text{ and } C \subset K_\varepsilon\},$$

where $A_\varepsilon = \{x \in X / d(x, A) \leq \varepsilon\}$. We denote

$$B_H(K, r) = \{C \in \mathcal{K}(X) \ / \ d_H(K, C) < r\}.$$

We will say that a subset $\mathcal{H}$ of $\mathcal{K}(X)$ is said to be *hereditary* if for every $K \in \mathcal{H}$ and every $C \in \mathcal{K}(X)$ with $C \subset K$ then we have $C \in \mathcal{H}$.

We will also need to use meeasures. For any Polish space X, we denote by $P(X)$ the space of all Borel probability measures on X with the weak* topology. We consider the Lévy metric ρ, defined by :

$$\rho(\mu, \nu) = \inf\{\varepsilon > 0 \ / \ \forall A \in \mathcal{B}(X), \mu(A) \leq \nu(A_\varepsilon) + \varepsilon \text{ and } \nu(A) \leq \mu(A_\varepsilon) + \varepsilon\}.$$

We denote

$$B_P(\mu, r) = \{\nu \in P(X) \ / \ \rho(\mu, \nu) < r\}.$$

If G is a Polish group and $\mu, \nu \in P(G)$, we denote by $\mu * \nu$ their *convolution product* defined by :

$$\mu * \nu(A) = \int_G \mu(Ax^{-1}) d\nu(x).$$

1.3 Haar-null sets

After the genral definitions, we are comming closer to Haar-null sets. Let's just define two sets :

Definition. *Let G be a Polish group and $A \subset X$ an universally measurable set. We let :*

$$T(A) = \{\mu \in P(G) \ / \ \forall g_1, g_2 \in G, \ \mu(g_1 A g_2) = 0\},$$

and

$$T_l(A) = \{\mu \in P(G) \ / \ \forall g \in G, \ \mu(gA) = 0\}.$$

And now, here are Haar-null sets :

Definition. *Let G be a Polish group and $A \subset X$ an universally measurable set. A is said to be* Haar-null *if $T(A)$ is not empty. A is said to be* genericaly Haar-null *if $T(A)$ is co-meager.*

Definition. *Let G be a Polish group and $A \subset X$ an universally measurable set. A is said to be* left Haar-null *if $T_l(A)$ is not empty. A is said to be* genericaly left Haar-null *if $T_l(A)$ is co-meager.*

2 Hereditary, dense G_δ sets and measures

After all these definitions, we will be able to start extending the Steinhaus theorem. Here comes the main theorem of the article, the one which will lead to the extension of the Steinhaus theorem :

Theorem. *Let G be an uncountable Polish group and let A be a universally measurable subset of G. Assume that $A^{-1}A$ is meager. Then A is generically left Haar-null.*

It is the contraposed which will interess us to extend the Steinhaus theorem. To prove this theorem, we will some steps. It will start we these three lemmas.

2.1 Preparation

From now, X will be a Polish space, d a compatible complete metric of X and $\mathcal{H}$ a hereditary, dense G_δ subset of $\mathcal{K}(X)$.

Lemma 1. *There exists a sequence $(\mathcal{U}_n)$ of open, dense and hereditary subsets of $\mathcal{K}(X)$ such that $\mathcal{H} = \bigcap_n \mathcal{U}_n$. We will say the sequence $(\mathcal{U}_n)$ is a* normal form *of $\mathcal{H}$.*

Proof. As $\mathcal{H}$ is a dense G_δ set, we can write $\mathcal{H} = \bigcap_n \mathcal{V}_n$ where each $\mathcal{V}_n$ is open and dense but not necessarily hereditary. For every n we define :

$$\mathcal{C}_n = \{K \in \mathcal{K}(X) \ / \ \exists C \subset K \text{ compact with } C \notin \mathcal{V}_n\}.$$

For every $\in \mathbb{N}$, we have that $\mathcal{C}_n$ is closed and $\mathcal{C}_n \cap \mathcal{H} = \varnothing$. We set $\mathcal{U}_n = \mathcal{K}(X) \backslash \mathcal{C}_n$ and $(\mathcal{U}_n)$ is the sequence desired. □

