Math Institut Camille Jordan

Laboratoire de recherche en mathématiques Lyon/Saint-Étienne

Non parametric statistics and global sensitivity analysis tools in the study of (tail) dependence

Statistique non paramétrique et outils de l'analyse de sensibilité globale pour l'étude de la dépendance (asymptotique)

Cécile Mercadier

Habilitation à diriger des recherches

Université Claude Bernard Lyon 1

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Spécialité : **Mathématiques**

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Je suis très reconnaissante envers Claudia Klüppelberg, John Einmahl et Bertrand Iooss d'avoir accepté d'évaluer mon manuscrit. Je vous remercie très chaleureusement. Je suis gênée du temps que je vole aussi aux personnes composant mon jury de soutenance. Ma profonde gratitude va à Patrice Bertail, John Einmahl, Bertrand Iooss, Anne-Laure Fougères, Véronique Maume-Deschamps et Clémentine Prieur d'avoir accepté ce rôle. Votre carrière à vous toutes et tous, rapporteurs et/ou membres du jury, m'inspire sincèrement. Je suis donc honorée tant par vos encouragements lorsque je vous ai présenté ce projet d'HDR, que par votre évaluation positive de mon travail. Merci mille fois pour votre soutien. Je n'oublie pas que chacun des articles présentés est passé entre les mains de rapporteurs anonymes. Ils sont une autre caution pour mon travail, je les remercie également.

Cette rédaction et cette soutenance ont été effectuées pendant ma délégation CNRS. Par delà notre système que je salue, je voudrais grandement remercier trois collègues, à l'intersection de l'Institut Camille Jordan et du département de mathématiques de Lyon 1, Thibault Espinasse, Anne-Laure Fougères et Clément Marteau. Sans leur accord vis-à-vis des conséquences sur nos services d'enseignement, sans leurs encouragements en amont, je n'aurais peut-être pas entrepris encore cette synthèse. Jamais ils ne comptent leur temps, qu'il soit de relecture, de discussion, et de soutien en tout genre.

La contribution de mes collaboratrices et collaborateurs dans ce document est considérable. Si je ne vous cite pas nommément ici, vous l'êtes dans le premier chapitre de ce mémoire qui a pour but de raconter mon parcours. La recherche est indéniablement source d'émotions partagées. Malheureusement, la satisfaction folle qui m'envahit après une de nos trouvailles est trop vite balayée par ce qui vient ensuite. Il faut encore améliorer un argument, la bibliographie, la rédaction, les illustrations ou que sais-je, puis choisir une revue, répondre aux rapporteurs et à l'éditeur... Tout est prétexte pour passer à la suite, encore et plus vite. En écrivant ces mots, sachez que je savoure à nouveau. Merci beaucoup. Christian Genest et Johan Segers sont un peu à part dans cette liste de remerciements car je leur ai accordé moi-même un rôle à part. Si vous n'aviez pas mesuré que nos échanges étaient précieux, soyez sincèrement remerciés.

J'ai d'autres collègues en or ! J'aimerais remercier tous ceux de l'Institut Camille Jordan et du département de mathématiques de Lyon 1 pour les moments que nous partageons joyeusement au quotidien. J'ai une pensée appuyée pour mes collègues (ex-)toulousains avec lesquels mes liens ne se sont jamais défaits malgré le temps et la distance. J'adresse aussi un clin d'oeil aux extrémistes de France et d'ailleurs.
Au lieu de vous citer toutes et tous (ce qui prendrait un temps dont je ne dispose plus), je préfère vous démontrer mon plaisir de vous voir à Lyon, lors de mes visites ou lors de conférences.

Volontairement, ce mémoire n'aborde pas les volets enseignement et diffusion, bien que ces missions jouent un rôle tout aussi essentiel dans nos vies professionnelles. Je remercie les étudiantes et étudiants de leur confiance, en particulier ceux que j'ai guidés au mieux pendant mon mandat de responsable de la filière Master 1 Mathématiques appliquées et Statistique. Beaucoup d'entre eux, ainsi que certains ayant suivi mes UE majeures, me tiennent informée de la suite de leur parcours, ce qui fait agréablement écho à mon investissement.

Ça y est ... je suis arrivée à ce moment critique des remerciements qui cherche la bonne mesure entre pudeur et convivialité. Alors, je les terminerai main dans la main avec Olivier, Louise et Nina. Elles sont nos deux merveilles, transformant notre quotidien en aventures parentales exquises. Tous quatre adressons un immense merci à tous nos proches.

Vincent van Gogh, *les Oliviers* (1889).
National Gallery of Scotland, Édimbourg, Écosse.

Andaluces de Jaén,
aceituneros altivos,
decidme en el alma: ¿quién,
quién levantó los olivos?

No los levantó la nada,
ni el dinero, ni el señor,
sino la tierra callada,
el trabajo y el sudor.

Unidos al agua pura
y a los planetas unidos,
los tres dieron la hermosura
de los troncos retorcidos.

Levántate, olivo cano,
dijeron al pie del viento.
Y el olivo alzó una mano
poderosa de cimiento.

Andaluces de Jaén,
aceituneros altivos,
decidme en el alma: ¿quién
amamantó los olivos?

Vuestra sangre, vuestra vida,
no la del explotador
que se enriqueció en la herida
generosa del sudor.

No la del terrateniente
que os sepultó en la pobreza,
que os pisoteó la frente,
que os redujo la cabeza.

Árboles que vuestro afán
consagró al centro del día
eran principio de un pan
que sólo el otro comía.

¡Cuántos siglos de aceituna,
los pies y las manos presos,
sol a sol y luna a luna,
pesan sobre vuestros huesos!

Andaluces de Jaén,
aceituneros altivos,
pregunta mi alma: ¿de quién,
de quién son estos olivos?

Jaén, levántate brava
sobre tus piedras lunares,
no vayas a ser esclava
con todos tus olivares.

Dentro de la claridad
del aceite y sus aromas,
indican tu libertad
la libertad de tus lomas.

Miguel Hernández, *Aceituneros* (1937).

Contents

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Synthèse de mon parcours scientifique

Ce mémoire d'habilitation à diriger des recherches synthétise mes travaux qui s'inscrivent à l'interface de la théorie des valeurs extrêmes, de l'analyse de sensibilité et de l'inférence non paramétrique. Les liens entre les deux premiers thèmes ne sont pas évidents mais leur combinaison s'est révélée féconde. Avant de les exposer plus largement dans ce manuscrit, j'ai choisi de raconter mon parcours fait de nombreuses collaborations. Mes publications sont rappelées pages 53-54 et apparaissent dans le texte sous la forme [Mn]. Les autres références, numérotées simplement [n], sont listées à la fin de ce document.

Toulouse - Paris

J'ai effectué ma thèse au sein du laboratoire de Statistique et Probabilités de l'université Paul Sabatier, Toulouse 3, sous la direction de Jean-Marc Azaïs. Elle était intitulée *Extrema de processus stochastiques. Propriétés asymptotiques de tests d'hypothèses*, [M9]. Les recherches dans ce domaine sont parfois motivées par la modélisation des états de la mer comme la hauteur significative des vagues et la période de ces dernières afin de comprendre les risques qu'encourent les structures des navires. On peut consulter [141], [121] et [42] par exemple et plus généralement les nombreuses publications de Igor Rychlik ou Georg Lindgren à ce sujet. Pendant ma thèse, mes recherches étaient consacrées à l'étude de la distribution des extrema des processus stochastiques, parfois gaussiens, dépendant de un ou plusieurs paramètres. Elle comprenait également l'application de ces résultats pour l'étude de la statistique de test du rapport de vraisemblance, en collaboration avec Elisabeth Gassiat et Jean-Marc Azaïs, voir [M1] qui conduira plus tard aussi à [M2]. Pour la partie processus, j'ai obtenu des résultats de nature asymptotique (en temps) et des résultats non-asymptotiques, en introduisant des hypothèses de régularité dans un contexte non nécessairement gaussien. Les expériences numériques et la mise à disposition d'une boite à outils ont permis d'évaluer la pertinence des bornes proposées. Ces recherches m'ont, entre autres, amené à participer à la conférence internationale EVA organisée cette année-là à Gotheborg. Lors de ces journées, j'avais eu énormément de plaisir à présenter ce qui deviendrait [M10]. J'ai un souvenir très précis de la présence dans la salle de Robert Adler. Il est l'un des fondateurs de l'analyse géométrique des processus stochastiques, tout comme mon directeur de thèse et notre regretté Mario Wschebor, voir [2] et [4]. J'ai aussi le souvenir d'avoir été piquée par les exposés auxquels j'avais assisté. Je découvrais à Gotheborg tout ce que la théorie des valeurs extrêmes pouvait englober. L'ambiance de cette conférence était vraiment bonne, et je me promettais de reprendre largement la bibliographie associée, dès que la situation me le permettrait.

L'année suivante, je découvrais l'Histoire des mouvements étudiants à l'université Paris-Nanterre en plein CPE. L'équipe de Modal'X, qui comprenait déjà Patrice Bertail et Philippe Soulier, m'a recruté sur un poste d'ATER à temps plein. J'avais donc beaucoup d'heures de cours à assurer, sauf pendant les huit semaines de mobilisation aigüe. Pendant ces mois parisiens, j'étais parvenue à répondre à une question posée par Bernard Bercu lors de ma soutenance de thèse. Il s'agissait de prouver la convergence presque sûre des extrêmes d'un processus Gaussien non stationnaire, faiblement dépendant. Ce travail est resté un preprint... la concurrence ayant été plus rapide.

La Statistique dans mon laboratoire de recherche

Je suis recrutée comme MCF à Lyon 1, et son Institut Camille Jordan fraichement renommé et restructuré. A l'époque, mon chef d'équipe est mon collègue Stéphane Attal. L'inititulé du poste associé à mon recrutement n'est d'ailleurs pas Statistique mais Probabilités Appliquées, démontrant que la Statistique dans ce laboratoire de mathématiques n'avait pas fait sa place jusque-là. Ce changement était aussi lancé par le fait que l'ICJ intègrait désormais les membres du laboratoire du sud de Lyon dont André Goldman assurait la direction et qui comptait parmi ses membres Gabriela Ciuperca par exemple. Nous collaborons pour estimer le paramètre dit de second ordre dans [M4] permettant d'appréhender la vitesse de convergence en théorie des valeurs extrêmes univariées.
Au moment de mon recrutement, on m'annonce que l'équipe de Statistique de l'ICJ sera renforcée, à commencer par le recrutement d'un poste de Professeur peu de temps après, et les promesses sont tenues. Dans l'ordre d'arrivée à l'Institut Camille Jordan, la Statistique se trouve au fil du temps représentée par : Anne-Laure Fougères, Irène Gannaz, Samuela Leoni-Aubin, Marianne Clausel-Lesourd, Jean-Baptiste Aubin, Céline Vial, Thibaut Espinasse, Céline Helbert, François Wahl, Mathieu Sart, Clément Marteau, Yohann De Castro et Gilles Cohen. En parallèle de ces recrutements, Véronique Maume-Deschamps, Esterina Masiello et Pierre Ribereau ont changé d'affiliation et ont rejoint l'ICJ. Ne m'autorisant pas à produire davantage de noms dans ce paragraphe, j'associe ici les collègues de la région qui participent à notre séminaire RSL.

Des recherches inspirées par des thèmes appliqués

Il est fréquent de croiser des repères de niveaux d'eau historiques le long des rivières. Lorsqu'on se pose la question de savoir quelle était la probabilité de dépasser cette hauteur, ou de la hauteur associée à une probabilité très faible d'être dépassée, on évoque ici la notion de niveau retour, introduite historiquement par les hydrologues. Plusieurs modèles permettent de fournir une estimation. Parmi eux, le modèle de Weibull qui caractérise la queue de distribution par une fonction à variations lentes et un paramètre appelé indice de queue de Weibull. Dans [M16], aux côtés de Philippe Soulier, la vitesse de convergence, au sens minimax, d'un estimateur à noyaux de ce paramètre est obtenue en prouvant que les bornes supérieure et inférieure coïncident. On fournit également un choix automatique de la fraction de données, considérées comme extrêmes, sur laquelle repose notre estimateur.
En dehors de l'hydrologie, ce sont aussi les domaines de l'assurance, de la finance et de la gestion des risques qui ont apporté de nombreuses questions à la théorie des valeurs extrêmes. Voir [58] et [13]. Les produits d'assurance et financiers reposent sur des modèles complexes ayant des entrées dépendantes et des queues de distributions plus ou moins lourdes. On se demande alors si la dépendance est gommée ou non asymptotiquement par la présence d'un facteur dominant. Dans [M6], nous avons progressé avec Anne-Laure Fougères sur la caractérisation des comportements asymptotiques de ces combinaisons pondérées. Dans [M8], et en collaboration avec Laurens de Haan et Chen Zhou, nous revenons à un cadre univarié et traitons deux problèmes majeurs dans l'application de l'analyse des valeurs extrêmes aux séries chronologiques financières, à savoir la correction du biais et le traitement de la dépendance temporelle.
De façon plus surprenante, c'est l'industrie automobile qui m'apportera plus tard une nouvelle collaboration. Certains de ses défis majeurs sont la réduction des émissions de gaz à effet de serre, la dépendance aux combustibles fossiles et la pollution locale. La calibration d'un moteur consiste à déterminer expérimentalement son réglage optimal. Afin de visiter les états possibles, une stratégie consiste à répartir les points uniformément dans toute la région expérimentale pour couvrir tout l'espace d'entrée. Cette technique s'appelle le *space-filling* design. C'est dans [M18] que mes deux collègues Céline Helbert et François Wahl me permettront d'apporter quelques considérations, basées sur la théorie des valeurs extrêmes univariées, sur le comportement asymp-

totique de leur indice renormalisé.

Les contrats de recherche

Je mentionne ci-dessous les projets de recherche auxquels j'ai pris part :
- Projet européen SEAMOCS 2005-2009 porté par Georg Lindgren de l'Université de Lund, *Applied stochastic models for ocean engineering, climate and safe transportation.*
- Projet Université Toulouse 3-CNES-Thalès SA 2008-2010 *Outils de mesure d'intégrité.* Coordonné par Jean-Marc Azaïs (Toulouse 3), Jean-Christophe Levy (THALES) et Suard Norbert (CNES). Collaboration scientifique avec Sébastien Gadat et Agnès Lagnoux.
- Projet ANR 2009-2011 *Approches Spatio-temporelles pour la modélisation du risque* porté par Véronique Maume-Deschamps.
- Co-responsable avec Anne-Laure Fougères du Contrat de Collaboration avec Secteur R&D d'EDF Chatou, avec Marta Nogaj à la tête du projet côté partenaire industriel finançant un post-doc et deux stagiaires de M2 entre 2011 et 2012.
- Co-responsable avec Anne-Laure Fougères du Contrat de Collaboration avec Secteur R&D d'EDF Clamart, avec Marie Gallois, Anne Dutfoy et Sylvie Parey côté partenaire industriel, finançant une thèse entre 2013 et 2016.
- Projet LEFE-MANU (MULTIRISK) entre 2014 et 2016. Coordonné par Clémentine Prieur.

Les deux contrats avec EDF susmentionnés m'ont permis de co-encadrer, avec Anne-Laure Fougères, le travail de Juan-Juan Cai en postdoc et la thèse de Quentin Sebille. La première était motivée par l'analyse du risque d'inondation d'une centrale nucléaire implantée sur le littoral français. Nos données journalières étaient extraites d'un atlas de houle et consistaient en la surcote maximale et la force du vent maximale. Quelques mois plus tard, Gilles Nicolet y intègrera une covariable circulaire, la direction du vent, pendant son stage de M2. Dans [M3], nous nous sommes intéressées à deux variables quantitatives mesurées au cours du temps, formant ainsi une série temporelle bivariée supposée stationnaire. Nous devions estimer une probabilité de défaillance, définie comme la probabilité que les deux variables soient grandes simultanément. Nous avons mis en concurrence trois méthodes basées sur la théorie univariée des valeurs extrêmes pour appréhender la valeur des niveaux retour. De son côté, la théorie multivariée fournira des estimateurs prenant ou non en compte la dépendance asymptotique. De nouveau, nous avons mis en concurrence plusieurs méthodes d'estimation. Dans [M17], nous avons accompagné Quentin à modéliser les précipitations extrêmes en France à l'aide de modèles spatiaux de valeurs extrêmes permettant d'inclure la dépendance spatiale de nos données. De façon concrète, il a entrepris durant sa thèse de répondre aux questions qu'on peut formuler sous la forme : 1) Quel est la probablilité que sur un ensemble de stations météorologiques un événement soit observé dont l'intensité dépasse celle du niveau retour centennal ? 2) Quelle est la probabilité conditionnelle pour qu'une ou plusieurs positions connaissent un dépassement du niveau retour centennal sur un jour donné, sachant que c'est le cas sur une autre partie du réseau de stations météorologiques ? Pour répondre au premier point, il s'est approprié la notion de processus max-stables, en portant une attention toute particulière à un modèle hiérarchique qu'il implémentera sous la forme d'un package R. Pour le second, il a apporté une réponse grâce aux processus de Pareto.

Un aperçu rapide du contenu de ce manuscrit

Tout ce que je viens de mentionner n'apparaît pas davantage dans ce mémoire. J'ai choisi d'y présenter ce qui compte parmi mes publications les plus récentes,

[M14] C. Mercadier and O. Roustant. “The tail dependograph”. In: *Extremes* 22.2 (2019)
[M13] C. Mercadier and P. Ressel. “Hoeffding–Sobol decomposition of homogeneous co-survival functions: from Choquet representation to extreme value theory application”. In: *Dependence Modeling* 9.1 (2021)
[M15] C. Mercadier, O. Roustant, and C. Genest. “Linking the Hoeffding–Sobol and Möbius formulas through a decomposition of Kuo, Sloan, Wasilkowski, and Woźniakowski”. In: *Statistics and Probability Letters* 185 (2022)
[M12] C. Mercadier. “Testing copula-based dependence hypotheses: a proofreading based on functional decompositions”. *In Review.* 2023
[M11] C. Mercadier. *Sensitivity Analysis Tools for Dependence and Asymptotic Dependence.* (2023). URL: https://CRAN.R-project.org/package=satdad

ou qui s’inscrivaient très facilement dans cette histoire globale,

[M7] A.-L. Fougères, C. Mercadier, and J. P. Nolan. “Dense classes of multivariate extreme value distributions”. In: *Journal of Multivariate Analysis* 116.C (2013)
[M5] A.-L. Fougères, L. De Haan, and C. Mercadier. “Bias correction in multivariate extremes”. In: *The Annals of Statistics* 43.2 (2015)

afin d’assurer une certaine continuité dans la lecture.

