Comments on:

Hyperlinear and Sofic Groups: A brief Guide

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Abstract

This is a comment on the article by Vladimir Pestov, which presents two classes of countable discrete groups : *sophic* and *hyperlinear* groups, which can be defined in very similar terms while their origins are really different.

1 Preliminary Definitions

The definitions we will use at first for our two classes of groups, though really different from the original ones, are based on ultrafilters and ultraproducts which we will define here, but will only serve our definitions.

1.1 Ultrafilters

Definition 1. An *ultrafilter* $\mathcal{U}$ on a set X is a subset of $\mathcal{P}(X)$ with the following properties:

1. $\emptyset \notin \mathcal{U}$
2. if A is a subset of X, then either $A \in \mathcal{U}$ or $X \backslash A \in \mathcal{U}$
3. if A, B are subsets of X, with $A \subset B$ and $A \in \mathcal{U}$, then $B \in \mathcal{U}$
4. if A, B are subsets of X, with A and $B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$

1.2 Ultraproducts

First we will give two definitions of ultraproducts, in case of groups, and normed spaces. Then we see how to define it properly on metric groups, based on the other two definitions.

Definition 2. The *algebraic ultraproduct* of a family of groups $(G_i)_{i\in X}$ through an ultrafilter $\mathcal{U}$ on X is the *quotient group* of their cartesian product $\prod (G_i)_{i\in X}$ by the normal subgroup $\mathcal{N}_{\mathcal{U}} = \{(g_i)_{i\in X} \mid \{i \mid g_i = e_i\} \in \mathcal{U}\}$.

Definition 3. The *normed spaces ultraproduct* of a family of normed spaces $(E_i, N_i)_{i\in X}$ through an ultrafilter $\mathcal{U}$ on X is the quotient of a proper subspace of their cartesian product $\mathcal{E} = \{u \in \prod (E_i)_{i\in X} \mid \sup_{i\in X} (N_i(u_i)) < \infty\}$ by the closed linear subspace $\mathcal{N} = \{u \mid \forall\varepsilon > 0,\ \{i \mid N_i(u_i) < \varepsilon\} \in \mathcal{U}\}$.

Here the quotient is on a subspace of the cartesian product where we can still define a norm, quotiented by a subspace we can call *infinitesimals*. We can even think of it as threads u whose limit along $\mathcal{U}$ is 0, meaning for every $\varepsilon > 0$, $\{i \mid N_i(u_i) < \varepsilon\} \in \mathcal{U}$.

Given a definition of ultraproducts of groups, and a definition of ultraproducts of normed spaces, it is natural to wonder if it gives us a definition of ultraproducts of metric groups. But then we have to consider bi-invariant metrics on our groups, or in general the space $\mathcal{N}$ by which we want to quotient will not be a normal subgroup and thus we will not have a group structure on the quotient. Here is the good definition :

Definition 4. The *ultraproduct of a family of metric groups* with bi-invariant metrics $(G_i, d_i)_{i\in X}$ through an ultrafilter $\mathcal{U}$ on X is the quotient of the subspace of their cartesian product $\mathcal{G} = \{g \in \prod (G_i)_{i\in X} \mid \sup_{i\in X} (d_i(g_i, e_i)) < \infty\}$ by the subgroup $\mathcal{N} = \{(g_i)_{i\in X} \mid \forall\varepsilon > 0,\ \{i \mid d_i(g_i, e_i) < \varepsilon\} \in \mathcal{U}\}$. It is a metric group with the metric $d(g\mathcal{N}, h\mathcal{N}) = \lim_{i\to\mathcal{U}} d_i(g_i, h_i)$.

We can see the link with both definitions given for groups or metric spaces.

Though not all metric groups admit bi-invariant metrics, there are lots of examples two of which will be used in our definitions of sofic and hyperlinear groups:

Example 1. The symmetric group of finite rank, S_n, with the bi-invariant metric given by its Hamming distance $d_H(\sigma, \tau) = \frac{1}{n}\#\{i \mid \sigma(i) \neq \tau(i)\}$.