This lemma gives us an information on the srtucture of $\mathcal{H}$. The next one will describe a bit how opens behave in $\mathcal{K}(X)$:

Lemma 2. *Let $\mathcal{U} \subset \mathcal{K}(X)$ be open, dense and hereditary. Also let $x_0, \dots, x_n$ be disctinct points in X and $r > 0$. Then there exists $y_0, \dots, y_n$ disctinct points in X such that $d(x_i, y_i) < r$ for all $i \in \{0, \dots, n\}$ and $\{y_0, \dots, y_n\} \in \mathcal{U}$.*

Proof. We can suppose that $B(x_i, r) \cap B(x_j, r) = \varnothing$ if $i \neq j$ (if it's not the case, we just have to take a smaller r). Let

$$\mathcal{V} = \left\{ K \in \mathcal{K}(x) \ / \ K \subset \bigcup_{i=0}^{n} B(x_i, r) \text{ and } \forall i, K \cap B(x_i, r) \neq \varnothing \right\}.$$

We have that $\mathcal{V}$ is open, so there exists $K \in \mathcal{V} \cap \mathcal{U}$. For every $i \in \{0, \dots, n\}$, we select $y_i \in K \cap B(x_i, r)$. We have that $\{y_0, \dots, y_n\} \subset K \in \mathcal{U}$, so $\{y_0, \dots, y_n\} \in \mathcal{U}$ and the y_i are the one we were looking for. □

Another Lemma linked to Lemma 2 and which leads directly to the key proposition.

Lemma 3. *Let $\mathcal{U} \subset \mathcal{K}(X)$ be open, dense and hereditary. Also let $\varepsilon > 0$. Then the set*

$$G_{\mathcal{U},\varepsilon} = \{\mu \in P(X) \ / \ \exists K \in \mathcal{U}, \ \mu(K) \geq 1 - \varepsilon\}$$

is co-meager in $P(X)$.

Proof. We will show that for every $V \subset P(X)$ open there exists $W \subset V$ such that $W \subset G_{\mathcal{U},\varepsilon}$. This will show that $G_{\mathcal{U},\varepsilon}$ contains a dense open set and so that it is co-meager. Let $V \subset P(X)$ be an open set. As finitely supported measures are dense in $P(X)$, we can take $\nu = \sum_{i=0}^{n} a_i \delta_{x_i}$ and $r > 0$ such that :
(1) $a_i > 0$ and $\sum_{i=0}^{n} a_i = 1$,
(2) $B_P(\nu, r) \subset V$.
The fact (1) just wants to say that ν is a probability measure, and the (2) comes from the density. By Lemma 2, there exists $y_0, \dots, y_n$ distinct points in X with $F = \{y_0, \dots, y_n\} \in \mathcal{U}$ and $d(x_i, y_i) < \frac{r}{2}$. We set $\mu = \sum_{i=0}^{n} a_i \delta_{y_i}$ and we have :
(3) $\rho(\mu, \nu) \leq \frac{r}{2}$.
As $\mathcal{U}$ is open, there exists $\theta > 0$ such that :
(4) $\theta < \min\{\frac{\varepsilon}{3}, \frac{r}{3}\}$,
(5) $B_H(F, 2\theta) \in \mathcal{U}$.
The fact (5) comes from $F \in \mathcal{U}$ and $\mathcal{U}$ is open and the (4) from the fact we could take θ as smaller as we want. Let $W = B_P(\mu, \theta)$. By (2)-(4), we have $W \subset V$. In fact, we have also that $W \subset G_{\mathcal{U},\varepsilon}$ which ends the proof. □

We are now comming to the proposition which is the key in the proof of the theorem.

Proposition 1. *The set*

$$G_{\mathcal{H}} = \{\mu \in P(X) \ / \ \forall \varepsilon > 0, \exists K \in \mathcal{H},\ \mu(K) \geq 1 - \varepsilon\}$$

is co-meager in $P(X)$.