Au centre de ce texte se trouve la *stable tail dependence function* qui caractérise la dépendance asymptotique dans les modèles multivariés. J’ai entrepris son étude aux côtés de Anne-Laure Fougères, Laurens de Haan, John Nolan et Paul Ressel . Elle dispose de propriétés mathématiques intéressantes et non triviales, rappelées dans le Chapitre 3. Cette fonction, à valeurs réelles positives, est définie sur un espace de dimension égale au nombre de variables conjointement étudiées. Elle devient additive sur les composantes asymptotiquement indépendantes. Or cette recherche de structure additive est aussi obtenue en analyse globale de sensibilité via les indices d’importance superset. C’est la raison pour laquelle j’ai commencé à m’intéresser avec Olivier Roustant à ce champ de recherche. Rapidement, on a espéré pouvoir l’appliquer à la notion d’indépendance, travail initié avec Christian Genest. Après quelques tentatives, nous avons finalement essayé de comprendre les points communs et les différences entre deux décompositions fonctionnelles phares. Munie de cet éclaicissement, je me suis rendue compte que je pouvais dégager un cadre très général permettant de traiter divers tests d’hypothèses sur les copules, cadre qui s’appuierait sur la décomposition fonctionnelle la plus adéquate à l’hypothèse testée. Ces éléments sont décrits dans le Chapitre 4. Plusieurs notions étudiées et présentées dans ce manuscrit le sont également d’un point de vue inférentiel, à l’aide de mesures principalement non paramétriques. C’est l’objet du Chapitre 5. Je termine ce document avec quelques perspectives de recherche.

2

A synopsis of my scientific career

This habilitation thesis summarizes my work at the interface of extreme value theory, sensitivity analysis, and non-parametric inference. The connections between the first two themes are not obvious, but their combination has proven fruitful. Before presenting them in more detail in this manuscript, I would like to share my collaborative experiences. My publications are listed on pages 53-54 and are referenced in the text as [Mn]. Other references, simply numbered [n], are listed at the end of this document.

Toulouse - Paris

I completed my thesis within the Statistics and Probability Laboratory of Paul Sabatier University, Toulouse 3, under the supervision of Jean-Marc Azaïs. Its title was *Extrema de processus stochastiques. Propriétés asymptotiques de tests d'hypothèses*, [M9]. Research in this field is sometimes motivated by modeling the states of the sea, such as the significant wave height and period, in order to understand the risks that ships' structures face. For example, see [141], [121] et [42], as well as the numerous publications by Igor Rychlik and Georg Lindgren on this topic. During my thesis, my research was devoted to studying the distribution of extrema of stochastic processes, sometimes Gaussian, depending on one or several parameters. It also included the application of these results to the study of the likelihood ratio test statistic, in collaboration with Elisabeth Gassiat and Jean-Marc Azaïs, see [M1], which later also led to [M2]. For the process part, I obtained asymptotic (in time) and non-asymptotic results by introducing regularity hypotheses in a non-necessarily Gaussian context. Numerical experiments and the publication of a toolbox helped evaluate the relevance of the proposed bounds. These studies, among other things, led me to participate in the EVA international conference held this year in Gothenburg. During these days, I had great pleasure in presenting what would become [M10]. I have a very precise memory of the presence of Robert Adler in the room. He is one of the founders of the geometric analysis of stochastic processes, just like my thesis director and our late Mario Wschebor, see [2] and [4]. I also remember being intrigued by the presentations I attended. In Gothenburg, I discovered everything the theory of extreme values could encompass. The atmosphere of this conference was really good, and I promised myself to extensively review the associated bibliography as soon as possible.
The following year, I discovered the History of Student Movements at Paris-Nanterre University during the CPE protests. The Modal'X team, which already included Patrice Bertail and Philippe Soulier, recruited me for a full-time ATER position. So I had a lot of teaching hours to cover, except during the eight weeks of acute mobilization. During those months in Paris, I managed to answer a question asked by Bernard Bercu during my thesis defense. It was about proving the almost sure convergence of extremes of a non-stationary, weakly dependent Gaussian process. This work remained as a preprint... another submission was published before your work.

Statistics in my research laboratory

I was hired as an associate professor at Lyon 1, and its newly renamed and restructured Camille Jordan Institute (ICJ). At that time, my team leader was my colleague Stéphane Attal. Interestingly, the title of the position associated with my recruitment was not Statistics but Applied

Probability, demonstrating that Statistics had not yet found its place in this mathematics laboratory. This change was also felt by the fact that the ICJ now encompassed members of the laboratory from the south of Lyon, whose director was André Goldman and whose members included Gabriela Ciuperca, for example. We collaborate to estimate the so-called second-order parameter in [M4]. This index is involved in the rate of convergence associated with extrapolations in univariate extreme value theory.
At the time of my recruitment, I was informed that the Statistics team at ICJ would be strengthened, starting with a Professor position shortly thereafter, and the promises were kept. In the order of arrival at the Camille Jordan Institute, Statistics is over time being represented by: Anne-Laure Fougères, Irène Gannaz, Samuela Leoni-Aubin, Marianne Clausel-Lesourd, Jean-Baptiste Aubin, Céline Vial, Thibaut Espinasse, Céline Helbert, François Wahl, Mathieu Sart, Clément Marteau, Yohann De Castro et Gilles Cohen. In addition to these recruitments, Véronique Maume-Deschamps, Esterina Masiello, and Pierre Ribereau changed their affiliation and joined the ICJ. Without allowing me to produce more names in this paragraph, I mention here colleagues from the region who participate in our RSL seminar.

Researches inspired by applied topics

It is common to come across historical water level markers along rivers. When one asks what the probability was of exceeding this threshold, or the height associated with a very low probability of being exceeded, one refers here to the notion of a return level, historically introduced by hydrologists. Several models could be used to estimate it. Among them, the Weibull model characterizes the tail distribution by a slowly varying function and a parameter called the Weibull tail index. In [M16], with Philippe Souiler, the convergence rate, in the minimax sense, of a kernel estimator of this parameter is obtained by proving that the upper and lower bounds coincide. An automatic choice of the fraction of data, considered as extreme, on which our estimator is based is also provided.
In addition to hydrology, the fields of insurance, finance, and risk management have brought numerous questions to extreme value theory, as illustrated in [58] and [13]. Insurance and financial products rely on complex models with dependent inputs and distributions with more or less heavy tails. One then wonders if the dependence is asymptotically erased by the presence of a dominant factor. In [M6], we made progress with Anne-Laure Fougères on characterizing the asymptotic behaviors of these weighted combinations. In [M8], in collaboration with Laurens de Haan and Chen Zhou, we go back to a univariate framework and address two major problems in the application of extreme value analysis to financial time series, namely the correction of bias and the treatment of serial dependence.
More surprisingly, it is the automotive industry that will later bring me a new collaboration. Some of its major challenges are the reduction of greenhouse gas emissions, dependence on fossil fuels, and local pollution. The calibration of an engine consists of experimentally determining its optimal settings. In order to explore possible states, one strategy is to evenly distribute points throughout the experimental region to cover the entire input space. This technique is called the *space-filling* design. It is in [M18] that my two colleagues Céline Helbert and François Wahl allowed me to bring some considerations, based on univariate extreme value theory, on the asymptotic behavior of their renormalized index.

Research contracts

Below are the research projects in which I have participated:
- European project SEAMOCS 2005-2009 led by Georg Lindgren from Lund University, *Applied stochastic models for ocean engineering, climate and safe transportation.*
- University of Toulouse 3-CNES-Thalès SA project 2008-2010 *Integrity measurement tools.* Coordinated by Jean-Marc Azaïs (Toulouse 3), Jean-Christophe Levy (THALES) and Suard

Norbert (CNES). Scientific collaboration with Sébastien Gadat and Agnès Lagnoux.
- ANR project 2009-2011 *Spatio-temporal approaches for risk modeling* led by Véronique Maume-Deschamps.
- Co-responsible with Anne-Laure Fougères for the Collaboration Contract with EDF Chatou R&D Sector, with Marta Nogaj leading the project on the industry partner side financing a post-doc and two M2 interns between 2011 and 2012.
- Co-responsible with Anne-Laure Fougères for the Collaboration Contract with EDF Clamart R&D Sector, with Marie Gallois, Anne Dutfoy and Sylvie Parey on the industry partner side financing a PhD thesis between 2013 and 2016.
- LEFE-MANU (MULTIRISK) project 2014-2016. Coordinated by Clémentine Prieur.

The two contracts with EDF mentioned above allowed me to co-supervise, with Anne-Laure Fougères, the work of Juan-Juan Cai in postdoc and the thesis of Quentin Sebille. The first was motivated by the analysis of the flooding risk of a nuclear power plant located on the French coast. Our daily data were extracted from a wave atlas and consisted of the maximum surge and maximum wind force. A few months later, Gilles Nicolet integrated a circular covariate, the wind direction, during his M2 internship. In [M3], we were interested in two quantitative variables measured over time, forming a stationary bivariate time series. We had to estimate a probability of failure, defined as the probability that the two variables are simultaneously large. We compared three methods based on univariate extreme value theory to estimate the value of return levels. On the other hand, multivariate theory provided estimators taking into account or not the asymptotic dependence. Again, we compared several estimation methods. In [M17], we assisted Quentin in modeling extreme precipitation in France using spatial models of extreme values that allowed for including the spatial dependence of our data. In concrete terms, during his thesis, he set out to answer questions that can be formulated as following: 1) What is the probability that an event with an intensity exceeding that of the centennial return level will be observed in a set of meteorological stations? 2) What is the conditional probability that one or more positions will experience an exceedance of the centennial return level on a given day, given that this is the case in another part of the meteorological station network? To answer the first point, he appropriated the notion of max-stable processes, paying particular attention to a hierarchical model that he implemented in the form of an R package. For the second point, he provided an answer through the use of Pareto processes.

A quick overview of the contents of this manuscript

This manuscript does not include the content mentioned earlier. I decided to present the most recent publications,

[M14] C. Mercadier and O. Roustant. "The tail dependograph". In: *Extremes* 22.2 (2019)
[M13] C. Mercadier and P. Ressel. "Hoeffding–Sobol decomposition of homogeneous co-survival functions: from Choquet representation to extreme value theory application". In: *Dependence Modeling* 9.1 (2021)
[M15] C. Mercadier, O. Roustant, and C. Genest. "Linking the Hoeffding–Sobol and Möbius formulas through a decomposition of Kuo, Sloan, Wasilkowski, and Woźniakowski". In: *Statistics and Probability Letters* 185 (2022)
[M12] C. Mercadier. "Testing copula-based dependence hypotheses: a proofreading based on functional decompositions". *In Review.* 2023
[M11] C. Mercadier. *Sensitivity Analysis Tools for Dependence and Asymptotic Dependence.* (2023). URL: https://CRAN.R-project.org/package=satdad

or the publications that easily fit into this overall story,

[M7] A.-L. Fougères, C. Mercadier, and J. P. Nolan. “Dense classes of multivariate extreme value distributions”. In: *Journal of Multivariate Analysis* 116.C (2013)
[M5] A.-L. Fougères, L. De Haan, and C. Mercadier. “Bias correction in multivariate extremes”. In: *The Annals of Statistics* 43.2 (2015)

in order to ensure a certain continuity in the reading.

At the core of this text is the stable tail dependence function that characterizes the asymptotic dependence in multivariate models. I studied it alongside Anne-Laure Fougères, Laurens de Haan, John Nolan, and Paul Ressel. It has interesting and non-trivial mathematical properties, which are recalled in Chapter 3. This function, taking non-negative real values, is defined on a space of dimension equal to the number of jointly studied variables. It becomes additive on asymptotically independent components. This search for additive structure is also obtained in global sensitivity analysis through superset importance indices. This is why I started to take an interest in this research field with Olivier Roustant. We hoped to apply it to the notion of independence, a task initiated with Christian Genest. After several attempts, we finally tried to understand the similarities and differences between two well-known functional decompositions. Equipped with this clarification, I realized that I could develop a very general theory for treating various testing hypotheses on copulas, a framework based on the most appropriate functional decomposition for the hypothesis under study. These elements are described in Chapter 4. Several notions studied and presented in this manuscript are also examined from an inferential perspective using mainly non-parametric measures. This is the subject of Chapter 5. I conclude this document with some research perspectives.

3

The stable tail dependence function

Stable tail dependence functions (stdf) play a central role to describe the asymptotic dependence between components of a random vector. Modeling the tail dependence remains a main challenge in multivariate extreme value theory. This chapter focuses on theoretical considerations and results obtained on the stdf in [M5], [M13], [M7] and [M11]. Before handling its empirical treatment, that is postponed to Chapter 5, the first part below recalls the main assumptions. The latter will thus ensure existence of the stdf, consistency within estimation and characterization of the associated asymptotic behavior including bias identification. The second part presents four bivariate settings under which calculations have been taken as far as possible. This list, fully transcribed here, is still used as simulation models, see [170] and the references therein. In the third part, the main characteristics of the stdf are presented: multivariate monotonicity and homogeneity. By playing with a change of norm and a change in the measure of reference, the stdf is linked in the last part to the concept of the scale function. The latter has allowed us to construct a new model, as well as to prove that certain classes of distributions are dense among multivariate extreme value models.

3.1 Notation and assumptions

The extremal dependence structure can be described *via* the stdf ℓ, firstly introduced by [96]. For any arbitrary dimension d, consider a multivariate vector $\mathbf{X} = (X^{(1)}, \ldots, X^{(d)})$ with continuous marginal cumulative distribution functions (cdf) $F_1, \ldots, F_d$. The stdf is defined for each positive reals $x_1, \ldots, x_d$ as

$$\lim_{t\to\infty} t\,\mathbb{P}\{1 - F_1(X^{(1)}) \le t^{-1}x_1 \text{ or } \ldots \text{ or } 1 - F_d(X^{(d)}) \le t^{-1}x_d\} = \ell(x_1, \ldots, x_d)\,.$$

The previous limit can be rewritten as

$$\lim_{t\to\infty} t\,[1 - F\{F_1^{-1}(1 - t^{-1}x_1), \ldots, F_d^{-1}(1 - t^{-1}x_d)\}] = \ell(x_1, \ldots, x_d)\,, \tag{3.1}$$

where F denotes the multivariate cdf of the vector $\mathbf{X}$, and $F_j^{-1}(t) = \inf\{z \in \mathbb{R} : F_j(z) \ge t\}$ for any $j = 1, \ldots, d$. *The limit* (3.1) *exists and is non degenerate* is an assumption equivalent to the classical assumption of existence of a multivariate domain of attraction for the componentwise maxima. We recall that the assumption F *is in the domain of attraction of an extreme value distribution with cdf* G supposes the existence for $j = 1, \ldots, d$ of sequences $a_n^{(j)} > 0$, $b_n^{(j)}$ of real numbers and a cdf G with nondegenerate marginals such that

$$\lim_{n\to\infty} \mathbb{P}(\max\{X_1^{(1)}, \ldots, X_n^{(1)}\} \le a_n^{(1)}x_1 + b_n^{(1)}, \ldots, \max\{X_1^{(d)}, \ldots, X_n^{(d)}\} \le a_n^{(d)}x_d + b_n^{(d)}) = G(\mathbf{x})$$

for all points $\mathbf{x}$ where G is continuous. Denote by G_j the jth marginal cdf of G. It is possible to show that the domain of attraction condition can be expressed as the condition (3.1) along with the convergence of the marginal distributions to the G_j's, and that

$$\ell(\mathbf{x}) = -\log G\left(\{-\log G_1\}^{-1}(x_1), \ldots, \{-\log G_d\}^{-1}(x_d)\right)\,. \tag{3.2}$$

The stdf is not the unique tool for capturing the asymptotic dependence. One can refer for instance to the Pickands function (see [61], [169] and [21]), the spectral measure ν, the exponent measure $\mu^\star$ (see Section 3.3.3 for their links with ℓ or [50] and [56] for first results on their estimation), the extreme value copula C_G (see Section 4.4.1 for its relation to ℓ, see also [61] and [33]), the tail dependence function that would be defined using *and* instead of *or* in the first probability above (denoted by R in [49], [45] or [30] under a bivariate setting and by $T^{\mathbf{X}}_{\text{upper}}$ in [90] under a multivariate framework). More details on multivariate extreme value theory can be found in [86], [46], [62], [6] and [156]. Several conditions are now described. The first two have been introduced by [85] whereas the third one comes from ([M5], Section 2).

- the *first order condition* consists of assuming that the limit given in (3.1) exists, and that the convergence is uniform on any $[0,T]^d$, for $T>0$. Note that pointwise convergence in (3.1) entails uniform convergence on the square $[0,T]^d$. See for instance of [84].

- the *second order condition* consists of assuming the existence of a positive function α, such that $\alpha(t)\to 0$ as $t\to\infty$, and a non null function M such that for all $\mathbf{x}$ with positive coordinates,

$$\lim_{t\to\infty}\frac{1}{\alpha(t)}\left\{t\left[1-F\{F_1^{-1}(1-t^{-1}x_1),\ldots,F_d^{-1}(1-t^{-1}x_d)\}\right]-\ell(\mathbf{x})\right\}=M(\mathbf{x})\,, \tag{3.3}$$

  uniformly on any $[0,T]^d$, for $T>0$.

- the *third order condition* consists of assuming the existence of a positive function β, such that $\beta(t)\to 0$ as $t\to\infty$, and a non null function N such that for all $\mathbf{x}$ with positive coordinates,

$$\lim_{t\to\infty}\frac{1}{\beta(t)}\left\{\frac{t\left[1-F\{F_1^{-1}(1-t^{-1}x_1),\ldots,F_d^{-1}(1-t^{-1}x_d)\}\right]-\ell(\mathbf{x})}{\alpha(t)}-M(\mathbf{x})\right\}=N(\mathbf{x})\,, \tag{3.4}$$

  uniformly on any $[0,T]^d$, for $T>0$. It implicitly requires that N is not a multiple of the function M.

The functions M and N involved in the second and third order conditions satisfy some usual properties, see e.g. [85]. More specifically, one can show that there exists non positive reals ρ and ρ' such that α (resp. β) is a regularly varying function of order ρ (resp. ρ'), i.e. $\alpha(tz)/\alpha(t)\to z^\rho$ and $\beta(tz)/\beta(t)\to z^{\rho'}$ when $t\to\infty$, for each positive z.

The framework described in this section is especially relevant for a situation of asymptotic dependence. It is therefore useful to distinguish this context from that of asymptotic independence, for which we would have $\ell(\mathbf{x})=\sum_{i=1}^d x_i$ in (3.1) or equivalently $G(\mathbf{x})=\prod_{i=1} G_i(x_i)$ in (3.2). In this case, the function M is the limit of the joint tail of the distribution, and in dimension 2, the coefficient of tail dependence η introduced by [119] and [118] equals $1/(1-\rho)$, where ρ is defined above. This distinction is still an ongoing challenge, as exemplified by recent research on automatic recognition between these two regimes using convolutional neural networks in [3]. Tests for asymptotic independence might also be derived from the statements of Section 5.2.2.