Example 2. The unitary group of rank n, U_n (matrices in $GL_n(\mathbb{C})$ such that $U^*U = I_n$), with its Hilbert-Schmidt distance which is bi-invariant :

$$d_{HS}(u, v) = ||u - v||_2 = \sup_{X \in \mathbb{C}^n, ||X||_2 < 1} ||(U - V)X||_2$$

2 Sofic and Hyperlinear Groups

2.1 First definition using ultraproducts

Though it is not the first way they were introduced, sofic and hyperlinear groups can be defined using ultraproducts:

> **Definition 5.** A group G is *sofic* if it is isomorphic to a subgroup of a metric ultraproduct of symmetric groups of finite rank with their normalized Hamming distance.

> **Definition 6.** A group G is *hyperlinear* if it is isomorphic to a subgroup of a metric ultraproduct of unitary groups of finite rank with their normalized Hilbert-Schmidt distance.

Remark. A group is *linear* if it is isomorphic to a subgroup of a unitary group of finite rank, thus hyperlinear is a generalization of linear. The term *sofic* was introduced by Benjy Weiss, whereas their first use is due to Gromov.

We know that S_n embeds as a subgroup in U_n with permutation matrices. Does it allow us to deduce that every sofic group is hyperlinear?

> **Theorem 1.** *[Elek and Szabò] Every sofic group is hyperlinear.*

The proof uses the fact that $d_{Ham}(\sigma, \tau) = \frac{1}{2}(d_{HS}(A_\sigma, A_\tau))^2$, which gives us enough links between the metrics to identify an ultraproduct of symmetric groups with its Hamming distance with the ultraproduct of the same symmetric groups embedded in the ultraproduct of unitary groups with the metric induced by the Hilbert-Schmidt distance.

The converse of this theorem is an open question: we do not know if there exist hyperlinear groups that are not sofic.

2.2 Definition without Ultraproducts

One might wonder whether the ultrafilter really influences the definition or not. In fact we can give a definition of sofic (and hyperfinite) groups without mentioning any ultraproduct.

Theorem 2. *[Elek and Szabò] A group G is sofic if and only if for every finite subset $F \subset G$ and for each $\varepsilon > 0$, there exists a $n \in \mathbb{N}$ and a map $\theta : F \to S_n$ such that :*

1. *if g, h, $gh \in F$, then $d_{Ham}(\theta(g)\theta(h), \theta(gh)) < \varepsilon$*
2. *if $e \in F$ then $d_{Ham}(\theta(e), Id) < \varepsilon$*
3. *for all x,$y \in F$, $x \neq y$ implies $d_{Ham}(x, y) > \frac{1}{4}$*

A map θ satisfying 1 and 2 is called a (F, ε)-almost homomorphism.

We can define hyperlinear groups in the same fashion, with almost homomorphisms. And it fact it has been done before the definition in case of sofic groups:

Theorem 3. *[Radulescu] A group G is hyperlinear if and only if for every finite subset $F \subset G$ and for each $\varepsilon > 0$, there exists a $n \in \mathbb{N}$ and a map $\theta : F \to U_n$ such that :*

1. *if g, h, $gh \in F$, then $d_{HS}(\theta(g)\theta(h), \theta(gh)) < \varepsilon$*
2. *if $e \in F$ then $d_{HS}(\theta(e), Id) < \varepsilon$*
3. *for all x,$y \in F$, $x \neq y$ implies $d_{HS}(x, y) > \frac{1}{4}$*

Those two definitions are easier to manipulate than the one with ultrafilters, and one can easily prove some groups are sofic (and hyperlinear) with them:

Example 3 (Finite groups are sofic)**.** Indeed, every finite group of rank n can be seen as a subgroup of S_n, and the morphism given by the usual embedding (Cayley homomorphism C_G defined by: $C_G(g_i) = \sigma_i$ such that for every j, $\sigma_i(j) = k$ where $g_k = g_i g_j$) satisfies 3 because $d_{Ham}(\sigma_i, \sigma_j) = \frac{1}{n}\#\{k \mid g_i g_k \neq g_j g_k\} = 1$ if $i \neq j$.

2.3 Definition with graphs

The first definition of a sofic group (without the name) is due to Gromov, and was stated in terms of graphs in an attempt to solve this conjecture:

Conjecture 1 (Gottschalk Surjunctivity Conjecture)**.** For every countable group G and every finite set A, the shift system A^G contains no proper closed G-invariant subset X isomorphic to A^G itself (as a compact G-space).

