ON THE ARTICLE
A GEOMETRIC INTRODUCTION TO FORKING AND THORN-FORKING
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Abstract. Independence relations, which are ternary relations satisfying some specific properties, have been studied in several different context, such as in o-minimal theories, stable theories or simple theories, with forking for instance. One of the aim of the article *A Geometric Introduction To Forking and Thorn-forking* by Hans Adler is to study independence relations in a more general context, and to find weak strict independence relations. This will lead us to define thorn-forking.

Contents

We will work in a "big" saturated model $\mathfrak{M}$, i.e. a model which is big enough for the study and allows us to work only in itself. We will write $(A_1, ..., A_n) \equiv_C (B_1, ..., B_n)$ if there is an automorphism fixing C pointwise and mapping A_i to B_i for all i. AB stands for $A \cup B$.

1. First step toward independence relations

1.1. What is an independence relation?

Definition 1.1. *A ternary relation* $\downarrow$ *between subsets of* $\mathfrak{M}$ *is an* independence relation *if it satisfies the following properties :*

- ***Invariance :***
  If $A \downarrow_C B$ *and* $(A', B', C') \equiv (A, B, C)$*, then* $A' \downarrow_{C'} B'$.
- ***Monotonicity :***
  If $A \downarrow_C B$, $A' \subseteq A$ *and* $B' \subseteq B$*, then* $A' \downarrow_C B'$.
- ***Base monotonicity :***
  If $D \subseteq C \subseteq B$ *and* $A \downarrow_D B$*, then* $A \downarrow_C B$.
- ***Transitivity :***
  If $D \subseteq C \subseteq B$, $B \downarrow_C A$ *and* $C \downarrow_D A$*, then* $B \downarrow_D A$.
- ***Normality :***
  If $A \downarrow_C B$ *then* $AC \downarrow_C B$.

- ***Extension :***
  If $A \downarrow_C B$ and $B \subseteq \hat{B}$, then there is $A' \equiv_{BC} A$ such that $A' \downarrow_C \hat{B}$.
- ***Finite character :***
  If $A_0 \downarrow_C B$ for all finite $A_0 \subseteq A$, then $A \downarrow_C B$.
- ***Local character :***
  For every A, there is a cardinal $\kappa(A)$ such that for any set B there is a subset $C \subseteq B$ of cardinality $|C| < \kappa(A)$ such that $A \downarrow_C B$.

Remark 1.2. For every relation $\downarrow$ satisfying monotonicity, transitivity and normality, $B \downarrow_{CD} A$ and $C \downarrow_D A$ implies $BC \downarrow_D A$.

Proof. By normality, $BCD \downarrow_{CD} A$ and $CD \downarrow_D A$. Then, by transitivity, $BCD \downarrow_D A$ so that $BC \downarrow_D A$ by monotonicity. □

The trivial relation $\downarrow^0$ such that $A \downarrow_C B$ holds for all A, B and C is an independance relation. To compare the independence relations, we introduce the following notion : a ternary relation $\downarrow$ is weaker than a relation $\downarrow^{\#}$ if $A \downarrow_C^{\#} B$ implies $A \downarrow_C B$. The trivial relation is always the weakest independence relation. However the study of this relation does not bring any information. Therefore, we wish to study more complex independence relations called *strict*.

Definition 1.3. *Recalling that $Aut(\mathfrak{M}/B)$ is the set of automorphisms of $\mathfrak{M}$ fixing B pointwise, the* algebraic closure *of B, $acl(B)$, is the set of the tuples of $\mathfrak{M}$ having finite orbit under the action of $Aut(\mathfrak{M}/B)$.*

Definition 1.4. *An independence relation $\downarrow$ is* strict *if it satisfies* ***anti-reflexivity****:*
If $a \downarrow_B a$ then $a \in acl(B)$.

The trivial relation is not strict.
Ternary relations can also satisfy the following useful properties :

- **Full existence**
  For any A, B and C there is $A' \equiv_C A$ such that $A' \downarrow_C B$.
- **Symmetry**
  $A \downarrow_C B$ is equivalent to $B \downarrow_C A$.

Remark 1.5. (1) Any relation satisfying invariance, extension and symmetry also satisfies normality.
(2) Any relation satisfying invariance, monotonicity, transitivity, normality, full existence and symmetry also satisfies extension.