It is important that the assumptions above are checked in real-data applications. The pioneering work was proposed by [77] and studied in [9] through the notion of extreme value copula. Actually, such assumptions are also close to the notion of multivariate regular variation, see the equivalent statements in [147]. Testing methods for such hypothesis have been recently developed in [57] or [53] for instance.

3.2 Theoretical bivariate examples

The aim of ([M5], Section 4) is to furnish several bivariate distributions that satisfy the third order condition (3.4). As already mentioned, these examples and our codes are still used in the recent literature as data generating processes. We start by focusing on heavy tailed margins. In this case, a first possible step to get the pointwise convergence is to obtain, for well chosen positive reals p and q, an expansion (for t tending to infinity) of the form

$$t\mathbb{P}(X > t^p x \,\text{or}\, Y > t^q y) = T_1(x,y) + \alpha(t)T_2(x,y) + \alpha(t)\beta(t)T_3(x,y) + o(\alpha(t)\beta(t)) \,,$$

with $T_1(1,1) > 0$. One can then identify each term involved in (3.4) as follows

$$\ell(x,y) = T_1(a(x), b(y)), \quad M(x,y) = T_2(a(x), b(y)), \quad \text{and } N(x,y) = T_3(a(x), b(y)),$$

where

$$a(x) = x^{-p}\{T_1(1,+\infty)\}^p, \quad b(x) = x^{-q}\{T_1(+\infty,1)\}^q \,.$$

Applying Corollary 5.18 in [148], one can check that in such a framework a form of the bivariate extreme value distribution G is given by

$$G(x,y) = \exp\left(-\frac{T_1(x,y)}{T_1(1,1)}\right) \,.$$

3.2.1 Powered norm densities

Following the idea of ([148], p. 276 and 286), consider first a norm $\|\cdot\|$, and a cone $\mathcal{D}$ of $\mathbb{R}^2$, that is to say a set such that if $(x,y) \in \mathcal{D}$, then $(tx,ty) \in \mathcal{D}$ for every positive t. Without loss of generality, suppose that $(1,1) \in \mathcal{D}$. Let (X,Y) be a bivariate random vector with probability density function given by

$$f(x,y) := \frac{c\mathbf{1}_{\mathcal{D}}(x,y)}{(1 + \|(x,y)^T\|^\alpha)^\beta} \,,$$

where c is a normalizing positive constant and where α and β are some positive real numbers such that $\alpha\beta > 2$. Set $A_{\mathcal{D}}(x,y) := \{(u,v) \in \mathcal{D} : u > x \text{ or } v > y\}$ and define $p := (\alpha\beta - 2)^{-1}$. One can check that, for $j = 1,2,3$,

$$T_j(x,y) = \iint_{A_{\mathcal{D}}(x,y)} \frac{c\, c_j \, du dv}{\|(u,v)^T\|^{\alpha(\beta+j-1)}} \,,$$

where $c_1 = 1$, $c_2 = -\beta$ and $c_3 = \beta(\beta+1)/2$. The functions M and N are homogeneous with order given through $\rho = \rho' = -\alpha p$.

Let us discuss some particular choices of the norm:

- For the L^1-norm and $\alpha = 1$, the model coincides with the bivariate Pareto of type II distribution, denoted by BPII(β), and referred to as $\mathrm{MP}^{(2)}(II)(0,1,\beta-2)$ in ([113], p. 604). In this case, $p = q = (\beta-2)^{-1}$, and $\ell(x,y) = x + y - (x^{-p} + y^{-p})^{-1/p}$. The latter stdf is known as the negative logistic model, introduced by [101], see also ([8], p. 307).

- When the Euclidean norm is chosen, one recovers the bivariate Cauchy distribution for $\alpha = 2$, $\beta = 3/2$ and $p = 1$. On the positive quadrant, that means for $\mathcal{D} = \mathbb{R}^2_+$, we have $c = 2/\pi$, $T_1(u,v) = c(u^{-2} + v^{-2})^{1/2}$ and $a(x) = b(x) = c/x$. On the whole plane, which means that $\mathcal{D} = \mathbb{R}^2$, we get $c = 1/(2\pi)$, $T_1(u,v) = c\left\{u^{-1} + v^{-1} + (u^{-2} + v^{-2})^{1/2}\right\}$ and $a(x) = b(x) = 2c/x$. This can also be seen as a particular case of the following item.

- The Student distributions with Pearson correlation coefficient θ arise choosing the norm $\|(x,y)^T\| = \nu^{-1/2}(x^2 - 2\theta xy + y^2)^{1/2}$, for a positive real number ν, $\alpha = 2$, $\beta = (\nu+2)/2$

and $p = \nu^{-1}$. In this case, the integral form of the function T_1 can not be totally simplified, and one classically writes the stdf as

$$\ell(x,y) = (x+y)\left[\frac{y}{x+y}F_{\nu+1}\left\{\frac{(y/x)^{1/\nu}-\theta}{\sqrt{1-\theta^2}}\sqrt{\nu+1}\right\} + \frac{x}{x+y}F_{\nu+1}\left\{\frac{(x/y)^{1/\nu}-\theta}{\sqrt{1-\theta^2}}\sqrt{\nu+1}\right\}\right] ,$$

where $F_{\nu+1}$ is the cdf of the univariate Student distribution with $\nu+1$ degrees of freedom. This dependence structure is also obtained for some elliptical models. See for instance [107], ([115], p. 1813) and next paragraph.

- Other choices for the norm would lead to other distributions. Note that one can also relax the symmetry condition, considering for instance the Mahalanobis pseudo-norm defined by $\|(x,y)^T\|^2 = (x/\sigma)^2 - 2\rho(x/\sigma)(y/\tau) + (y/\tau)^2$ for a real number ρ such that $|\rho| < 1$ and some positive real numbers σ and τ.

3.2.2 Elliptical distributions

Consider the usual representation of the centered elliptical distribution $(X,Y)^T = R\mathbf{A}\mathbf{U}$, in terms of a positive random variable R, a 2×2 matrix $\mathbf{A}$ such that $\mathbf{\Sigma} = \mathbf{A}\mathbf{A}^T$ is of full rank, and a bivariate random vector $\mathbf{U}$ independent of R, uniformly distributed on the unit circle of the plane. Assume that R has a probability density function denoted by g_R. One can then express the probability density function of (X,Y) as

$$f(x,y) := \frac{1}{|\det\mathbf{A}|}g_R\left\{(x,y)\mathbf{\Sigma}^{-1}(x,y)^T\right\} .$$

A sufficient condition to satisfy (3.4) is to assume that the distribution of R belongs to the Hall and Welsch class [88], viz.

$$\mathbb{P}(R > r) = cr^{-1/\gamma}\left\{1 + D_1 r^{\rho/\gamma} + D_2 r^{(\rho+\rho_1)/\gamma} + o(r^{(\rho+\rho_1)/\gamma})\right\} ,$$

with positive real c, non null reals D_1 and D_2, and negative reals ρ and ρ_1. One can check that, for $j = 1,2,3$,

$$T_j(x,y) = \frac{c}{2\pi\gamma|\det\mathbf{A}|}\iint_{\{(u,v):u>x \text{ or } v>y\}}\frac{dudv}{\{(u,v)\mathbf{\Sigma}^{-1}(u,v)^T\}^{1+1/(2\gamma)+p_j}} ,$$

where $p_1 = 0, p_2 = -\rho/(2\gamma)$ and $p_3 = -(\rho+\rho_1)/(2\gamma)$.

Assuming for simplicity that $\mathbf{\Sigma} = \begin{pmatrix} 1 & \theta \\ \theta & 1 \end{pmatrix}$, the stdf can be written as

$$\ell(x,y) = (x+y)\left[\frac{y}{x+y}F_{1/\gamma+1}\left\{\frac{(y/x)^{\gamma}-\theta}{\sqrt{1-\theta^2}}\sqrt{1/\gamma+1}\right\} + \frac{x}{x+y}F_{1/\gamma+1}\left\{\frac{(x/y)^{\gamma}-\theta}{\sqrt{1-\theta^2}}\sqrt{1/\gamma+1}\right\}\right] ,$$

which is the form already obtained for the Student distribution in Section 3.2.1 for $\nu = 1/\gamma$. See [44]. Note finally that for a general matrix $\mathbf{\Sigma}$ and the special case $g_R(r) = c(1+r^\alpha)^{-\beta}$, one recovers the Mahalanobis pseudo-norm already mentioned in the previous section.

When dealing with margins that are *not* heavy tailed, the calculus are done directly from (3.3). The last two examples of bivariate distributions have short and light tailed margins respectively.

3.2.3 Archimax distributions

Consider the bivariate cdf defined for each $0 \le u, v \le 1$ by $F(u,v) = \left\{1 + \ell(u^{-1}-1, v^{-1}-1)\right\}^{-1}$ given in terms of a stdf ℓ. This distribution has standard uniform univariate margins and corresponds to a particular case of Archimax bivariate copulas introduced in [32], in which the function $\phi(t) = t^{-1} - 1$ is the Clayton Archimedean generator with index 1. Expanding the left-hand side term of (3.3) leads to, as t tends to infinity,

$$t \left\{1 - F\left(1 - t^{-1}x,\, 1 - t^{-1}y\right)\right\} = \ell(x,y) + t^{-1}M(x,y) + t^{-2}N(x,y) + o\left(t^{-2}\right) \ ,$$

where, when the notation $\partial_{ij}\ell$ stands for $\partial^2\ell/(\partial x_i \partial x_j)$,

$$\begin{aligned} M(x,y) &:= x^2\partial_1\ell(x,y) + y^2\partial_2\ell(x,y) - \ell^2(x,y) \\ N(x,y) &:= x^4/2\partial_{11}^2\ell(x,y) + x^2y^2\partial_{12}^2\ell(x,y) + y^4/2\partial_{22}^2\ell(x,y) \\ &\qquad + \ell^3(x,y) + \left(x^3 - 2x^2\ell(x,y)\right)\partial_1\ell(x,y) + \left(y^3 - 2y^2\ell(x,y)\right)\partial_2\ell(x,y) \ . \end{aligned}$$

This allows to identify $\rho = \rho' = -1$. For multivariate extensions, see [34], [36] and [M11].

3.2.4 Symmetric logistic distributions

Consider the cdf defined by $F(x,y) = \exp\left\{-\left(e^{-x/r} + e^{-y/r}\right)^r\right\}$ for each $x, y \in \mathbb{R}$, which corresponds to the bivariate extreme value distribution with Gumbel univariate margins $F_1(x) = F_2(x) = \exp\{-e^{-x}\}$ and symmetric logistic stdf $\ell(x,y) = (x^{1/r} + y^{1/r})^r$, where $0 < r \le 1$. This distribution has been introduced in [164], see e.g. ([8], p. 304).
Expanding the left hand term of (3.1) leads to $\ell(x,y) + t^{-1}M(x,y) + t^{-2}N(x,y) + o\left(t^{-2}\right)$ where

$$\begin{aligned} M(x,y) &:= \frac{1}{2}\ (xx^{1/r} + yy^{1/r})\{\ell(x,y)\}^{1-1/r} - \frac{1}{2}\{\ell(x,y)\}^2 \\ N(x,y) &:= \frac{1}{3}(x^2x^{1/r} + y^2y^{1/r})\{\ell(x,y)\}^{1-1/r} + \frac{1-r}{8r}(xy)^{1/r}(x-y)^2\{\ell(x,y)\}^{1-2/r} \\ &\qquad + \frac{1}{3!}\{\ell(x,y)\}^3 - \frac{1}{2}\ (xx^{1/r} + yy^{1/r})\{\ell(x,y)\}^{2-1/r} \ . \end{aligned}$$

This allows to identify $\rho = \rho' = -1$. The identification of second and third order terms has previously been derived by [118].

3.3 Homogeneous co-survival functions

Now that initial distributions F and associated stdf ℓ have been identified on specific examples, let us turn back to a general multivariate framework. The idea here is to present the two main properties that characterize a stdf. This section mainly relies on the developments of [147], [86], [151] and ([M13], Section 2.1), the writing of which was motivated so that each argument finds its place with the greatest clarity as well as rigor.

3.3.1 Multivariate monotonicity property

Consider $f : \mathbb{R}_+^d \to \mathbb{R}_+$ and let μ denote a non-negative Radon measure on $[0,\infty]^d \setminus \{\boldsymbol{\infty}\}$. If for any $\mathbf{x} \in \mathbb{R}_+^d$,

$$f(\mathbf{x}) = \mu([\mathbf{x}, \boldsymbol{\infty}]^c) \ ,$$

then f is said the *co-survival function* of μ. One should only keep in mind that the notion of Radon measures ensures that f is well defined and finite for any $\mathbf{x} \in \mathbb{R}_+^d$.

Similar to distribution functions, also co-survival functions are essentially characterized by a special multivariate monotonicity property. First, we introduce a notation. Let $V_1, \ldots, V_d$ be non-empty sets, $V = V_1 \times \cdots \times V_d$, and let $f : V \to \mathbb{R}$ be any function. Then for $\mathbf{x}, \mathbf{z} \in V$ set

$$D_{\mathbf{z}}^{\mathbf{x}} f := \sum_{\mathcal{A} \subseteq \{1,\ldots,d\}} (-1)^{|\mathcal{A}|} f(\mathbf{z}_{\mathcal{A}}, \mathbf{x}_{-\mathcal{A}}) \ , \tag{3.5}$$

where $(\mathbf{z}_{\mathcal{A}}, \mathbf{x}_{-\mathcal{A}})$ stands for the concatenated vector with values from $\mathbf{z}$ (resp. $\mathbf{x}$) at coordinates in $\mathcal{A}$ (resp. $-\mathcal{A} = \{1, \ldots, d\} \setminus \mathcal{A}$). Moreover, for a non-empty subset $\mathcal{B} \subsetneq \{1, \ldots, d\}$ and for $\mathbf{y}_{-\mathcal{B}} \in \prod_{j \in -\mathcal{B}} V_j$, let us define on $\prod_{j \in \mathcal{B}} V_j$

$$f(\cdot, \mathbf{y}_{-\mathcal{B}})(\mathbf{z}_{\mathcal{B}}) := f(\mathbf{z}_{\mathcal{B}}, \mathbf{y}_{-\mathcal{B}}) \, .$$

If $V_j \subseteq \overline{\mathbb{R}}$ for all j, the function f is called $\mathbf{1}_d$*−alternating*, if $D_{\mathbf{z}}^{\mathbf{x}} f \leq 0$ for $\mathbf{x} \leq \mathbf{z}$ (both in V), and if this inequality also holds whenever some of the variables are fixed, for the function of the remaining variables, i.e. if for each non-empty subset $\mathcal{B} \subsetneq \{1, \ldots, d\}$, for each $\mathbf{y}_{-\mathcal{B}} \in \prod_{j \in -\mathcal{B}} V_j$ and any $\mathbf{x}_{\mathcal{B}} \leq \mathbf{z}_{\mathcal{B}}$ both in $\prod_{j \in \mathcal{B}} V_j$, we have

$$D_{\mathbf{z}_{\mathcal{B}}}^{\mathbf{x}_{\mathcal{B}}} f(\cdot, \mathbf{y}_{-\mathcal{B}}) \leq 0 \, .$$

See [149] or [152] for a detailed presentation of this concept, that has been first introduced in [150] under the name *fully d-max-decreasing*.

By Theorem 3 in [151] one knows that f *is the co-survival function of* μ is equivalent to assuming f $\mathbf{1}_d$-alternating, left continuous, and $f(\mathbf{0}) = 0$. Moreover, for any $\mathbf{0} \leq \mathbf{x} < \mathbf{z}$ in $\mathbb{R}_+^d$

$$D_{\mathbf{z}}^{\mathbf{x}} f = -\mu([\mathbf{x}, \mathbf{z}[)$$

by an application of the inclusion/exclusion principle.

3.3.2 Homogeneity assumption

Let $\varphi : \mathbb{R}_+^d \to \mathbb{R}_+$ be a homogeneous function, that is $\varphi(t\mathbf{x}) = t\varphi(\mathbf{x})$ for any positive t and vector $\mathbf{x}$. Assume now that φ is the co-survival function of μ. Then the measure μ is homogeneous: $\mu(tA) = t\mu(A)$ for any positive t and measurable subset A (and reciprocally). Note that any homogeneous $\mathbf{1}_d$-alternating function $\varphi : \mathbb{R}_+^d \to \mathbb{R}$ is automatically continuous, non-negative, with $\varphi(\mathbf{0}) = 0$, see ([151], Corollary 1). Classical examples of homogeneous co-survival functions to keep in mind are the power mean values, defined for $0 < r \leq 1$ by $\mathbf{x} \mapsto \left(\sum_{i=1}^d x_i^{1/r}\right)^r$ for $\mathbf{x} \in [0, \infty[^d$. The latter have been already encountered in the bivariate setting, see Section 3.2.4.

We consider the unit cube $C = \{\mathbf{w} = (w_1, \ldots, w_d) \in [0,1]^d | \max(w_1, \ldots, w_d) = 1\}$. An important example of a homogeneous measure is given by the image $\lambda_{\mathbf{w}}$ of the Lebesgue measure λ on $\mathbb{R}_+$ under the mapping $s \mapsto s/\mathbf{w}$, where $\mathbf{w} \in C$. The co-survival function of $\lambda_{\mathbf{w}}$ is then

$$\lambda_{\mathbf{w}}([\mathbf{x}, \boldsymbol{\infty}]^c) = \lambda(\{s \in \mathbb{R}_+ | s/\mathbf{w} \ngeq \mathbf{x}\}) = \lambda(\{s \in \mathbb{R}_+ | s < \max_{i=1,\ldots,d}(x_i w_i)\}) = \max(\mathbf{x} \cdot \mathbf{w}) \, .$$

These functions play a decisive role in the following, since they are the *building stones* of all homogeneous co-survival functions. More precisely, consider the set of all normalized functions discussed above

$$K := \{\psi : \mathbb{R}_+^d \to \mathbb{R} | \psi \text{ is } \mathbf{1}_d\text{-alternating, homogeneous and } \psi(\mathbf{1}) = 1\} \, .$$

Then K is obviously convex and compact (with respect to pointwise convergence). It turns out that K is even a simplex, with

$$\{\mathbf{x} \mapsto \max(\mathbf{x} \cdot \mathbf{w}) | \mathbf{w} \in C\} = \mathrm{ex}(K)$$

as its set of extreme points, and this set is closed (so compact as well) ; see ([151], Theorem 4 (ii)). In other words, K is a so-called Bauer simplex, i.e. for each $\psi \in K$ the representing probability measure on $\mathrm{ex}(K)$ guaranteed by Krein–Milman's theorem, is unique. The resulting integral representation is also called Choquet representation. See ([151], Theorem 2) for the original statement of this spectral representation and next part for its explicit formula for stdf.