The shift is the action of G on A^G, where A is a finite set with its discrete topology, given by $(g \cdot x)_h = x_{g^{-1}h}$.

Gromov solved the conjecture in the case of sofic groups:

Theorem 4. *[Gromov] The Gottschalk surjunctivity conjecture is true for sofic groups.*

Here is the definition he used:

Theorem 5. *[Elek and Szabò] A group G with a finite set of generators V is sofic if and only if it satisfies the following Gromov condition:*

For every $N \in \mathbb{N}$, ans every $\varepsilon > 0$, there is a finite graph Γ V-edge coloured such that for at least $(1-\varepsilon)|\Gamma|$ vertices x of Γ, the N-ball around x is isomorphic to the N-ball in G.

A subgraph of the Cayley graph does not work in general (does not work with free groups in particular, because of border effects), but it does with amenable groups, using Folner condition, and thus every amenable group is sofic.

2.4 Yet another definition

We can also give another definition, in terms of group actions and dynamical systems.

Definition 7. If X is a space with a finitely additive measure μ, a *near action* of a group G on (X, μ) is a map from G to measure preserving maps defined almost-everywhere on X, with $\theta(g)\theta(h) = \theta(gh)$ almost everywhere. Such an action is *essentially free* if for every $g \in G$, $\theta(g)$ has almost no fixed point.

Theorem 6. *[Elek and Szabò] A group G is sofic if and only if it admits an essentially free near-action on a set X equipped with a finitely additive measure μ defined on the algebra of sets $\mathcal{P}(X)$.*

Remark. Every countable discrete group admit an essentially free near-action on a Cantor space with a sigma-additive measure, but defined only on its Borel subsets.

3 Examples, Open questions

3.1 Known sofic groups

- finite groups. See example 3
- residually finite groups

  Using definition of theorem 2 and knowing that for every finite part A of a residually finite group G we can find an homomorphism $\theta : G \to H$ to a finite group injective on A, we can show that $C_H \circ \theta$ is a (F, ε)-almost homomorphism on F.

- non abelian free groups

  They are residually finite.

- amenable groups

  Using Folner condition and definition of theorem 2, we can build an almost-homomorphism:

  Let G be an amenable group, $\varepsilon > 0$ and $A \subset G$ finite. Folner condition assure that there exist $B \subset G$ finite such that for all $a \in A$, $\#\{aB\Delta B\} < \varepsilon \frac{\#(B)}{\#(A)}$.

  Then we consider the map Φ defined by, if $a \in A$, $g \in B$: $\Phi(a)(g) = ag$ if $ag \in B$, extended on B arbitrarily to be a bijection. $C_B \circ \Phi$ is a (A, ε)-almost homomorphism of G into $S_{\#(B)}$.

- initially subamenable groups (LEA : Locally embeddable into amenable groups)

  A group G is *initially subamenable* if looking at finite parts of G one can not assure it is not amenable. Meaning for every finite part A of G, there is an amenable group F with a finite subset B that has the same partial multiplication than A.

  One can define in the same fashion LEF groups (locally embeddable into finite groups).

  Using either definition in theorem 2 or 5 one can see that we only need to look at finite parts of a group to assure its soficity, thus the result.

Remark. There are non LEA groups, LEA groups not LEF.

3.2 Open questions

However, even among non-LEA groups, we don't know if there are non-sofic groups.

$$\begin{array}{ccccccccc} \text{finite} & \Rightarrow & \text{residually finite} & \Rightarrow & \text{LEF} & & & & \\ \Downarrow & & \Downarrow & & \Downarrow & & & & \\ \text{amenable} & \Rightarrow & \text{residually amenable} & \Rightarrow & \text{LEA} & \Rightarrow & \text{sofic} & \Rightarrow & \text{hyperlinear} \end{array}$$

Many open questions remain:

- Is every hyperlinear group sofic?
- Is every group hyperlinear?
- Is every group sofic?

A positive answer to the last question, along with theorem 5, would be a mean to solve Gottschalk surjonctivity conjecture.

Conversely, hyperlinear groups are the only groups on which Connes' embedding conjecture for groups (related to tracial ultraproducts and Von Neumann algebra) is true. An answer (either positive or negative) to the second question would solve this part of Connes' embedding conjecture.

References

[Pestov] V. Pestov : *Hyperlinear and sofic groups: a brief guide*, arXiv.