Proof. Let $(\mathcal{U}_n)$ be a normal form of $\mathcal{H}$. For every $n, m \in \mathbb{N}$, let

$$G_{n,m} = \left\{\mu \in P(X) \ / \ \exists K \in \mathcal{U}_n,\ \mu(K) \geq 1 - \frac{1}{m+1}\right\}.$$

By Lemma 3, we have that $G_{n,m}$ is co-meager. So $\bigcap_{n,m} G_{n,m}$ is also co-meager. It is clear that $G_{\mathcal{H}} \subset \bigcap_{n,m} G_{n,m}$. In fact, we have that $G_{\mathcal{H}} = \bigcap_{n,m} G_{n,m}$. □

2.2 The proof of the theorem

Theorem. *Let G be an uncountable Polish group and let A be a universally measurable subset of G. Assume that $A^{-1}A$ is meager. Then A is generically left Haar-null.*

Proof. We take a sequence $\mathcal{C}_n$ of closed nowhere dense sets such that :
(1) $1 \notin \mathcal{C}_n$ for all $n \in \mathbb{N}$,
(2) $A^{-1}A \subset \bigcup_n \mathcal{C}_n$.
For all $n \in \mathbb{N}$, we let :

$$\mathcal{U}_n = \{K \in \mathcal{K}(G) \ / \ K^{-1}K \cap \mathcal{C}_n = \varnothing\}.$$

It is clear that $\mathcal{U}_n$ is hereditary. As the function $f : \mathcal{K}(G) \to \mathcal{K}(G)$ defined by $f(K) = K^{-1}K$ is continuous, $\mathcal{U}_n$ is open. In fact $\mathcal{U}_n$ is also dense in $\mathcal{K}(G)$ for every $n \in \mathbb{N}$. Then we let

$$\mathcal{H} = \bigcap_n \mathcal{U}_n$$

which is a hereditary, dense G_δ subset of $\mathcal{K}(G)$ with $(\mathcal{U}_n)$ as a normal form. Let

$$B_1 = \{\mu \in P(G) \ / \ \forall \varepsilon > 0, \exists K \in \mathcal{H}, \ \mu(K) \geq 1 - \varepsilon\}$$

, it is co-meager in $P(G)$, by Proposition 1. We admit the fact that the sets of non atomic measures in $P(G)$ is co-meager in $P(G)$. So

$$B_2 = \{\mu \in P(G) \ / \ \mu \text{ is non-atomic and } \mu \in B_1\}$$

is also co-meager in $P(G)$ (as intersection of two co-meager sets). And we have that $B_2 \subset T_l(A)$ and $T_l(A)$ is co-meager in $P(G)$. □

3 Consequences of the theorem

This theorem can give informations on analytic subgroups of Polish groups :

Definition. *We say that a subset A of a Polish space X is analytic if there exists a continuous map $f : \mathbb{N}^{\mathbb{N}} \mapsto X$ with $f(\mathbb{N}^{\mathbb{N}}) = A$.*

Corollary 1. *Let G be an uncountable Polish group and let H an analytic subgroup of G with empty interior. Then H is generically left Haar-null.*

It gives also the extension of the Steinhaus theorem we were looking for :

Corollary 2. *Let G be an uncountable Polish group and let A be an analytic subset of G. If A is not generically left Haar-null, then $1 \in Int(A^{-1}AA^{-1}A)$.*

It is in fact a direct consequence of the theorem and Pettis' lemma.

4 The Haar-null sets

We only talked about left Haar-null sets, but what about the Haar-null sets. In fact, if we add a condition we have the same result that the theorem but with $T(A)$.This condition is the following :

(C) For every analytic and meager subset A of G, the conjugate saturation

$$[A] = \{x \in G \ / \ \exists g \in G, \exists a \in A, x = gag^{-1}\}$$

of A is meager.

Proposition 2. *Let G be an uncountable Polish group that satisfies (C). If A is an analytic subset of G such that $A^{-1}A$ is meager. Then A is generically Haar-null.*

The proof is similar to the proof of the theorem, we build a co-meager set B_2 of non-atomic probability measure on G. And we show that $B_2 \subset T(A)$.