3.3.3 Stdf Choquet representation and choice of the norm

A homogeneous co-survival function ℓ with Choquet representation

$$\ell(\mathbf{x}) = \ell(\mathbf{1}) \int_C \max(\mathbf{x} \cdot \mathbf{w}) d\nu(\mathbf{w}), \quad \mathbf{x} \in \mathbb{R}_+^d \tag{3.6}$$

is a stable tail dependence function iff its associated (and unique) probability measure ν on C satisfies the d constraints

$$\int_C w_i d\nu(\mathbf{w}) = 1/\ell(\mathbf{1}) \qquad \forall\, i = 1, \ldots, d\,.$$

It is easily seen that ℓ is the co-survival function of the measure $\mu := \ell(\mathbf{1}) \int_C \lambda_{\mathbf{w}} d\nu(\mathbf{w})$. Recall that it means that $\ell(\mathbf{x}) = \mu([\mathbf{x}, \boldsymbol{\infty}]^c)$. This measure μ is for instance denoted by Λ in [49]. It is closely related to the so-called *exponent measure* $\mu_\star$ introduced in ([147], Section 5.4.1) for instance. In fact, for any $\mathbf{x} \in \mathbb{R}_+^d$

$$\mu([\mathbf{x}, \boldsymbol{\infty}]^c) = \mu_\star([\mathbf{0}, 1/\mathbf{x}]^c)\,.$$

This means that $\mu_\star$ is the image of μ under the mapping $\mathbf{x} \mapsto 1/\mathbf{x}$, so that μ is directly homogeneous (as is ℓ) when $\mu_\star$ is inversely homogeneous: $\mu_\star(tA) = t^{-1}\mu_\star(A)$ for any positive t and any measurable set A of $[0, \infty]^d \setminus \{\mathbf{0}\}$.

Whereas the characterization of stdfs was shown relatively late ([151], Theorem 6), their integral representation was known long before: it goes back essentially to [86] and [104]. Most of the use of their integral representation has been done under the L_1 or L_2-norm on $\mathbb{R}_+^d$. But as emphasized by de Haan and Resnick, it is an arbitrary choice. As seen in the previous section, the extreme points of K (functions $\mathbf{x} \mapsto \max(\mathbf{x} \cdot \mathbf{w})$ for $\mathbf{w} \in C$) combined with the max-norm is also a convenient choice. In particular, it is really helpful for proving results from ([M13], Section 3.2), presented here in Chapter 4.

3.4 Multivariate Fréchet distributions

The stdf ℓ characterizes the dependence structure of the limiting distribution G, also referred to as the max-attractor of F. See (3.2) and above display there. The focus in [M7] is on parametric and semiparametric models of extreme value distributions G. This topic has been initiated by [82], [166], [66] and [164]. Different reviews of parametric multivariate extreme value models are given by [37], [102], [114] and ([8], Section 9.2.2), among others. Our presentation is done in terms of Fréchet margins, but other choices are possible and would lead to equivalent expressions. To illustrate these choices through the literature, one can refer e.g. to [166] or [63], who worked with Gumbel marginal distributions, whereas [87] or [108] chose Fréchet margins, and [140] or [164] studied exponential margins.

3.4.1 The scale function

Consider multivariate extreme value distribution functions with Fréchet margins and a common shape parameter. We keep ℓ to denote the stdf. As a consequence,

$$G(\mathbf{x}) = \exp\left(-\ell\left(\left\{\frac{\boldsymbol{\sigma}}{\mathbf{x} - \boldsymbol{\mu}}\right\}^{\xi}\right)\right) \qquad \forall\, \mathbf{x} > \boldsymbol{\mu}$$

where the shape ξ and the scales σ_i are some positive real numbers, and where the locations μ_i are real numbers. In particular, for any $i = 1, \ldots, d$ and $x_i > \mu_i$, the ith marginal is

$$G(x_i, \boldsymbol{\infty}_{-i}) = \exp\left(-\frac{\sigma_i^{\xi}}{(x_i - \mu_i)^{\xi}}\right)\,.$$

From (3.6), one knows the Choquet representation of ℓ in terms of the measure ν and the max-norm. Let us proceed to a change of measure (ν becomes H) with respect to another norm ($\|\cdot\|_\infty$ becomes $\|\cdot\|_1$). Set $W = \{\mathbf{w} \in \mathbb{R}_+^d, w_1 + \ldots + w_d = 1\}$ as the simplex associated with the L^1-norm. Let us define a positive Radon measure H on W by

$$H(\mathcal{B}) := \ell(\mathbf{1}) \int_C \mathbf{1}\{\frac{\boldsymbol{\sigma}^\xi \cdot \mathbf{w}}{w_1 + \ldots + w_d} \in B\}(w_1 + \ldots + w_d) d\nu(\mathbf{w})$$

for any borel set $\mathcal{B}$ of W, so that $\displaystyle\int_W w_i dH(\mathbf{w}) = \sigma_i^\xi$.

We define in ([M7], Section 2.2) on $\mathbb{R}_+^d$ the *scale function* as follows

$$\sigma(\mathbf{u}) = \left(\int_{\mathbb{W}_+} \max(\mathbf{w} \cdot \mathbf{u}^\xi) H(d\mathbf{w}) \right)^{1/\xi},$$

which allows to write the distribution function as $G(\mathbf{x}) = \exp\left(-\sigma^\xi((\mathbf{x} - \boldsymbol{\mu})^{-1})\right)$, for any $\mathbf{x} > \boldsymbol{\mu}$. We write $G = \mathrm{Fr}(\xi, \boldsymbol{\mu}, \sigma(\cdot))$.

The main difference between the stdf and the scale function is that some information from the margins G_i is contained in the scale function. For instance, the stable tail dependence function evaluated at each unit basis vector is equal to one, whereas the scale function at the unit basis vectors equals the corresponding margin scale. As a consequence, one may see the scale function as an unnormalized version of the stable tail dependence function. At first glance, the notion of scale function would seem perhaps unnatural. However, we had three reasons to work with this representation. The first one is that it makes it easier to construct classes of multivariate extreme value distributions by combining other ones ; see ([M7], Section 2.3). Renormalizing complicates these constructions and masks the essential structure. The second reason comes from the estimation point of view. When one constrains the search to guarantee a (normalized) stable tail dependence function, we force exact agreement at each unit axis vector. This is somewhat artificial, since in any practical problem, the marginals are normalized based on a sample, and thus the scaling of components is inexact. This enforced match at the margins might cause a poorer fit globally. The third reason is based on the result of [83], see also ([M7], Remark 1), who proved that for any vector following such an extreme value distribution, its max-projection along any direction is univariate Fréchet, and conversely. More precisely, one can check that the scale of the univariate max-projection is given by the scale function evaluated at this direction. As a consequence, the estimation of the dependence is reduced to a sequence of univariate estimations through the estimation of max-projection scales. Note that such a method was previously used (with min-projections and under exponential margins) by [140] and several other authors.

3.4.2 Generalized logistic mixtures

Recall that a positive multivariate stable distribution with index α is the law of a positive random vector $\mathbf{S} = (S_1, \ldots, S_d)^T$ with Laplace transform

$$\mathbb{E}[e^{-<\mathbf{u},\mathbf{S}>}] = \exp(-c_\alpha \gamma^\alpha(\mathbf{u})) \, , \mathbf{u} \in \mathbb{R}_+^d$$

where $c_\alpha = \sec(\pi\alpha/2)$ and

$$\gamma^\alpha(\mathbf{u}) = \int_{\mathbb{S}_+} <\mathbf{u}, \mathbf{s}>^\alpha \Lambda(d\mathbf{s}) \, . \tag{3.7}$$

In the previous display, $\alpha \in (0, 1)$ and $\mathbb{S}_+$ is the first orthant of the unit sphere in the Euclidean norm, and Λ denotes a positive and finite measure on $\mathbb{S}_+$. We will say that $\mathbf{S} = (S_1, \ldots, S_d)^T$ is a positive α-stable random vector with sum-stable spectral measure Λ.

Theorem 1 *([M7], Theorem 1) Let $\alpha \in (0,1)$ and $\mathbf{S} = (S_1, \dots, S_d)^T$ be a positive α-stable random vector with sum-stable spectral measure Λ. Let $Z_1, \dots, Z_d$ be independent and identically distributed univariate Fréchet $(\xi/\alpha, \mu = 0, \sigma = 1)$ and $\mathbf{Z} = (Z_1, \dots, Z_d)^T$. Assume that $\mathbf{S}$ and $\mathbf{Z}$ are independent. Then the random vector*

$$\mathbf{X} := \mathbf{S}^{\alpha/\xi} \cdot \mathbf{Z} = (S_1^{\alpha/\xi} Z_1, \dots, S_d^{\alpha/\xi} Z_d)^T$$

is $\mathrm{Fr}(\xi, \boldsymbol{\mu} = \mathbf{0}, \sigma(\cdot))$ *with scale function for* $\mathbf{u} \in \mathbb{R}_+^d$

$$\sigma^\xi(\mathbf{u}) = c_\alpha \gamma^\alpha(\mathbf{u}^{\xi/\alpha}),$$

where the right hand side is given by (3.7).

Several models have been defined combining positive stable distributions and extreme value distributions. For earlier results, see ([37], Section 4.2), as well as [95], [39], [165]. Note also that [63] unified the results in the previous papers and used them to construct structured models, e.g. max-stable time series. The key point is to produce dependent Fréchet distributions by mixing independent Fréchet components with dependent sum-stable scales, the two ingredients being independent. In the latter paper, the focus is on the fact that in these models, both conditional and unconditional distributions are extreme value distributions. The generalized logistic mixture or generalized logistic model presented in ([M7], Theorem 1) allows more general dependence in the terms of the mixture distribution. It is available for any dimension, differentiable and the general expression of its density in provided by ([M7], Proposition 2).

- When the measure Λ is discrete with a single mass, say $\Lambda = \lambda \delta_{s^\star}$, then the construction leads to the symmetric logistic model with dependence parameter $r = \alpha$.
  - If $\boldsymbol{\sigma}$ is given then $\lambda = c_\alpha^{-1} \|\boldsymbol{\sigma}\|_2^\alpha$ and $s^\star = \boldsymbol{\sigma}/\|\boldsymbol{\sigma}\|_2$.
  - If Λ is given then $\sigma_i^\xi = c_\alpha \lambda \{s_i^\star\}^\alpha$.

  In both cases, $\ell(\mathbf{x}) = \left(\sum_{i=1}^d x_i^{1/\alpha}\right)^\alpha$, already mentioned in Sections 3.2.4 and 3.3.2.

- When the measure Λ is discrete with several masses, say $\Lambda = \sum_{k=1}^m \lambda_k \delta_{s^{[k]}}$, then the resulting stdf is a mixture of asymmetric logistic terms having a common dependence parameter α. More precisely, $\ell(\mathbf{x}) = \sum_{k=1}^m \left(\sum_{i=1}^d (x_i \beta_{i,k})^{1/\alpha}\right)^\alpha$ with $\beta_{i,k} = c_\alpha \lambda_k \sigma_i^{-\xi} \{s_i^{[k]}\}^\alpha$ and $\sigma_i^\xi = c_\alpha \sum_{h=1}^m \lambda_h \{s_i^{[h]}\}^\alpha$. Such asymmetric model differs from the well-known family first introduced by [165]. The latter has much more flexibility on the dependence parameter, but the generalized logistic mixture offers a user-controlled number of terms m.

- When Λ admits a continuous density, we obtain a larger class of asymmetric logistic mixtures.

3.4.3 Combinations and denseness property

In ([M7], Lemma 1, 2 and 3), we describe how scale functions σ and spectral measures H combine such multivariate extreme value distributions when this is done under a list of operations: componentwise maximum $\max(\mathbf{Y} \cdot \mathbf{Z})$, up to some power $\mathbf{Y}^p$, scalar scale $c\mathbf{Y}$, vector scale $(c_1 Y_1, \dots, c_d Y_d)$, matrix max-product $A \times_{\max} \mathbf{Y}$, sum stable scale $S^{1/\xi}\mathbf{Y}$ (for S a positive sum stable random variable), concatenation $(\mathbf{Y}^T, \mathbf{Z}^T)$. While several of these facts were known, it was useful to collect them in one place, expand the list, and see how the scale function is a useful way to represent combinations of max-stable laws. Note in passing that the matrix max-product finds several generalizations in the literature, as for instance max-linear causal graphical models or max-linear Bayesian networks studied in [167] or infinite max-linear models recently introduced in [109].

In ([M7], Lemma 6, 7 and 9), the class of multivariate Fréchet distributions is shown to be closed under most of the previous combinations for three particular classes: 1) when the spectral measure H is discrete, 2) for generalized logistic mixtures, and 3) when H admits a piecewise polynomial spectral density. The latter requires some tools for integration over a simplex, see [5], [131] and [130].

If two multivariate Fréchet distributions have similar scale functions or similar spectral measures, then their cumulative distribution functions are uniformly close, as stated in ([M7], Theorem 2 and 3) and following [128]. It allows to prove the denseness of the three multivariate Fréchet classes already mentioned. Thus these results stated in ([M7], Proposition 4, 5 and 6) offer three different approximations of any multivariate Fréchet distribution, and therefore of any multivariate extreme value distribution G after well chosen marginal transformations. In practice, there is no abstract reason to choose one of these models over another. It is unlikely that one will be able to distinguish between these classes with real data, unless there is a massive data set. However, the choice of a model can be based on some physical understanding of the situation where the data is obtained, or on arguments such as parsimony, existence of a density, etc. For example, in higher dimensions, it may be preferable to use a generalized logistic mixture with a few terms that gives a smooth model, than a discrete spectral measure with many terms.

Concluding remarks

The stable tail dependence function provides a full characterization of the extremal dependence structure. Its study allows for the discussion of very exciting mathematical concepts. Each of the papers cited in this chapter has also led to data simulation procedures. The bivariate models of [M5] have been shared with the community *via* private communications. The statement of ([M7], Theorem 1) is a stochastic representation that allows these models to be generated as soon as a multivariate sum-stable model can be simulated, see [126], [132] and [129]. We also refer to [160], [16], [124] and [34] for pioneering simulation algorithms under various frameworks. For (a)symmetric logistic models, the well-known *evd* R-package [161], and more specifically its function `rmvevd` therein has been incredibly useful. However, creating the `asy` vector becomes very tedious as the dimension increases. In the recent *satdad* R-package [M11], this issue is addressed through the writing of the `gen.ds` function, with the meaning `gen`erate a `d`ependence `s`tructure. The result is a list of several arguments that encodes a stdf. This package also includes the functions `rMevlog` and `rArchimax` that generate multivariate extreme value models and Archimax models both with (a)symmetric logistic stdf. An illustration ends the chapter.

First illustration of [M11]

```
> library(satdad, quietly = TRUE)
> ds12 <- gen.ds(d = 12, type = "alog", sub = list(1:2, 3:7, c(3,5,7), c(2,8:12)))
```

Taking into account the support indicated in the list `sub` above, the stdf coded in `ds12` has the form

$$
\begin{aligned}
\ell(x_1,\ldots,x_{12}) = & \Big((\beta_{1,1}\,x_1)^{1/\alpha_1} + (\beta_{2,1}\,x_2)^{1/\alpha_1}\Big)^{\alpha_1} \\
& + \Big((\beta_{3,2}\,x_3)^{1/\alpha_2} + (\beta_{4,2}\,x_4)^{1/\alpha_2} + (\beta_{5,2}\,x_5)^{1/\alpha_2} + (\beta_{6,2}\,x_6)^{1/\alpha_2} + (\beta_{7,2}\,x_7)^{1/\alpha_2}\Big)^{\alpha_2} \\
& + \Big((\beta_{3,3}\,x_3)^{1/\alpha_3} + (\beta_{5,3}\,x_5)^{1/\alpha_3} + (\beta_{7,3}\,x_7)^{1/\alpha_3}\Big)^{\alpha_3} \\
& + \Big((\beta_{8,4}\,x_8)^{1/\alpha_4} + (\beta_{9,4}\,x_9)^{1/\alpha_4} + (\beta_{10,4}\,x_{10})^{1/\alpha_4} + (\beta_{11,4}\,x_{11})^{1/\alpha_4} + (\beta_{12,4}\,x_{12})^{1/\alpha_4}\Big)^{\alpha_4}
\end{aligned}
$$

where the asymmetric coefficients $\beta_{\cdot,\cdot}$, randomly generated here, are saved in the list

```
> ds12$asy

## [[1]]
## [1] 1.0000000 0.4072452
##
## [[2]]
## [1] 0.7975571 1.0000000 0.9096378 1.0000000 0.6048126
##
## [[3]]
## [1] 0.2024429 0.0903622 0.3951874
##
## [[4]]
## [1] 0.5927548 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000
```

and where the dependence parameters $\alpha_\cdot$, also randomly generated here, are given by the vector

```
> ds12$dep

## [1] 0.04749820 0.40928894 0.77149036 0.06592641
```

The stdf is evaluated below at `rep(1,12)` and at `(1:12)/20` (arbitrarily chosen)

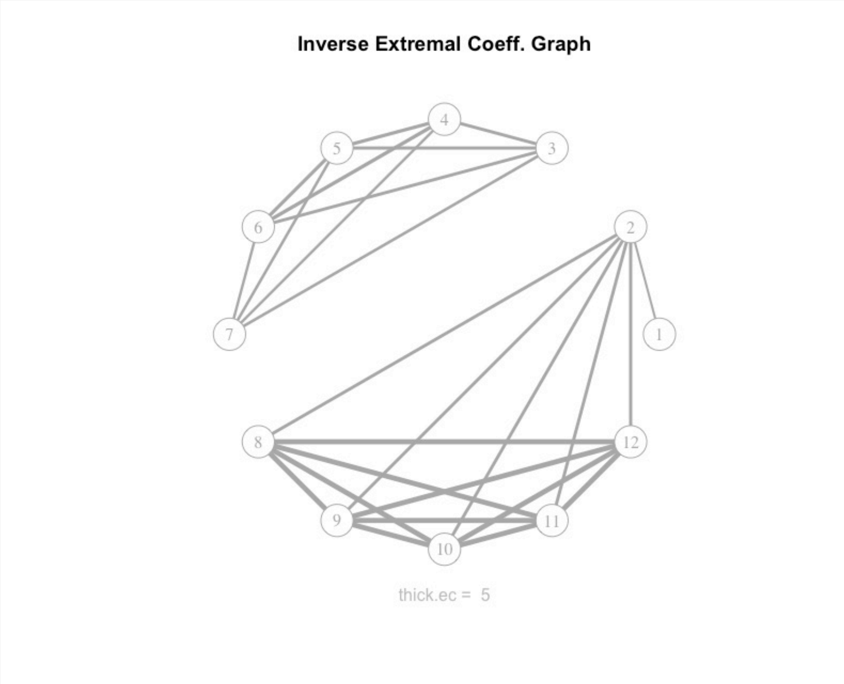
```
> ellMevlog(x = rep(1,12), ds = ds12)

## [1] 4.372033

> ellMevlog(x = (1:12)/20, ds = ds12)

## [1] 1.254858
```

The well-known collection of extremal coefficients is obtained by the evaluation of the stdf at specific points. For $\mathcal{A}$ a subset of $\{1,\ldots,d\}$, it is given by $\ell(\mathbf{1}_{\mathcal{A}},\mathbf{0}_{-\mathcal{A}}) \in [1,|\mathcal{A}|]$. For instance, the bivariate family associated with the 12-dimensional stdf can be computed with the command

```
> ec(ds = ds12, ind = 2)$ec
```

The quantities $2-\ell(\mathbf{1}_{\{i,j\}},\mathbf{0}_{-\{i,j\}})$ are plotted in the Inverse extremal coefficients graph.

```
> graphs(ds = ds12, which = "iecgraph")
```



Two random samples are now generated.