Proof. (1) If $A \downarrow_C B$, then by symmetry $B \downarrow_C A$. However $A \subseteq AC$ so there is $B' \equiv_{AC} B$ so that $B' \downarrow_C AC$ by extension. As $(AC, B, C) \equiv (AC, B', C)$, $B \downarrow_C AC$ by invariance and finally, $AC \downarrow_C B$ by symmetry. □

Theorem 1.6. *Every independence relation $\downarrow$ is symmetric.*

Proof. The proof uses Morley sequences. □

Example 1.7. We consider T the theory of non-empty undirected forests that branch infinitely in every node. In this theory, $acl(A)$ is the set of all nodes on a path between two nodes of A. The following relation is a strict independence relation :
$A \downarrow_C B \Leftrightarrow$ every path from A to B meets $acl(C)$.

We will try to find the weakest strict independence relation.

1.2. **Algebraic independence.** To find an independence relation, we introduce the following relation :

Definition 1.8. *The relation $A \downarrow^a_C B$, which holds if and only if the equation $acl(AC) \cap acl(BC) = acl(C)$ holds is called the* algebraic independence.

Proposition 1.9. *The relation $A \downarrow^a_C B$ satisfies the* full existence condition *and all axioms for strict independence relations except* base monotonicity.

It satisfies base monotonicity if and only if for all algebraically closed sets A, B, C such that $C \subseteq B$, the equation $B \cap acl(AC) = acl(B \cap AC)$ holds.

Proof. By compactness. □

2. Forking

To obtain independence relations, properties can be forced such as the extension axiom with *forking.*

Definition 2.1. *For any relation $\downarrow$, we define the relation $\downarrow^*$ of forking independence : $A \downarrow^*_C B$ holds if and only if for all $\hat{B} \supseteq B$ there is $A' \equiv_{BC} A$ such that $A' \downarrow_C \hat{B}$.*

$A \downarrow^*_C B$ implies $A \downarrow_C B$ and $\downarrow = \downarrow^*$ if and only if $\downarrow$ satisfies the extension axiom. From $\downarrow$ to $\downarrow^*$, only possibly finite character and local character can be lost but extension is obtained.

Lemma 2.2. *If $\downarrow$ is a relation satisfying invariance and monotonicity, then $\downarrow^*$ satisfies invariance, monotonicity and extension. If, moreover, $\downarrow$ satisfies one of the following property, then $\downarrow^*$ also satisfies it : base monotonicity, transitivity, normality, anti-reflexivity, full existence.*

Definition 2.3. *If M is a model, a* n-type over $B \subset M$ *is a maximal set of $\mathcal{L}(B)$-formulas such that all finite subsets of formulas are satisfied by M.*
A type is complete *if it contains φ or the negation of φ for all $\mathcal{L}(B)$-formula φ.*
$S^*(B)$ is the class of complete types over B in arbitrarily long sequences of distinct formal variables.

Definition 2.4. *A model $\mathcal{M}_\mathcal{S}$ is* κ-saturated *if all 1-type over $A \subset \mathcal{M}_\mathcal{S}$ such that $|A| < \kappa$ is realised in $\mathcal{M}_\mathcal{S}$, i.e. for all 1-type over A, $p(x)$, there is $a \in \mathcal{M}_\mathcal{S}$ such that $\mathcal{M}_\mathcal{S}$ satisfied $p(a)$ ($\mathcal{M}_\mathcal{S} \vDash p(a)$).*

Proof.

- **Invariance**
  It comes directly from the invariance of $\downarrow$.

- **Monotonicity**
  If $A \downarrow^*_C B$, $A_0 \subseteq A$ and $B_0 \subseteq B$, then, by definition of $\downarrow^*$, for all $\hat{B} \supseteq B$ there is $A' \equiv_{BC} A$ such that $A' \downarrow_C \hat{B}$. Let $A'_0 \subseteq A'$ correspond to $A_0 \subseteq A$. Then, as $(A', B, C) \equiv (A, B, C)$, $(A'_0, B_0, C) \equiv (A_0, B_0, C)$ by restriction and $A'_0 \downarrow_C B'$ for all $B' \subseteq \hat{B}$ by monotonicity of $\downarrow$. Thus, for all $B' \supseteq B_0$, there is $\hat{B} \supseteq B$ and $A'_0 \equiv_{B_0 C} A_0$ such that $B' \subseteq \hat{B}$ and $A'_0 \downarrow_C B'$: $A_0 \downarrow^*_C B_0$.