The first has standard Fréchet margins and ℓ as stdf.

```
> X_fr12 <- rMevlog(n = 1000, ds = ds12)
```

The second is sampled from the Archimax copula $C(u_1,\ldots,u_{12}) = \psi\left(\ell(\psi^{-1}(u_1),\ldots,\psi^{-1}(u_{12}))\right)$ for the function $\psi(x) = 1/(x+\sigma)^a$, where the positive shape a and positive scale σ are chosen at random.

```
> a <- runif(1, 0.01, 5)
> sigma <- runif(1, 0.01, 5)
```

```
> a ; sigma

## [1] 1.445411
## [1] 3.305163

> X_archimax12 <- rArchimaxMevlog(n = 1000,
+             ds = ds12, dist = "gamma", dist.param = c(a, sigma))
```

4

Functional decompositions for (tail) dependence

The stdf being infinite-dimensional it is desirable to build finite-dimensional summaries. The well-known family of extremal coefficients can be easily calculated from the stdf and has an interesting probabilistic interpretation. The first objective of this chapter is to propose another family of indices that, when measured on a pair, provide information about the asymptotic distribution associated with that pair as well as with any set of indices containing that pair. This work, based on a specific functional decomposition, has been proposed in [M14] and refined in [M13]. It is the subject of the first two parts of the current chapter. The third part, that reflects the extension obtained in [M15], investigates the understanding of a generic form of functional decompositions. The result therein allows then to explore a rewrite of testing hypotheses for dependence, as it has been done in [M12] and presented here in the last part.

4.1 Global sensitivity analysis

Global sensitivity analysis is a branch of control theory and computer sciences. This type of analysis measures how sensitive the outcome quantity $f(\mathbf{x})$ is to the variation of individual input variables x_i. A variance decomposition determines which of the multiple input parameters are responsible for most of the variation in the outcome. We refer to [98], [97] or [41] for a presentation of methods and algorithms. This section presents an overview of functional decomposition, computational issues and consequences in terms of variance.

4.1.1 The Hoeffding–Sobol Decomposition

The functional decomposition mentioned here has a long story that is nicely described in [168]. To provide a short presentation, let us start by quoting [91]. His pioneering work uses L^2 projections to decompose and study U-statistics. But it is in [92] that the author proposes a recursive construction, based on conditional expectations, of what can be called the Hoeffding decomposition. Its first terms, depending on combinations of measurable functions of only one variable, corresponds to the Hajek projection. [48] seems to be the first reference with a clear statement and proof of the Hoeffding decomposition. It appears also in [159], with its own proof. This work had a major impact in the field of global sensitivity analysis. This explains why the name Sobol is now attached to the first one.

Let $f : [0,1]^d \to \mathbb{R}$ be a function in $L^2([0,1]^d, \lambda)$ where $\lambda = \prod_{i=1}^d \lambda_i$ is a product of probability measures on $[0,1]$. One way to understand the structural form of the d-variables function f is to decompose it into functions of increasing complexity. This is precisely what allows the functional analysis of variance (FANOVA). It relies on the Hoeffding-Sobol decomposition

$$f(\mathbf{x}) = \sum_{\mathcal{A}\subseteq\{1,\dots,d\}} f_{\mathcal{A}}(\mathbf{x}) \tag{4.1}$$

where

$$f_{\mathcal{A}}(\mathbf{x}) = \sum_{\mathcal{B}\subseteq\mathcal{A}} (-1)^{|\mathcal{A}\setminus\mathcal{B}|} \int f(\mathbf{x}) d\lambda_{-\mathcal{B}}(\mathbf{x})$$

for $d\lambda_{\mathcal{A}}(\mathbf{x}) = \prod_{i\in\mathcal{A}} d\lambda_i(x_i)$ and $-\mathcal{B} = \{1,\dots,d\}\setminus\mathcal{B}$. See [91], [159], [168]. The terms $f_{\mathcal{A}}$ only depend on the components of $\mathbf{x}$ associated with $\mathcal{A}$. They can be interpreted as follows. Set $\mathbf{U} \sim \lambda$.

The constant $f_\emptyset = \int f d\lambda = \mathbb{E}[f(\mathbf{U})]$ is the global mean. The term $f_i(\mathbf{x}) = \mathbb{E}[f(\mathbf{U}_{-i}, x_i)] - f_\emptyset$ represents the main effect of component $\{i\}$, $f_{ij}(\mathbf{x}) = \mathbb{E}[f(\mathbf{U}_{-\{ij\}}, \mathbf{x}_{\{ij\}})] - f_i(\mathbf{x}) - f_j(\mathbf{x}) - f_\emptyset$ captures the second-order interaction from the pair of components $\{i, j\}$, and so on. In the previous formulae, the mathematical expectation refers to the integral with respect to $d\lambda$ in $f_\emptyset$, to $\otimes_{t\neq i} d\lambda_t$ in the definition of f_i, and to $\otimes_{t\neq i,j} d\lambda_t$ in that of f_{ij}.
This decomposition has found many applications in statistics, notably in global sensitivity analysis; see [100], [35], [67], [99], [153], [133] and [144]. In particular, the Hoeffding–Sobol decomposition has been used extensively to compare the performance of Monte Carlo versus quasi-Monte Carlo integration in high-dimensional integration problems, where the notion of effective dimension reduces the dimension of the domain of integration. For more details, we refer to [122], [135] and [134].

4.1.2 A variance decomposition

The Hoeffding–Sobol decomposition has the advantage of a variance decomposition. The global variance is given by $\sigma^2(f) = \int (f - f_\emptyset)^2 d\lambda = \mathrm{var}[f(\mathbf{U})]$. Set $\sigma^2_\emptyset(f) = 0$ and $\sigma^2_{\mathcal{A}}(f) = \int f^2_{\mathcal{A}} d\lambda_{\mathcal{A}} = \mathrm{var}[f_{\mathcal{A}}(\mathbf{U})]$. For instance, $\sigma^2_i(f)$ is the variance of $f(\mathbf{U})$ due to the ith component only. Similarly, $\sigma^2_{ij}(f)$ is the variance due to the combined effect of components $\{i, j\}$. From orthogonality arguments (see [48] for instance), the term $f_{\mathcal{A}}$ is centered (except for the empty set) and the FANOVA expression relies on the equality

$$\sigma^2(f) = \sum_{\mathcal{A}\subseteq\{1,\ldots,d\}} \sigma^2_{\mathcal{A}}(f) \; .$$

Interest in the individual variances $\sigma^2_{\mathcal{A}}(f)$, and more particularly their ratio to the total variance $\sigma^2_{\mathcal{A}}(f)/\sigma^2(f)$, traces back to [159] and [137].

One combination of such variances is of prime interest. It is defined by

$$\Upsilon^2_{\mathcal{A}}(f) = \sum_{\mathcal{B}\supseteq\mathcal{A}} \sigma^2_{\mathcal{B}}(f)$$

and referred to as the superset importance of the subset $\mathcal{A}$. See [94] and [122] for pioneering discussion. This coefficient is positive and smaller than or equal to $\sigma^2(f)$. This index makes it possible to discover additive structures in multivariate functions. Indeed, under continuity assumptions on λ and f, $\Upsilon^2_{ij}(f) = 0$ implies that f does not simultaneously depend on i and j in its decomposition and, thus, f is additive with respect to x_i and x_j.

4.1.3 Computational aspects

From ([122], Formula 9), there exists for $\Upsilon^2_{\mathcal{A}}(f)$ an integral expression that relies on f only. It thus makes possible to compute these indices without identifying the terms of the decomposition $\{f_{\mathcal{B}}, \mathcal{B} \supseteq \mathcal{A}\}$. For instance, in a pairwise setting,

$$\Upsilon^2_{ij}(f) = \frac{1}{4}\int_{[0,1]^{d+2}} \left\{f(\mathbf{x}) - f(\mathbf{x}_{-i}, v) - f(\mathbf{x}_{-j}, w) + f(\mathbf{x}_{-\{ij\}}, v, w)\right\}^2 d\lambda(\mathbf{x}) d\lambda_i(v) d\lambda_j(w) \; .$$

But, even if the superset importance coefficient $\Upsilon^2_{\mathcal{A}}(f)$ has an integral formula, it is not easy to compute explicitly. Hopefully, any $\Upsilon^2_{\mathcal{A}}(f)$ admits a Monte Carlo approximation. For the subset $\mathcal{A}$ being a pair $\{i, j\}$, its statistical properties have been investigated in [65]: it is unbiased and asymptotically normal when the true value is not zero. Moreover, it is asymptotically efficient in a class of models with exchangeable variables, indicating that it has the smallest within-class variance. Furthermore, its Monte Carlo estimate vanishes when the theoretical value is zero: the Monte Carlo error of estimation under this specific case is always zero, which is remarkable (recall that f is known here).

4.2 New measures for the tail dependence

A sensitivity analysis is the search for the input into a system that will have the greatest impact on the systems output, often extremely complex. This search can be applied to subsets of components whose variations will most significantly impact variations in the output. In other words, given a pre-established size of subset, which group of input parameters will have the most impact on the outcome? This question, which is generally addressed using global variances, is in fact linked to the extreme value theory if the indicator observed becomes a high quantile or a probability of failure. We refer to the works of [136], [116], [163], [19] and [123].

This section reflects the advances obtained in [M14], [M13] and [M11]. It is again at the crossroads of two domains: the global sensitivity analysis introduced in Section 4.1 and the *multivariate* extreme value theory presented in Chapter 3. Recall that ℓ stands for the stable tail dependence function (stdf) of a random vector $\mathbf{X}$. In such a setting, we have natural entries represented by $X_1, \ldots, X_d$ the d variables jointly studied. The idea is to define a pseudo-response such as $\ell(\mathbf{U})$ at some generic random vector $\mathbf{U}$, with the intention of ranking dependence in the asymptotic joint structure. Note that [38] studies the influence of inputs distributions on renormalized variance indices in relation with stochastic orders. Our entries may have the same distribution, our goal is rather to understand the structure of the function ℓ that captures the tail dependence of the random vector $\mathbf{X}$.

4.2.1 The tail superset importance coefficients

The main goal here is to introduce new indices for multivariate extreme value analysis. Roughly speaking, our indices are derived from the decomposition of the variance of $\ell(\mathbf{U})$. By looking at their values, it is possible to understand part of the support of the asymptotic dependence structure.

By homogeneity of ℓ, the stdf can be restricted to $[0,1]^d$ without loss of information. By continuity, ℓ is square integrable on $[0,1]^d$. Recall that the continuity is also useful to deduce that a term of variance zero is null. Assume also that the measure λ is continuous with support $[0,1]^d$. Set $\mathbf{U} \sim \lambda$ and $\{\ell_{\mathcal{A}}, \mathcal{A} \subseteq \{1,\ldots,d\}\}$ the Hoeffding-Sobol decomposition terms of ℓ. A *tail superset importance coefficient* has been introduced in ([M14], Section 3.2) as a superset importance index

$$\Upsilon^2_{\mathcal{A}}(\ell) = \sum_{\mathcal{B}\supseteq\mathcal{A}} \sigma^2_{\mathcal{B}}(\ell) = \sum_{\mathcal{B}\supseteq\mathcal{A}} \mathrm{var}[\ell_{\mathcal{B}}(\mathbf{U})]$$

for the stdf ℓ. Even though it is associated with a random vector $\mathbf{X}$, the function ℓ is deterministic. Because ℓ is generally unknown, we provide here a new modeling tool. Under asymptotic independence, all tail superset importance coefficients vanish. However, these indices are also helpful in less exaggerated situations. For instance, Υ^2_{ij} vanishes if the margins X_i and X_j are part of asymptotically independent groups. As pointed out in ([M13], Formula 7), the integral representation of [122] for superset importance coefficient can be expressed as

$$\Upsilon^2_{\mathcal{A}}(\ell) = 2^{-|\mathcal{A}|} \int_{[0,1]^{d+|\mathcal{A}|}} (D^{\mathbf{x}_{\mathcal{A}}}_{\mathbf{z}_{\mathcal{A}}} \ell(\cdot, \mathbf{z}_{-\mathcal{A}}))^2 d\lambda_{\mathcal{A}}(\mathbf{x}) d\lambda(\mathbf{z})$$

where $D^{\mathbf{x}}_{\mathbf{z}}$ has been already defined in (3.5).

Another formula exists for $\Upsilon^2_{\mathcal{A}}(\ell)$. In [M13], we take advantage of the fact that ℓ is a homogeneous co-survival function. Recall (3.6) from Section 3.3.2 where the probability measure ν on the unit cube $C = \{\mathbf{w} \in [0,1]^d | \max(\mathbf{w}) = 1\}$ satisfies the d constraints $\int_C w_i d\nu(\mathbf{w}) = 1/\ell(\mathbf{1})$ for any $i = 1, \ldots, d$. The expression of the Hoeffding-Sobol decomposition terms $\ell_{\mathcal{A}}$ and variances $\sigma^2_{\mathcal{A}}(\ell)$ are obtained in ([M13], Theorem 1) as integrals of rank-one tensors. Formulae depending on the spectral measure are also derived for cumulated variances. In particular, the tail superset importance coefficients $\Upsilon^2_{\mathcal{A}}(\ell)$ are also written as integrals of rank-one tensors in

([M13], Corollary 1). More precisely, if $K_i(w,v;s,t) := \lambda_i([0,(s/w)\wedge(t,v)\wedge 1])$ and if $K_i(w;s)$ stands for $K_i(w,w;s,s)$, then

$$\begin{aligned}\Upsilon^2_{\mathcal{A}}(\ell) =& \ell(\mathbf{1})^2\int_C d\nu(\mathbf{w})\int_C d\nu(\mathbf{v})\int_0^1 ds\int_0^1 dt \\ & \prod_{i\notin\mathcal{A}} K_i(w_i,v_i;s,t)\prod_{i\in\mathcal{A}}\left(K_i(w_i,v_i;s,t)-K_i(w_i;s)K_i(v_i;t)\right) .\end{aligned}$$

Such integral representations improve the numerical performance of Monte Carlo estimates, as illustrated in ([M13], Section 2.3).

4.2.2 The tail dependograph

The features of the FANOVA graph, introduced by [127], are also highlighted in ([M14], Section 3.3) for asymptotic dependence modeling. Applied to the stable tail dependence function, we refer to it as the *tail dependograph* since it graphically represents, on an undirected but weighted graph, the strength of asymptotic dependence within a vector. On this graph, the edge thickness between two vertices is proportional to the force of tail dependence in the corresponding bivariate model but also cumulates the information of the asymptotic dependence structure for any multivariate model that contains this pair.

An *undirected and weighted graph* is a list $G = (V, \mathbf{g})$ of the set of vertices $V = \{1, \ldots, d\}$ and the collection $\mathbf{g} = \{g_{1;2}, g_{1;3}, \ldots, g_{1;d}, \ldots, g_{d-1;d}\}$ of the $d(d-1)/2$ edge weights. Graphically, we represent the vertex number inside a bubble and plot a segment between $\{i\}$ and $\{j\}$ whose width varies proportionally to $g_{i;j}$. For a *dependency graph* of a random vector $\mathbf{X}$, the set V represents the components of $\mathbf{X}$ and $\mathbf{g}$ contains some positive dependence measures between pairs. An undirected and weighted graph G will be known as a *tail dependency graph* if the following holds: the vectors $\mathbf{X}_I = \{X_i, i \in I\}$ and $\mathbf{X}_J = \{X_j, j \in J\}$ are asymptotically independent when there is no edge from any vertex in I to any vertex in J, for any two subsets of vertices I and J from V: $g_{i;j} = 0$, $\forall i \in I, \forall j \in J$.

Recall that the random vectors $\mathbf{X}_I$ and $\mathbf{X}_J$ are asymptotically independent if

$$\ell(\mathbf{x}_I,\mathbf{x}_J,\mathbf{0}_K) = \ell(\mathbf{x}_I,\mathbf{0}_J,\mathbf{0}_K) + \ell(\mathbf{0}_I,\mathbf{x}_J,\mathbf{0}_K) \quad \forall(\mathbf{x}_I,\mathbf{x}_J)\in[0,1]^{|I|+|J|} . \tag{4.2}$$

Proposition 2 *([M14], Section 3) The* tail dependograph *is defined as a tail dependency graph with edge weights given by the pairwise tail superset importance coefficients $\Upsilon^2_{ij}(\ell)$. Tail independence from the tail dependograph is concordant with (4.2).*

4.2.3 The upper bound of the tail superset importance coefficients

The question addressed here is as simple as the proof to obtain it was difficult. Coming from a reviewer of [M14], it is about understanding the upper bound of these new indices. Indeed, the bounds for any ℓ are as follows

$$\max(\mathbf{x}) \le \ell(\mathbf{x}) \le \sum_i^d x_i$$

for all $\mathbf{x} \in [0,1]^d$. A natural question is whether these bounds transfer in some way to the new tail measures. The lower bound, that is 0, is trivial. The spectral representation (3.6) is the main ingredient in deriving a sharp upper bound for the new quantities.

Theorem 3 *([M13], Theorem 2) Let ℓ be a d-variate stable tail dependence function. Then, for any non-empty $\mathcal{A} \subseteq \{1, \ldots, d\}$*

$$\Upsilon^2_{\mathcal{A}}(\ell) \leq \Upsilon^2_{\mathcal{A}}(\ell^{\vee,\mathcal{A}}) = \frac{2(|\mathcal{A}|!)^2}{(2|\mathcal{A}|+2)!}$$

where $\ell^{\vee,\mathcal{A}}(\mathbf{x}_{\mathcal{A}}) = \max_{i\in\mathcal{A}} x_i$.
If ℓ is a d-variate stdf with equality $\Upsilon^2_{\mathcal{A}}(\ell) = \Upsilon^2_{\mathcal{A}}(\ell^{\vee,\mathcal{A}})$ for a given $\emptyset \neq \mathcal{A} \subseteq \{1, \ldots, d\}$, then its projection on the variables $\mathbf{x}_{\mathcal{A}}$ is equal to $\ell(\mathbf{x}_{\mathcal{A}}, \mathbf{0}_{-\mathcal{A}}) = \ell^{\vee,\mathcal{A}}(\mathbf{x}_{\mathcal{A}}) = \max_{i\in\mathcal{A}} x_i$.
In particular, if $\Upsilon^2_{\{1,\ldots,d\}}(\ell) = \frac{2(d!)^2}{(2d+2)!}$ for a d-variate stdf ℓ then $\ell(\mathbf{x}) = \max(\mathbf{x})$.

The optimization problem dealt with in previous ([M13], Theorem 2) might be looked at in the broader perspective of maximizing a convex functional over a compact convex set (which need not be a simplex). Bauer's maximum principle ensures that the maximal value is attained in an extreme point. It does however give no hint to localize such a point nor to its uniqueness. Our statement answers completely the question: it asserts the existence, the uniqueness and the location (and so finds the maximal value) of the maximization problem. The proof of ([M13], Theorem 2) also relies on the following preliminary result.

Proposition 4 *([M13], Proposition 1) Let $f : \mathbb{R}^d_+ \to \mathbb{R}$ be $\mathbf{1}_d$-alternating and let $\mathcal{A}$ be a non-empty subset of $\{1, \ldots, d\}$. Then,*

$$\Upsilon^2_{\mathcal{A}}(f) \leq \Upsilon^2_{\mathcal{A}}(f^{[\mathcal{A}]})$$

with $f^{[\mathcal{A}]}(\mathbf{z}_{\mathcal{A}}) := f(\mathbf{z}_{\mathcal{A}}, \mathbf{0}_{-\mathcal{A}})$.

The following statement is included in the proof of ([M13], Theorem 2).

Corollary 5 *([M13], Corollary 2) For any d-variate stable tail dependence function ℓ given as (3.6)*

$$\Upsilon^2_{\{1,\ldots,d\}}(\ell) \leq \ell(\mathbf{1})^2 \left(\int_C \prod_{i=1}^d w_i^{1/d} d\nu(\mathbf{w}) \right)^2 \frac{2(d!)^2}{(2d+2)!}.$$

The knowledge of the upper bound for $\Upsilon^2_{\mathcal{A}}(\ell)$ induces a normalized version of these indices. The quantities

$$\Upsilon^2_{\mathcal{A}}(\ell) \times \frac{(2|\mathcal{A}|+2)!}{2(|\mathcal{A}|!)^2}$$

all belong to the interval $[0, 1]$, that does not depend on the cardinality of $\mathcal{A}$.

4.3 Linking the Hoeffding–Sobol and Möbius formulas

The Hoeffding–Sobol decomposition, widely used in Section 4.1 and 4.2, and the Möbius decomposition are two common ways of expressing a real-valued function f of $d \geq 2$ variables acting on a domain $D \subseteq \mathbb{R}^d$ into a sum running over all subsets of $\{1, \ldots, d\}$, as described by (4.1). Both formulas are well-known and useful in statistics because the terms in the representation (4.1) become gradually simpler (in a specific sense) as the size $|\mathcal{A}|$ of the set $\mathcal{A}$ decreases from d to zero. While many might have suspected that the two decompositions have a common origin, the purpose of [M15] is to elucidate this connection.

4.3.1 Preliminary remarks

By comparison, the Möbius decomposition is relatively unknown and should not be confused with the celebrated Möbius inversion formula. The Möbius decomposition was originally suggested by [43] as a way to construct rank-based tests of independence among the components of a continuous random vector. More specifically, an application of Möbius' formula leads to a representation of an empirical copula, and hence also of the empirical copula process, into a finite number of components which, under the null hypothesis of independence, are asymptotically independent. Cramér–von Mises statistics derived from the sub-processes prove to be very powerful, both asymptotically and in finite samples, as illustrated in [78], [74] and [73]. This approach has since been extended to the problem of testing for independence between random vectors, see [10] and [110], and more recently to testing for dependence between arbitrary random variables [71].

In [117] conditions are given to guarantee the existence and uniqueness of the general decomposition (4.1). They describe how their result applies to the Hoeffding–Sobol formula. The Möbius formula is also of the form (4.1) but its existence and uniqueness do not follow from the result of [117], whose conditions are too restrictive. They rely on projections $\mathbf{P}_1, \ldots, \mathbf{P}_d$ which induce the decomposition upon setting, for each $\mathcal{A} \subseteq \{1, \ldots, d\}$,

$$f_{\mathcal{A}} = \left(\prod_{i \in \mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i \right) (f) \,. \tag{4.3}$$

Their result holds provided that each projection eliminates the dependence on a specific input variable to which the function is applied.

4.3.2 Extending Kuo, Sloan, Wasilkowski and Woźniakowski functional decomposition

The following result relaxes assumptions in [117] so that the resulting formula encompasses both the Hoeffding–Sobol and Möbius decompositions as special cases. For any $\mathcal{A} \subseteq \{1, \ldots, d\}$, the composition of operators is denoted by $\mathbf{P}_{\mathcal{A}} = \prod_{i \in \mathcal{A}} \mathbf{P}_i$ and $\mathbf{P}_{-\mathcal{A}} = \mathbf{P}_{\{1,\ldots,d\} \setminus \mathcal{A}}$.

Proposition 6 *([M15], Proposition 1) Let $\mathcal{F}$ be a linear space of real-valued functions acting on a domain D of $\mathbb{R}^d$ and let $\mathbf{P}_1, \ldots, \mathbf{P}_d$ be commuting and idempotent operators on $\mathcal{F}$. Then, the following statements hold true.*
Part A: *Any function $f \in \mathcal{F}$ can be written as (4.1) in which the term $f_{\mathcal{A}}$, defined by (4.3) is such that the annihilating property is satisfied, meaning that, for all $i \in \{1, \ldots, d\}$,*

$$\mathbf{P}_i(f_{\mathcal{A}}) = \begin{cases} 0 & \text{if } i \in \mathcal{A} \\ f_{\mathcal{A}} & \text{if } i \notin \mathcal{A}. \end{cases}$$

Part B: *Suppose that every function $f \in \mathcal{F}$ can be expressed as in (4.1) and that the annihilating property holds for the operators $\mathbf{P}_1, \ldots, \mathbf{P}_d$, which are further assumed to be linear. Then the term $f_{\mathcal{A}}$ is given by $f_{\mathcal{A}} = \mathbf{P}_{-\mathcal{A}}(f) - \sum_{\mathcal{B} \subsetneq \mathcal{A}} f_{\mathcal{B}}$, which is equivalent to $f_{\mathcal{A}} = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{A} \setminus \mathcal{B}|} \mathbf{P}_{-\mathcal{B}}(f)$, and hence also to formula (4.3).*

While ([M15], Proposition 1) achieves the main objective of providing a common representation for the Hoeffding–Sobol and Möbius decompositions, the need to impose linearity in Part B implies that existence and uniqueness are no longer obtained under a common set of assumptions. The result stated in ([M15], Proposition 2), which remains valid for non-linear operators, shows that one can find a unique assumption and can even bypass the idempotence assumption.

4.4 Exploring the functional decomposition of a copula

The Hoeffding–Sobol and Möbius formulas are thus two ways of decomposing a function of several variables as a sum of terms of increasing complexity. As they were developed and used by distinct research communities, their suspicious resemblance had never been investigated before [M15]. Beyond its intrinsic interest, the existence of the common setting revealed in ([M15], Proposition 1) opens the door to cross-fertilization in their respective domains of application. Dependence modeling with copulas, or other characterizations of dependence, is a domain where this has already happen in [M12].

4.4.1 Copulas

Identifying and modeling dependencies with copulas remain an important topic, which has become very popular over the last decades since it has been applied in almost every discipline. We refer to the following non-exhaustive list of testing procedures associated with copulas: [10], [18], [162], [47], [68], [106], [120], [69], [143], [145], [89] and [71].

Consider a d-dimensional random vector $\mathbf{X}$. We assume that the cumulative distribution function (c.d.f.) F of the representative vector $\mathbf{X}$ has continuous univariate margins denoted by $F_1, \ldots, F_d$. There exists then a unique copula $C : [0,1]^d \to [0,1]$, that is a d-dimensional c.d.f. with standard uniform margins such that $F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d))$ for all $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$. This representation, due to [158], illustrates that the copula C characterizes the dependence between the components of $\mathbf{X}$.

So far, the focus has been done in this manuscript on the stdf, which can be linked to the copula. Indeed, the first order condition (3.1) implies that for any positive real $x_1, \ldots, x_d$, as t tends to infinity,

$$t\left(1 - C(1 - t^{-1}x_1, \ldots, 1 - t^{-1}x_d)\right) \to \ell(x_1, \ldots, x_d) \; .$$

Note also that if G is the max-attractor of F, its copula is the extreme value copula

$$C_G(x_1, \ldots, x_d) = \exp\left(-\ell\left(-\log x_1, \ldots, -\log x_d\right)\right) \; .$$

4.4.2 Feedback on the extension for copulas

The Hoeffding–Sobol case is reconstructed from

$$\mathbf{P}_i(C)(\mathbf{x}) = \int_0^1 C(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_d) d\lambda_i(z) \; ,$$

whereas, for Möbius, one has

$$\mathbf{P}_i(C)(\mathbf{x}) = x_i \times C(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d) \; . \tag{4.4}$$

Applied to a copula, the projection $\mathbf{P}_{\mathcal{A}}$ associated to the Hoeffding–Sobol decomposition cancels the influence of variables in the subset $\mathcal{A}$ through integration. The terms are of increasing complexity with $|\mathcal{A}|$ in the sense that $C_{\mathcal{A}}$ depends on a vector $\mathbf{x}_{\mathcal{A}}$ whose length grows with $|\mathcal{A}|$. In that decomposition, $C_\emptyset$ does not depend on C.

In contrast, the projection $\mathbf{P}_{\mathcal{A}}$ associated with the Möbius decomposition erases the stochastic dependence between the variables with indices belonging to the set $\mathcal{A}$. More precisely, if a random vector $\mathbf{Z}$ has distribution $\mathbf{P}_{\mathcal{A}}(C)$, the subvectors $\mathbf{Z}_{\mathcal{A}}$ and $\mathbf{Z}_{-\mathcal{A}}$ are then independent. While there is no increasing complexity in the terms of the decomposition, in the sense that the value of $C_{\mathcal{A}}$ at any $\mathbf{x}$ depends on both $\mathbf{x}_{\mathcal{A}}$ and $\mathbf{x}_{-\mathcal{A}}$, it is noteworthy that only the stochastic dependence embodied within $\mathbf{x}_{\mathcal{A}}$ is retained. It can be said, therefore, that the terms in the Möbius decomposition are also increasing in complexity, but in the sense of probabilistic dependence. Further note that while the term $C_\emptyset$ does not vanish in this decomposition, it does not contain any information about dependence given that $C_\emptyset = \Pi$, the independence copula.

Beyond the relative degree of complexity of their terms, the Hoeffding–Sobol and Möbius decompositions each have their comparative advantage. One key feature of the Hoeffding–Sobol decomposition is that it provides orthogonal terms, so that the structure of the function of interest can be analyzed through variances. In contrast, the terms in the Möbius decomposition are not orthogonal. However, a strong point of the Möbius decomposition is the ease with which any term can be computed as a simple alternate combination of evaluations of the function C.

4.4.3 Functional decomposition for testing copula-based assumptions

Recall the pioneering idea of Deheuvels in [43] which reveals the independence through the Möbius decomposition of the empirical process. The null hypothesis is thus equivalent to the intersection of a finite set of assumptions since all secondary terms of the decomposition vanish. Let us be more precise by introducing some notation. Let $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$ be a collection of commuting idempotent functionals and set

$$\mathbf{M}_{\mathcal{A}} = \prod_{i \in \mathcal{A}} (\mathbf{I} - \mathbf{P}_i) \prod_{i \notin \mathcal{A}} \mathbf{P}_i \ , \tag{4.5}$$

so that $f_{\mathcal{A}} = \mathbf{M}_{\mathcal{A}}(f)$ in (4.3). If $\mathbf{P}_i$ is given by (4.4), the decomposition (4.1) is the Möbius decomposition, as already said. Set $\Pi(\mathbf{x}) = x_1 \cdots x_d$ the independence copula. If $(\mathcal{H})\, C = \Pi$ then one can check that $\mathbf{M}_{\emptyset}(C) = \prod_{i=1}^d \mathbf{P}_i = \Pi$ and that $\mathbf{M}_{\mathcal{A}}(C)$ vanishes whenever $\mathcal{A} \neq \emptyset$.

We generalize in [M12] the method by applying another functional decomposition, chosen in accordance with the structural assumption being tested. Indeed, under various interesting examples, the null hypothesis describing a structural form of dependence is *often* characterized by the stability of the copula under the action of a transformation $\mathbf{M}_{\emptyset} = \prod_{i=1}^d \mathbf{P}_i$. It leads to the test of $(\mathcal{H})\, C = \mathbf{M}_{\emptyset}(C)$ against its negation. From (4.1), $C - \mathbf{M}_{\emptyset}(C) = \sum_{\mathcal{A} \in \mathcal{P}_d^\star} \mathbf{M}_{\mathcal{A}}(C)$, where $\mathcal{P}_d^\star$ stands for the non-empty subsets of $\{1, \dots, d\}$. As a consequence, the summation $\sum_{\mathcal{A} \in \mathcal{P}_d^\star} \mathbf{M}_{\mathcal{A}}(C)$ vanishes when $(\mathcal{H})$ holds true. It is thus interesting to consider for any $\mathcal{A} \in \mathcal{P}_d^\star$ the null sub-hypothesis

$$(\mathcal{H}^{\mathcal{A}})\, \mathbf{M}_{\mathcal{A}}(C) = 0 \ .$$

A relevant question is to analyze whether any $(\mathcal{H}^{\mathcal{A}})$ holds true under the null hypothesis $(\mathcal{H})$. What is its link exactly with the intersection? Part of the question finds an answer.

Proposition 7 *([M12], Proposition 1) Let $\mathbf{P}_1, \dots, \mathbf{P}_d$ be a commuting collection of idemptotent operators on $\mathcal{F}$. Then, the null hypothesis satisfies the equality*

$$(\mathcal{H}) = \bigcap_{\mathcal{A} \in \mathcal{P}_d^\star} (\mathcal{H}^{\mathcal{A}}) \ .$$

The study based on a functional decomposition reveals that a collection of sub-hypotheses $(\mathcal{H}^{\mathcal{A}})$ hold true under $(\mathcal{H})$. In consequence, new test statistics are defined by combining the information extracted from $(\mathcal{H})$ with that extracted from any $(\mathcal{H}^{\mathcal{A}})$.

4.4.4 Examples of null hypotheses and associated maps

A kind of unification of various papers, as [43], [74], [73], [110] and [111] among others, has been provided by [M12]. All derive copula-based tests of the structure of dependence. The solution explained above was to dip them in the functional decomposition context of [117] (and its recent version of [M15] which removes the linearity assumption in the existence statement) in order to reveal a common pattern. The following summary identifies for some structural dependence null hypotheses $(\mathcal{H})$, their associated set of operators $\{\mathbf{P}_1, \dots, \mathbf{P}_d\}$ that allows to test $(\mathcal{H})\, C = \mathbf{M}_{\emptyset}(C)$ against $(\mathcal{K})\, C \neq \mathbf{M}_{\emptyset}(C)$.

($\mathcal{H}$) Independence among subvectors ([M12], Section 2.3.1)

$$\mathbf{P}_i(C)(\mathbf{x}) = C(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}}) \cdot C(\mathbf{1}_{\{i\}}, \mathbf{x}_{-\{i\}})$$

($\mathcal{H}$) Complete independence among all components ([M12], Remark 2)

$$\mathbf{P}_i(C)(\mathbf{x}) = x_i \cdot C(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_d)$$

($\mathcal{H}$) Weak associativity ([M12], Section 2.3.2)

$$\mathbf{P}_i(C)(\mathbf{x}) = C(x_1, \ldots, x_i, C(1, \ldots, 1, x_{i+1}, \ldots, x_d), 1, \ldots, 1)$$

($\mathcal{H}$) Specific Archimedean copula ([M12], Section 2.3.3)

$$\mathbf{P}_i(C)(\mathbf{x}) = \varphi^{[-1]} \left[\varphi\left(C(x_i, \mathbf{1}_{-i})\right) + \varphi\left(C(1_i, \mathbf{x}_{-i})\right) - \varphi\left(C(\mathbf{1})\right)\right]$$

($\mathcal{H}$) Specific Archimedean by blocks ([M12], Section 2.3.4)

$$\mathbf{P}_i(C)(\mathbf{x}) = \varphi^{-1}\left[\varphi(C(\mathbf{x}_{\{i\}}, \mathbf{1}_{-\{i\}})) - \varphi(C(\mathbf{1})) + \varphi(C(\mathbf{x}_{-\{i\}}, \mathbf{1}_{\{i\}}))\right]$$

($\mathcal{H}$) Specific Archimax copulas ([M12], Section 2.3.5)

$$\mathbf{P}_i(C)(\mathbf{x}) = \varphi^{-1}\left[\left\{\left(\ell\left(\varphi \mathbf{x}_{\{i\}}, \mathbf{0}_{-\{i\}}\right)\right)^{\theta} + \left(\varphi \circ C\left(\mathbf{1}_{\{i\}}, \mathbf{x}_{-\{i\}}\right)\right)^{\theta}\right\}^{1/\theta}\right]$$

($\mathcal{H}$) Max-stability ([M12], Section 2.4.1)

$$\mathbf{P}_i(C)(\mathbf{x}) = C^{r_i}(\mathbf{x}^{1/r_i})$$

($\mathcal{H}$) Exchangeability ([M12], Section 2.4.2)

$$\mathbf{P}_i(C)(\mathbf{x}) = C(\mathbf{x}_{\tau_i})$$

Concluding remarks

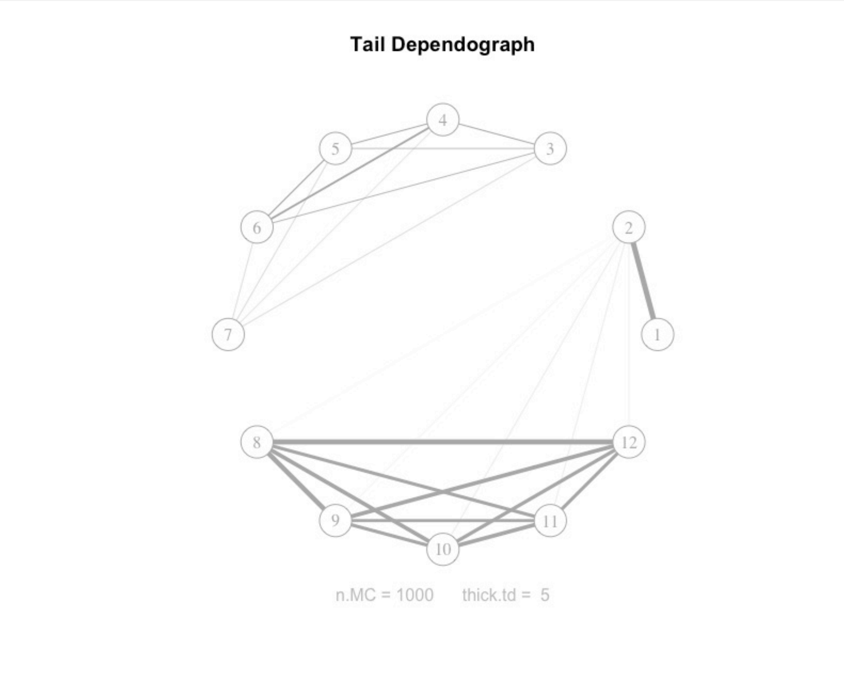
The use of functional decompositions to analyze structures of asymptotic as well as non-asymptotic dependencies has proven to be fruitful. In [M15], a common context was investigated for fixed projections (those associated with Hoeffding–Sobol and Möbius decompositions), while in [M12], the projections were identified for pre-specified statistical hypotheses, which led to the emergence of a general framework. Furthermore, this chapter had started with the global sensitivity analysis and its use in the multivariate extreme value theory. These new concepts are also included in the *satdad* R-package [M11]. We conclude this chapter by illustrating the calculation of tail importance superset coefficients and their plot through the tail dependograph on the asymptotic model created at the end of Chapter 3.