- **Extension**
  Suppose $\bar{a} \downarrow^*_C B$ where $\bar{a}$ is a possibly infinite tuple, and let $\hat{B} \supseteq B$ be any superset of B. We obtain that for all $M \supseteq \hat{B}$ there is $a' \equiv_{BC} a$ such that $a' \downarrow_C \hat{B}$.
  We claim that there is a type $p(x) \in S^*(\hat{B}C)$, extending $tp(\bar{a}/BC)$, such that for all cardinal κ there is a κ-saturated model $M \supseteq \hat{B}$ and $\bar{a}' \vDash p(\bar{x})$

such that $\bar{a}' \downarrow_C M$.
If not, then for each $p(\bar{x}) \in S^*(\hat{B}C)$ extending $tp(\bar{a}/BC)$ there is a cardinal $\kappa(p)$ such that no $\kappa(p)$-saturated model $M \supseteq \hat{B}$ there is a tuple $\bar{a}' \models p$ satisfying $\bar{a}' \downarrow_C M$. Let κ be the supremum of the cardinals $\kappa(p)$ and $M \supseteq \hat{B}$ be κ-saturated. Then there is no $\bar{a}' \equiv_{BC} \bar{a}$ such that $\bar{a}' \downarrow_C M$: we have found a contradiction with the definition of $\downarrow^*$.
Choosing $\bar{a}' \models p(\bar{x})$, where $p(\bar{x})$ is as in the claim, clearly $\bar{a}' \equiv_{BC} \bar{a}$ since $p(\bar{x})$ is an extension of $tp(\bar{a}/BC)$, and $\bar{a}' \downarrow^*_C \hat{B}$.

- **Base monotonicity**
If $C \subseteq C' \subseteq B$ and $A \downarrow^*_C B$, then, by definition of $\downarrow^*$, for all $\hat{B} \subseteq B$ there is $A' \equiv_{BC} A$ such that $A' \downarrow_C \hat{B}$. Applying base monotonicity for $\downarrow$, we obtain $A' \downarrow_{C'} \hat{B}$ and finally $A' \downarrow^*_{C'} B$.

- **Transitivity**
By invariance of $\downarrow$, we have the following equivalences :
$$\begin{aligned} A \downarrow^*_C B \quad &\Leftrightarrow \quad (\text{for all } \hat{B} \supseteq B \text{ there is } A' \equiv_{BC} A \text{ such that } A' \downarrow_C \hat{B}) \\ &\Leftrightarrow \quad (\text{for all } \hat{B} \supseteq B \text{ there is an } A' \text{ and an automorphism } f \\ &\qquad \text{fixing } B \text{ and } C \text{ pointwise such that } f(A') = A \text{ and } A' \downarrow_C \hat{B}) \end{aligned}$$
Consider $\hat{B} \supseteq B$ and f such as in the previous equivalence.
Then $(f(A), \hat{B}, f(C)) \equiv (A; f^{-1}(\hat{B}), C)$, f fixes C pointwise and $f(A) \downarrow_C \hat{B}$ so by invariance of $\downarrow$, $A \downarrow_C f^{-1}(\hat{B})$.

Thus, with $\hat{B}' = f^{-1}(\hat{B})$, we obtain :
$$A \downarrow^*_C B \quad \Leftrightarrow \quad (\text{for all } \hat{B} \supseteq B \text{ there is } \hat{B}' \equiv_{BC} \hat{B} \text{ such that } A \downarrow_C \hat{B}')$$

Suppose $D \subseteq C \subseteq B$, $B \downarrow^*_C A$ and $C \downarrow^*_D A$ hold, and $\hat{A} \supseteq A$. We need to show that $B \downarrow_D \hat{A}^*$ for some $\hat{A}^* \equiv_{AD} \hat{A}$. Let $\hat{A}' \equiv_{AD} \hat{A}$ be such that $C \downarrow_D \hat{A}'$ and let $\hat{A}^* \equiv_{AC} \hat{A}'$ be such that $B \downarrow_C \hat{A}^*$. Then $\hat{A}^* \equiv_{AD} \hat{A}$ and $C \downarrow_D \hat{A}^*$. By transitivity of $\downarrow$, we get $B \downarrow_D \hat{A}^*$.

- **Normality**
It comes directly from the normality of $\downarrow$.

- **Anti-reflexivity**
$a \downarrow^*_B a$ implies $a \downarrow_B a$, which implies $a \in acl(B)$ by anti-reflexivity of $\downarrow$.

- **Full existence**
Consider three subsets A,B,C. $\downarrow$ satisfies full existence so $A \downarrow^*_C \varnothing$.Since $\downarrow^*$ satisfies extension there is $A' \equiv_C A$ such that $A' \downarrow^*_C B$.

□

Theorem 2.5. *If $\downarrow$ satisfies invariance, monotonicity, base monotonicity, transitivity, normality and finite character and $\downarrow^*$ satisfies local character, then $\downarrow^*$ is an independence relation.*

Although there are natural examples in which local character can be lost, the existence of a relation $\downarrow$ satisfying invariance, monotonicity and finite character, for which $\downarrow^*$ does not have finite character is still an open question.