Consider the 12-dimensional dependence structure generated at the end of Chapter 3. The collection of bivariate tail superset importance coefficients is estimated below.

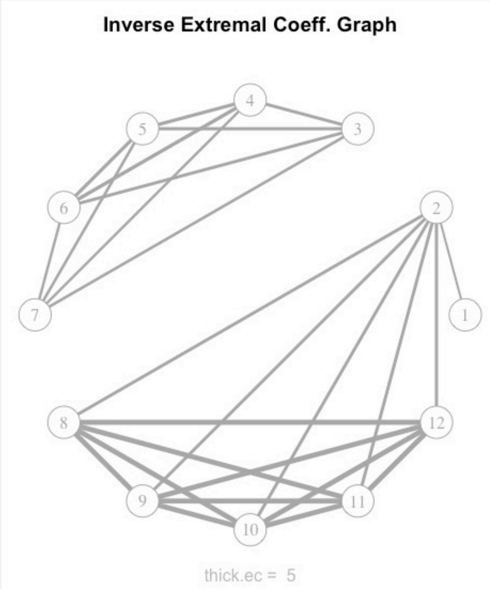
```
> tsic(ds = ds12, ind = 2)$tsic

##  [1] 1.665236e-03 2.476284e-32 2.344396e-32 2.075690e-32 2.362885e-32
##  [6] 2.540379e-32 1.253549e-32 1.305318e-32 1.539511e-32 1.345994e-32
## [11] 1.491440e-32 2.602008e-32 2.462725e-32 2.493540e-32 2.382606e-32
## [16] 2.625428e-32 2.227552e-05 9.369008e-05 2.035716e-05 4.187246e-05
## [21] 9.017091e-05 2.238604e-04 1.922358e-04 1.707785e-04 1.093489e-04
## [26] 1.482812e-32 1.488975e-32 1.702214e-32 1.719470e-32 1.559233e-32
## [31] 3.295230e-04 4.743226e-04 7.927839e-05 1.397763e-32 1.435973e-32
## [36] 1.541977e-32 1.477882e-32 1.447067e-32 3.882164e-04 1.130791e-04
## [41] 1.601141e-32 1.569094e-32 1.752750e-32 1.527185e-32 1.527185e-32
## [46] 8.666063e-05 1.461858e-32 1.421182e-32 1.564163e-32 1.380507e-32
## [51] 1.458160e-32 1.587583e-32 1.744122e-32 1.691121e-32 1.555535e-32
## [56] 1.597443e-32 1.205515e-03 1.403054e-03 1.136166e-03 1.040342e-03
## [61] 1.089571e-03 8.862386e-04 9.601708e-04 1.279907e-03 9.065016e-04
## [66] 1.200806e-03
```

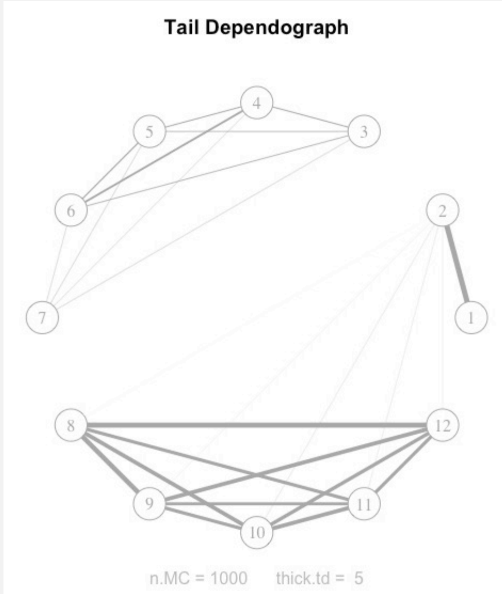
These pairwise indices are plotted (renormalized, among others, by their maximal value) in the so-called Tail dependograph.

```
> graphs(ds = ds12)
```



To compare the two graphs, we plot them side by side.





— The first common feature is the ability of both bivariate collections to separate the support of the associated stdf into two blocks.

— The second common feature is that within a common homogeneous group ($\{3, 5, 7\}$ or $\{8, \ldots, 12\}$ for instance), the graphs both illustrate this resemblance.

— The first difference is the relief given by the two graphs. On the left, the values can be hierarchized, but one has to observe the edges' thickness carefully to obtain an order. On the right, the ordering appears more evident.

— The second difference is that the resulting order is not the same, which is obvious since we are not measuring the same characteristics. It comes from the fact that the tail dependograph shares a structural analysis of the stdf. The probabilistic interpretation is lost except for homogeneous comparison.

- Let us focus indeed on the component 2. Its strong link with 1 on the tail dependograph only reflects that $\beta_{2,1} = 0.4072452$ is still large, considering that $\alpha_1 = 0.0474982$ (the smallest value among the dependence parameters). If $\beta_{2,1}$ decreases to 0, its link to the group 8-12 will obviously predominate in comparison with its link to 1. Recall that $\beta_{2,4} = 0.5927548$ and $\alpha_4 = 0.0659264$.
- We provide another illustration by focusing now on the other group. The dissimilarities between $\{4, 6\}$ and $\{3, 5, 7\}$ appear more clearly in the tail dependograph. Note that in these groups, the asymmetric coefficients and the dependence parameters go in the same direction. In other words : $\beta_{4,2} = \beta_{6,2} = 1 > \beta_{j,2}$ for $j = 3, 5, 7$ and $\alpha_2 = 0.4092889 < 0.7714904 = \alpha_3$.

5

Non-parametric statistics

So far, the work presented can be described as *modeling*, through the study of the stdf, its structure analysis by a finite number of coefficients or the rewriting of testing hypotheses. In addition to modeling, we have used several mathematical concepts to study the properties of each abstract object. In this chapter, the non-parametric counterpart of what has been presented hitherto is examined.

Given a sample of multivariate observations, how can we estimate the tail of the underlying multivariate distribution? By definition, the stdf is easily linked to the probability that at least one of the variables is large. Under a general setting, the estimation of the tail margins would be first required. However, under standard Pareto margins, for instance, if $\mathbf{x} \sim t\mathbf{y}$ for $\mathbf{y}$ in the positive orthant and with large postive t,

$$\mathbb{P}(\mathbf{X} \in [\mathbf{0}, t\mathbf{y}]^c) \simeq t^{-1} \|\mathbf{y}^{-1}\| \, \ell(\mathbf{y}^{-1}/\|\mathbf{y}^{-1}\|) \ ,$$

so the estimation of such probability relies on that of ℓ. The approximation above only relies on (3.1). The usual balance between bias and variance occurs in multivariate extreme value theory with respect to the size of k, the number of largest observations considered as sufficiently large. The first part below provides the asymptotic bias reduction of the non-parametric estimator of the stdf for large k. In the second part, estimators for the new global sensitivity measures for tail dependence are defined and studied. Finally, new test statistics derived from the functional decomposition are introduced and analyzed in the last part of this chapter, both from a theoretical and numerical point of view.

5.1 Bias correction procedure

The nonparametric estimation of the extremal dependence structure has been widely studied in the bivariate case, see for instance [96], [51], [31], [1], [81] and [21]. Bias correction problems in the bivariate context received less attention than in the univariate setting, see [7] and [79], which consider the estimation of bivariate joint tails. As for the multivariate framework, [85] introduces the empirical estimator (5.1). General approaches under parametric assumptions on the stdf have been developed e.g. by [37], [103], [54], [55] and [52].

The paper [M5] proposes the first procedure correcting the bias for dimension greater than two. Note that our method does not consist only of applying the univariate bias procedure at several points. Indeed, the bias is not anymore a parametric function, so that the new feature is mainly the fact that we are able to estimate and then subtract a function with an unknown form.

5.1.1 First development

Let $\mathbf{X}_1 = (X_1^{(1)}, \dots, X_1^{(d)}), \dots, \mathbf{X}_n = (X_n^{(1)}, \dots, X_n^{(d)})$ be independent and identically distributed multivariate random vectors with cdf F and continuous marginal cdfs F_j for $j = 1, \dots, d$. Consider an intermediate sequence, that is to say a sequence $k = k(n)$ tending to infinity with n such that $k/n \to 0$. Denote by $X_{k,n}^{(j)}$ the kth order statistics among n realisations of the margins $X^{(j)}$. The empirical estimator of $\ell(\mathbf{x})$ is obtained from replacing F by its empirical version, t by n/k, and $F_j^{-1}(1-t^{-1}x_j)$ for $j=1,\ldots,d$ by its empirical counterpart $X^{(j)}_{n-[nt^{-1}x_j],n}$, so that

$$\hat{\ell}_k(\mathbf{x}) = \frac{1}{k}\sum_{i=1}^{n} \mathbf{1}_{\left\{X_i^{(1)} \geq X^{(1)}_{n-[kx_1]+1,n} \text{ or } \ldots \text{ or } X_i^{(d)} \geq X^{(d)}_{n-[kx_d]+1,n}\right\}} . \tag{5.1}$$

Under suitable conditions, it can be shown that the estimator $\hat{\ell}_k(\mathbf{x})$ has the following asymptotic expansion

$$\hat{\ell}_k(\mathbf{x}) - \ell(\mathbf{x}) \approx \frac{\mathbb{W}_\ell(\mathbf{x})}{\sqrt{k}} + \alpha(n/k)M(\mathbf{x}) ,$$

where $\mathbb{W}_\ell$ is a continuous centered Gaussian process, α is a function that tends to 0 at infinity, and M is a continuous function. Such notation is in correspondence with the setting described in Section 3.1. In particular $\sqrt{k}\{\hat{\ell}_k(\mathbf{x}) - \ell(\mathbf{x})\}$ can be approximated in distribution by $\mathbb{W}_\ell(\mathbf{x})$, provided that $\sqrt{k}\alpha(n/k)$ tends to 0 as n tends to infinity. This condition imposes a slow rate of convergence of the estimator $\hat{\ell}_k(\mathbf{x})$, so one would be interested in relaxing this hypothesis. As a counterpart, as soon as $\sqrt{k}\alpha(n/k)$ tends to a non null constant λ, an asymptotic bias appears and is explicitely given by $\lambda M(\mathbf{x})$. The aim of [M5] is to provide a procedure that reduces the asymptotic bias. The latter is estimated and then subtracted from the empirical estimator. This kind of approach has been considered in the univariate setting for the bias correction of the extreme value index with unknown sign by [28]. Refer also to [139], [64], [80], [27] and [138] for previous contributions on this univariate problem. Note finally that the case of time series has been studied in [M8].

Let us introduce the notation $\mathcal{R}(\mathbf{x}) := [\mathbf{x}, \infty]^c$. Recall that for $\mathbf{w} \in C$ the unit cube associated with the max-norm, $\lambda_{\mathbf{w}}$ stands for the image of the Lebesgue measure on $\mathbb{R}_+$ under the mapping $s \mapsto s/\mathbf{w}$. As already explained in Section 3.3, $\ell(\mathbf{x}) = \mu(\mathcal{R}(\mathbf{x}))$ where $\mu = \ell(\mathbf{1})\int_C \lambda_{\mathbf{w}} d\nu(\mathbf{w})$. For a positive T, let $D([0,T]^d)$ be the space of real valued functions that are right-continuous with left-limits.

Proposition 8 *([M5], Proposition 2) Suppose that the thrird order condition is satisfied, so that (3.3) and (3.4) hold. Therein, assume that the regular variation coefficients ρ and ρ' of the functions α and β are negative, the function M is differentiable and the function N is continuous. Suppose further that the first order partial derivatives of ℓ (denoted by $\partial_j\ell$ for $j=1,\ldots,d$) exist and that $\partial_j\ell$ is continuous on the set of points $\{\mathbf{x}=(x_1,\ldots,x_d) \in \mathbb{R}_+^d : x_j > 0\}$. Assume k is such that $\sqrt{k}\alpha(n/k) \to \infty$ and $\sqrt{k}\alpha(n/k)\beta(n/k) \to 0$. Then as n tends to infinity,*

$$\sqrt{k}\left\{\hat{\ell}_k(\mathbf{x}) - \ell(\mathbf{x}) - \alpha(n/k)M(\mathbf{x})\right\} \xrightarrow{d} \mathbb{W}_\ell(\mathbf{x}) , \tag{5.2}$$

in $D([0,T]^d)$ for every $T>0$ where

$$\mathbb{W}_\ell(\mathbf{x}) := \mathbb{G}_\ell(\mathbf{x}) - \sum_{j=1}^{d} \partial_j\ell(\mathbf{x})\mathbb{G}_\ell(x_j, \mathbf{0}_{-j}) , \tag{5.3}$$

and where the process $\mathbb{G}_\ell$ above is a continuous centered Gaussian process with covariance structure $\mathbb{E}[\mathbb{G}_\ell(\mathbf{x})\mathbb{G}_\ell(\mathbf{y})] = \mu(\mathcal{R}(\mathbf{x}) \cap \mathcal{R}(\mathbf{y}))$.

5.1.2 Bias corrected estimator

Equation (5.2) suggests a natural correction of $\hat{\ell}_k$ as soon as an estimator of $\alpha(n/k)M(\mathbf{x})$ is available. In order to take advantage of the homogeneity of ℓ, let us introduce a positive scale parameter a which allows to contract or to dilate the observed points. We denote

$$\hat{\ell}_{k,a}(\mathbf{x}) := a^{-1}\hat{\ell}_k(a\mathbf{x}) , \qquad \text{and} \qquad \hat{\Delta}_{k,a}(\mathbf{x}) := \hat{\ell}_{k,a}(\mathbf{x}) - \hat{\ell}_k(\mathbf{x}) .$$

Fixing a such that $a^{-\rho} - 1 = 1$, a natural estimator of the asymptotic bias of $\hat{\ell}_k(\mathbf{x})$ is thus $\hat{\Delta}_{k,2^{-1/\hat{\rho}}}(\mathbf{x})$, where $\hat{\rho}$ is an estimator of ρ. Recall that the unknown parameter ρ is the regular variation index of the function α involved in the second order condition (3.3). Let k_ρ, such that $k_\rho \gg k$, be an intermediate sequence that represents the number of order statistics used in the estimator $\hat{\rho}$. A first asymptotically unbiased estimator of $\ell(\mathbf{x})$ can be defined as

$$\mathring{\ell}_{k,1,k_\rho}(\mathbf{x}) := \hat{\ell}_k(\mathbf{x}) - \hat{\Delta}_{k,2^{-1/\hat{\rho}}}(\mathbf{x}) \, .$$

The previous construction can be easily generalized by correcting the estimator $\hat{\ell}_{k,a}$ instead of $\hat{\ell}_k$. Indeed, from (5.2) one can see that the asymptotic bias of $\hat{\ell}_{k,a}(\mathbf{x})$ is $\alpha(\frac{n}{k})a^{-\rho}M(\mathbf{x})$. Recall that when n tends to infinity, one has for any positive real b,

$$\frac{\hat{\Delta}_{k,b}(\mathbf{x})}{\alpha(\frac{n}{k})} \xrightarrow{\mathbb{P}} (b^{-\rho} - 1)M(\mathbf{x}) \, .$$

Thus, fixing b such that $b^{-\rho} - 1 = a^{-\rho}$ will help for canceling the asymptotic bias. It yields the following asymptotically unbiased estimator of ℓ

$$\mathring{\ell}_{k,a,k_\rho}(\mathbf{x}) := \hat{\ell}_{k,a}(\mathbf{x}) - \hat{\Delta}_{k,(a^{-\hat{\rho}}+1)^{-1/\hat{\rho}}}(\mathbf{x}) \, .$$

Theorem 9 *([M5], Theorem 3) Assume that the conditions of ([M5], Proposition 2) are fulfilled. Let k_ρ be an intermediate sequence such that $\sqrt{k_\rho}\alpha(n/k_\rho)(\hat{\rho} - \rho)$ converges in distribution. Suppose also that k is such that $k = o(k_\rho)$, $\sqrt{k}\alpha(n/k) \to \infty$ and $\sqrt{k}\alpha(n/k)\beta(n/k) \to 0$. Under these assumptions, as n tends to infinity,*

$$\sqrt{k}\left\{\mathring{\ell}_{k,a,k_\rho}(\mathbf{x}) - \ell(\mathbf{x})\right\} \xrightarrow{d} \mathring{\mathbb{W}}_{\ell,a}(\mathbf{x}) \, ,$$

in $D([0,T]^d)$ for every $T > 0$, where $\mathring{\mathbb{W}}_{\ell,a}$ is a continuous centered Gaussian process defined by

$$\mathring{\mathbb{W}}_{\ell,a}(\mathbf{x}) := \mathbb{W}_\ell(\mathbf{x}) - (1 + a^{-\rho})^{1/\rho}\, \mathbb{W}_\ell((1 + a^{-\rho})^{-1/\rho}\mathbf{x}) + a^{-1}\mathbb{W}_\ell(a\mathbf{x})$$

with covariance $\mathbb{E}[\mathring{\mathbb{W}}_{\ell,a}(\mathbf{x})\mathring{\mathbb{W}}_{\ell,a}(\mathbf{y})] = \mathbb{E}[\mathbb{W}_\ell(\mathbf{x})\mathbb{W}_\ell(\mathbf{y})]\left(1 + a^{-1/2} - (1 + a^{-\rho})^{1/(2\rho)}\right)^2$.