3. Thorn-forking

The algebraic independence studied in the first part turns out to satisfy all axioms for independence relations, except possibly monotonicity. As we succeeded in forcing the extension axiom, we will try to force base monotonicity with what is known as *thorn-forking.*

Definition 3.1. *The relation* $\downarrow^M$ *of* M-dividing independence *is defined by* $A \downarrow^M_C B$ *holds if and only if for any* C' *such that* $C \subseteq C' \subseteq acl(BC)$*, the equation*

$$acl(AC') \cap\ acl(BC') = acl(C')$$

also holds.

Definition 3.2. *The relation* $\downarrow^{th}$ *of thorn-forking independence is defined by* $\downarrow^{th} = \downarrow^{M*}$*. It means that* $A \downarrow^{th}_C B$ *holds if and only if for all* $\hat{B} \supseteq B$*, there is* $A' \equiv_{BC} A$ *such that* $A' \downarrow^M_C \hat{B}$*.*

Definition 3.3. *A complete theory is* rosy *if* $\downarrow^{th}$ *is an independence relation.*

Lemma 3.4. *If* $\downarrow$ *is any strict independence relation, then* $A \downarrow_C B$ *implies* $A \downarrow^{th}_C B$*.*

Proof. Suppose $\downarrow$ is any strict independence relation, $A \downarrow_C B$ and $\hat{B} \supseteq B$. By definition of $\downarrow^{th}$, we need to show that there is $A' \equiv_{BC} A$ such that $A' \downarrow^M_C \hat{B}$. By extension of $\downarrow$ there is $A' \equiv_{BC} A$ such that $A' \downarrow_C acl(\hat{B}C)$. For any D satisfying $C \subseteq D \subseteq acl(\hat{B}C)$ base monotonicity of $\downarrow$ gives $A' \downarrow_D acl(\hat{B}C)$. By extension and symmetry of $\downarrow$ there is a set $H \equiv_{A'D} acl(A'D)$ that satisfies $H \downarrow_D acl(\hat{B}C)$. We obtain $H = acl(A'D)$, so $acl(A'D) \downarrow_D acl(\hat{B}C)$. By anti-reflexivity of $\downarrow$, $acl(A'D) \cap acl(\hat{B}CD) \subseteq acl(D)$, so $acl(A'D) \cap acl(\hat{B}CD) = acl(D)$ and $A' \downarrow^M_C \hat{B}$. □

Lemma 3.5. *The relation* $\downarrow^M$ *of M-dividing independence always satisfies invariance, monotonicity, base monotonicity, transitivity, normality and finite character (i.e. all the axioms for independence relations except extension and local character). It also satisfies anti-reflexivity.*

Theorem 3.6. *The relation* $\downarrow^{th}$ *of thorn-forking independence is a strict independence relation if and only if it has local character, if and only if there is any strict independence relation at all. If* $\downarrow^{th}$ *is a strict independence relation, then it is the weakest.*

4. Satisfaction of the axioms for independence relations

Some of the axioms for independence relations, namely *invariance*, *monotonicity*, *base monotonicity*, *transitivity*, *normality* and *finite character*, are satisfied by $\downarrow^a$, $\downarrow^*$, $\downarrow^M$ and $\downarrow^{th}$. Therefore, we will call them *basic axioms for independence relations.* To study relations which can not be independence relation, it is crucial to know the implications between the other axioms (extension, local character) and full existence and symmetry.

Theorem 4.1. *We consider relations* $\downarrow$ *that satisfies basic axioms for independence relations.*

1. *If* $\downarrow$ *satisfies extension and local character, then it also satisfies symmetry and full existence.*
2. *If* $\downarrow$ *satisfies symmetry and full existence, then* $\downarrow$ *also satisfies extension. No other relations between extension, local character, symmetry and full existence hold in general.*

The 11 possible situations are :

	1	2	3	4	5	6	7	8	9	10	11
Extension	×	∘	×	×				×			
Symmetry	∘	×	×		×		×		×		
Full existence	∘	×		×		×	×			×	
Local character	×				×	×					×

where × means that the axiom is assumed and ∘ means that the axiom is implied by assumption.

Conclusion

We have studied the implications between the axioms for independence relations themselves, symmetry and full existence. As some axioms could break when trying to define independence relations, we forced some of them, such as the extension axiom with forking and base monotonicity with thorn-forking. The latter gave the weakest independence relation when there is any. However some questions still remain open : we still do not know whether there can really be a relation $\mathop{\smile\hskip -0.9em ^| \ }$ satisfying invariance, monotonicity and finite character, for which $\mathop{\smile\hskip -0.9em ^| \ }^*$ does not have finite character for instance.