5.1.3 In practice

For any model among those presented in Section 3.2, the parameters have been tuned in practice as follows: $n = 1000$, $a = 0.4$, $k_\rho = 99/100 \times n$ and ρ estimated using

$$\hat{\rho}_{k,a,r}(\mathbf{x}) := \left(1 - \frac{1}{\log r} \log \left| \frac{\hat{\Delta}_{k,a}(r\mathbf{x})}{\hat{\Delta}_{k,a}(\mathbf{x})} \right| \right) \wedge 0$$

with $a = r = 0.4$. Another way to estimate ρ could have been to use the techniques developed in the univariate setting, as in [M4], and combine the estimates associated with each margin. Note that the empirical estimator $\hat{\ell}_k$ behaves fairly well in terms of bias for small values of k. Besides, the bias is efficiently corrected by the estimator $\mathring{\ell}_k = \mathring{\ell}_{k,0.4,990}$. Since the bias almost vanishes along the range of k, one can think about reducing the variance through an aggregation in k (*via* mean or median) of $\mathring{\ell}_k$. This leads to consider the following estimator

$$\mathring{\ell}_{\text{agg}} := \text{Median}(\mathring{\ell}_k, k = 1, \cdots, \kappa_n) \, ,$$

where n is the sample size and κ_n is an appropriate fraction of n. If κ_n satisfies the condition imposed on k_n in ([M5], Theorem 3), then the aggregated estimators $\mathring{\ell}_{\text{agg}}$ would inherit the asymptotic properties of $\mathring{\ell}_k$. Indeed, all the estimators jointly converge, since they are based on a single empirical process. The fraction κ_n is arbitrarily fixed to $n-1$. Such a choice is open to criticism since it does not satisfy the theoretical assumptions mentioned in the previous remark. But it is motivated here by the fact that the bias happened to be efficiently corrected even for very large values of k. Note however that such a choice would not be systematically the right one. In presence of more complex models such as mixtures, κ_n should not exceed the size of the subpopulation with heaviest tail.

To illustrate the practical use of the bias correction procedure, the estimation of the probability $\mathbb{P}(X > 10^4 \text{ or } Y > 2 \times 10^4)$ is studied under two specific settings. Sampling bivariate data from BPII(β) with $\beta = 3$, see Section 3.2.1, the margins are assumed to be known on the left panel of Figure 5.1, whereas they are also estimated on the right. The true value is given by the black line, the bias corrected and aggregated estimate $\mathring{\ell}_{\text{agg}}$ is in red and the collection of original non-parametric estimates $\hat{\ell}_k$ are in green. More illustrations are provided in [M5].

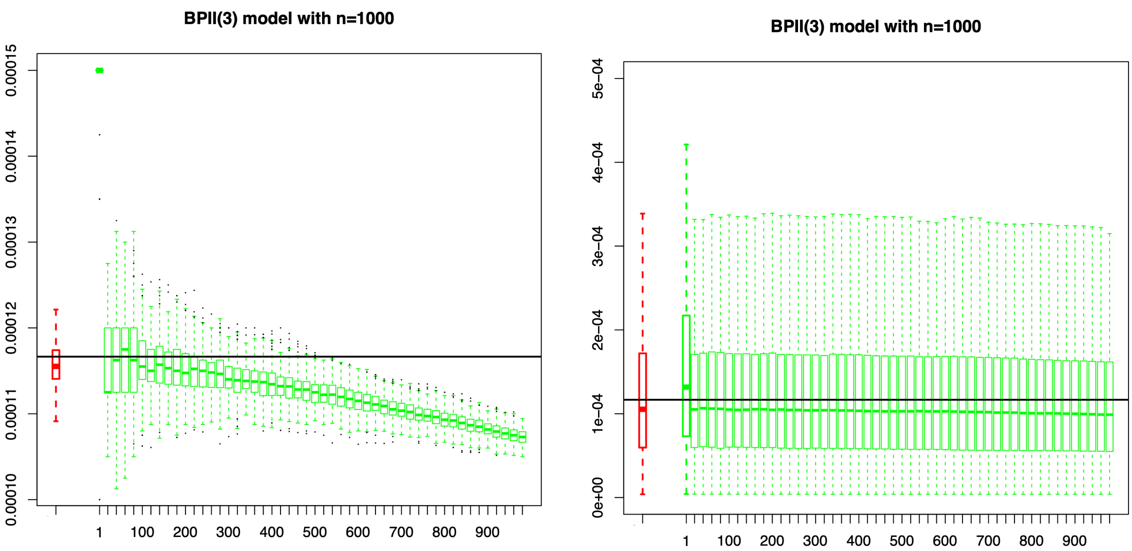


Figure 5.1: Boxplot (based on 500 replicates) for the estimation of $\mathbb{P}(X > 10^4 \text{ or } Y > 2 \times 10^4)$ when (X, Y) is drawn from the BPII(3) model with sample size $n = 1000$. In red: the bias corrected and aggregated estimate $\mathring{\ell}_{\text{agg}}$. In green: the collection of original non-parametric estimates $\hat{\ell}_k$.

5.2 The empirical tail measures

In this part, the empirical counterpart of the tail superset importance coefficients (see Section 4.2.1) and the tail dependograph (from Section 4.2.2) are introduced. These statements come from [M14]. The Hoeffding–Sobol decomposition of the function $\hat{\ell}_{k,n}$ is given by

$$\hat{\ell}_{k,n}(\mathbf{x}) = \sum_{\mathcal{A} \subseteq \{1,\dots,d\}} \hat{\ell}_{k,n;\mathcal{A}}(\mathbf{x})$$

where the subfunctions are centered and orthogonal as explained in Section 4.1.1. The framework under study is fairly unusual in global sensitivity analysis since most of the empirical computation can be done and all the terms have an explicit formula as stated below.

5.2.1 Rank-based expressions

The notation is update here so that i and j are no more used to describe the sample but a pair in $\{1, \dots, d\}$. More precisely, for $s = 1, \dots, n$ and $t = 1, \dots, d$, set $\tilde{R}_s^{(t)} := (n - R_s^{(t)} + 1)/k$

where $R_s^{(t)}$ denotes the rank of $X_s^{(t)}$ among $X_1^{(t)}, \dots, X_n^{(t)}$. The empirical stdf (5.1) is a tensor-product function since

$$\hat{\ell}_{k,n}(\mathbf{x}) = \frac{1}{k} \sum_{s=1}^{n} \left(1 - \prod_{t=1}^{d} \mathbf{1}\{x_t < \tilde{R}_s^{(t)}\} \right) = \frac{n}{k} - \frac{1}{k} \sum_{s=1}^{n} \prod_{t=1}^{d} \mathbf{1}\{x_t < \tilde{R}_s^{(t)}\}. \tag{5.4}$$

The proof of the next result relies on both (5.4) and ([M14], Lemma 2), that is a technical preliminary lemma for the the sensitivity analysis of tensor-product functions. The latter lemma is also very similar to ([38], Proposition 4.6).

Theorem 10 *([M14], Theorem 2) For any non-empty $\mathcal{A} \subseteq \{1, \dots, d\}$, the associated term in the decomposition is*

$$\hat{\ell}_{k,n;\mathcal{A}}(\mathbf{x}) = -\frac{1}{k} \sum_{s=1}^{n} \left\{ \prod_{t \in \mathcal{A}} \left(\mathbf{1}\{x_t < \tilde{R}_s^{(t)}\} - \lambda_t\left(\tilde{R}_s^{(t)}\right) \right) \prod_{t \notin \mathcal{A}} \lambda_t\left(\tilde{R}_s^{(t)}\right) \right\},$$

and its variance

$$\sigma_{\mathcal{A}}^2(\hat{\ell}_{k,n}) = \frac{1}{k^2} \sum_{s,s'=1}^{n} \prod_{t \in \mathcal{A}} \left\{ \lambda_t\left(\tilde{R}_s^{(t)} \wedge \tilde{R}_{s'}^{(t)}\right) - \lambda_t\left(\tilde{R}_s^{(t)}\right) \lambda_t\left(\tilde{R}_{s'}^{(t)}\right) \right\} \prod_{t \notin \mathcal{A}} \lambda_t\left(\tilde{R}_s^{(t)}\right) \lambda_t\left(\tilde{R}_{s'}^{(t)}\right)$$

is an estimate of $\sigma_{\mathcal{A}}^2(\ell)$. The superset part of the Hoeffding–Sobol decomposition, that contains the variables $\{x_t, t \in \mathcal{A}\}$, is given by

$$\sum_{\mathcal{B} \supseteq \mathcal{A}} \hat{\ell}_{k,n;\mathcal{B}}(\mathbf{x}) = -\frac{1}{k} \sum_{s=1}^{n} \prod_{t \in \mathcal{A}} \left(\mathbf{1}\{x_t < \tilde{R}_s^{(t)}\} - \lambda_t\left(\tilde{R}_s^{(t)}\right) \right) \prod_{t \notin \mathcal{A}} \mathbf{1}\{x_t < \tilde{R}_s^{(t)}\},$$

with variance

$$\Upsilon_{\mathcal{A}}^2(\hat{\ell}_{k,n}) = \frac{1}{k^2} \sum_{s,s'=1}^{n} \left\{ \prod_{t \in \mathcal{A}} \left(\lambda_t\left(\tilde{R}_s^{(t)} \wedge \tilde{R}_{s'}^{(t)}\right) - \lambda_t\left(\tilde{R}_s^{(t)}\right) \lambda_t\left(\tilde{R}_{s'}^{(t)}\right) \right) \prod_{t \notin \mathcal{A}} \lambda_t\left(\tilde{R}_s^{(t)} \wedge \tilde{R}_{s'}^{(t)}\right) \right\}.$$

Additionally, the constant term in the Hoeffding–Sobol decomposition can be written as

$$\hat{\ell}_{k,n;\emptyset} = \frac{n}{k} - \frac{1}{k} \sum_{s=1}^{n} \prod_{t=1}^{d} \lambda_t\left(\tilde{R}_s^{(t)}\right).$$

By construction, the inequality $\sigma_{\mathcal{A}}^2(\ell) \leq \Upsilon_{\mathcal{A}}^2(\ell)$ is verified by the empirical estimates whose expressions are stated above. This can be directly checked by observing that

$$\lambda_t\left(\tilde{R}_s^{(t)}\right) \lambda_t\left(\tilde{R}_{s'}^{(t)}\right) = \lambda_t\left(\tilde{R}_s^{(t)} \wedge \tilde{R}_{s'}^{(t)}\right) \lambda_t\left(\tilde{R}_s^{(t)} \vee \tilde{R}_{s'}^{(t)}\right) \leq \lambda_t\left(\tilde{R}_s^{(t)} \wedge \tilde{R}_{s'}^{(t)}\right).$$

Set now $\bar{R}_s^{(t)} := \tilde{R}_s^{(t)} \wedge 1$. All preceding expressions can be simplified thanks to the equalities $\lambda_t(\tilde{R}_s^{(t)}) = \bar{R}_s^{(t)}$ and $\lambda_t(\tilde{R}_s^{(t)} \wedge \tilde{R}_{s'}^{(t)}) = \bar{R}_s^{(t)} \wedge \bar{R}_{s'}^{(t)}$, when each probability measure λ_t is the standard uniform distribution. In particular, the pairwise empirical tail superset importance coefficient is

$$\Upsilon_{ij}^2(\hat{\ell}_{k,n}) = \frac{1}{k^2} \sum_{s,s'=1}^{n} \left\{ \prod_{t \in \{i,j\}} \left(\bar{R}_s^{(t)} \wedge \bar{R}_{s'}^{(t)} - \bar{R}_s^{(t)} \bar{R}_{s'}^{(t))} \right) \prod_{t \notin \{i,j\}} \bar{R}_s^{(t)} \wedge \bar{R}_{s'}^{(t)} \right\}.$$

Since these quantities are tractable, the empirical tail superset importance coefficients are defined as $\Upsilon_{\mathcal{A}}^2(\hat{\ell}_{k,n})$ and the empirical tail dependograph is defined as the dependency graph whose weights are given by the pairwise empirical tail superset importance coefficients $\Upsilon_{ij}^2(\hat{\ell}_{k,n})$.

5.2.2 Consistency and asymptotic distribution

The uniform convergence is valid for any subfunction of the Hoeffding–Sobol decomposition, as stated in the next result.

Corollary 11 *([M14], Corollary 1) Assume the first order condition so that the limit given in (3.1) holds uniformly on $[0,1]^d$. Let k be an intermediate sequence such that as n tends to infinity, $k \to \infty$ with $k = o(n)$. All the empirical subfunctions defined in ([M14], Theorem 2) are uniformly consistent*

$$\sup_{\mathbf{x}\in[0,1]^d} |\hat{\ell}_{k,n;\mathcal{A}}(\mathbf{x}) - \ell_{\mathcal{A}}(\mathbf{x})| \xrightarrow{\mathbb{P}} 0 \,.$$

As a consequence, the empirical variances $\sigma^2_{\mathcal{A}}(\hat{\ell}_{k,n})$ or $\Upsilon^2_{\mathcal{A}}(\hat{\ell}_{k,n})$ are consistent for their equivalent theoretical forms. In particular, the empirical tail dependograph is consistent.

The asymptotic distributional expansion (5.3) is inherited by $\hat{\ell}_{k,n;\mathcal{A}}$ and that of the empirical tail superset importance coefficients are also derived in ([M14], Proposition 3). More precisely, assume the second order condition is satisfied, so that (3.3) holds. Assume k is such that $\sqrt{k}\alpha(n/k) \to \lambda$.

- If $\Upsilon^2_{\mathcal{A}}(\ell) > 0$ then

$$\sqrt{k}\left(\Upsilon^2_{\mathcal{A}}(\hat{\ell}_{k,n}) - \Upsilon^2_{\mathcal{A}}(\ell)\right) \xrightarrow{d} \frac{1}{2^{|\mathcal{A}|-1}} \int_{[0,1]^{d+|\mathcal{A}|}} D^{\mathbf{x}_{\mathcal{A}}}_{\mathbf{z}_{\mathcal{A}}}\ell(\cdot, \mathbf{z}_{-\mathcal{A}})\, D^{\mathbf{x}_{\mathcal{A}}}_{\mathbf{z}_{\mathcal{A}}}\{\mathbb{W}_\ell + \lambda M\}(\cdot, \mathbf{z}_{-\mathcal{A}}) d\mathbf{x}_{\mathcal{A}} d\mathbf{z} \,.$$

- Whereas if $\Upsilon^2_{\mathcal{A}}(\ell) = 0$ then $k\Upsilon^2_{\mathcal{A}}(\hat{\ell}_{k,n}) \to \Upsilon^2_{\mathcal{A}}(\mathbb{W}_\ell + \lambda M)$.

The previous lines can be used to test the asymptotic independence between two variables or groups of variables. The illustrations for the new theoretical and empirical tail measures have been extracted from the text and placed at the end of each chapter. This has also been done in the current chapter, highlighting the use of the *satdad* package [M11].

5.3 Testing copula-based dependence hypotheses

This part refers back to the problem presented in Section 4.4.3. Recall that a general writing for several copula-based dependence hypotheses has been identified as $(\mathcal{H})\, \mathbf{M}_\emptyset(C) = 0$ for $\mathbf{M}_\emptyset$ and more generally $\mathbf{M}_{\mathcal{A}}$ defined as the composition of a collection of operators $\mathbf{P}_i$ chosen in accordance with the meaning of $(\mathcal{H})$. See Equation (4.5). Start from a copula estimator C_n, one might think in one of the four examples listed in the middle of the next part. It is natural to construct the testing process as $(\sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n))(\mathbf{x}), \mathbf{x} \in [0,1]^d)$ when considering $(\mathcal{H})$. This is precisely what is done in the literature. Nevertheless, since $(\mathcal{H})$ implies any sub-hypothesis $(\mathcal{H}^{\mathcal{A}})\, \mathbf{M}_{\mathcal{A}}(C) = 0$, as stated in Section 4.4.3, another choice is possible.

5.3.1 Empirical testing process

Consider $\mathbf{X}_1, \dots, \mathbf{X}_n$ a sample of d-variate observations of $\mathbf{X}$ where $\mathbf{X}_j$ stands for $(X_{j1}, \dots, X_{jd})$. Set $\mathbf{U}_j = (F_1(X_{j1}), \dots, F_d(X_{jd}))$ for $j \in \{1, \dots, n\}$ where, for $i = 1, \dots, d$, F_i denotes the true cumulative function of the ith margin. The empirical cumulative distribution function (cdf) based on $\mathbf{U}_1, \dots, \mathbf{U}_n$ is denoted by G_n and we set $\mathbb{G}_n = \sqrt{n}(G_n - C)$.
Introduce $D_{\emptyset,n} := \sqrt{n}(C_n - \mathbf{M}_\emptyset(C_n))$ and $D_{\mathcal{A},n} := \sqrt{n}\mathbf{M}_{\mathcal{A}}(C_n)$. We study in [M12] the concatenated empirical testing process

$$\left(D_{\emptyset,n}, \{D_{\mathcal{A},n}\}_{\mathcal{A}\in\mathcal{P}^\star_d}\right) \,.$$

We assume regular conditions, so that the empirical process $\mathbb{G}_n$ converges weakly in $\ell^\infty([0,1]^d)$ to a tight centered Gaussian process $\mathbb{G}_C$ concentrated on

$$\mathcal{C}_0 = \Big\{ h \in C([0,1]^d) \text{ such that } h(\mathbf{1}) = 0 \text{ and} \\ h(\mathbf{x}) = 0 \text{ if some components of } \mathbf{x} \text{ are equal to } 0 \Big\} .$$

Let us also introduce

$$\mathbb{W}_C(\mathbf{x}) = \mathbb{G}_C(\mathbf{x}) - \sum_{i=1}^{d} \partial C_i(\mathbf{x}) \mathbb{G}_C(x_i, \mathbf{1}_{-i}), \qquad \mathbf{x} \in [0,1]^d . \tag{5.5}$$

Theorem 12 *([M12], Theorem 3.1) Consider* $(\mathcal{H})\, C = \mathbf{M}_\emptyset(C)$*, depending through (4.5) on a set of operators* $\mathbf{P}_1, \ldots, \mathbf{P}_d$*. It is assumed, at least when* $(\mathcal{H})$ *holds true, that*

- $\{\mathbf{P}_1, \ldots, \mathbf{P}_d\}$ *are commuting and idempotent maps.*
- *The associated maps* $\{\mathbf{M}_\mathcal{A}\}_{\mathcal{A} \in \mathcal{P}_d}$ *are Hadamard-differentiable at* C *tangentially to* $\mathcal{C}_0$*.*

Consider an empirical copula C_n *such that, as* n *tends to infinity, the empirical copula process* $\sqrt{n}(C_n - C)$ *converges weakly in* $\ell^\infty([0,1]^d)$ *to* $\mathbb{W}_C$ *given in (5.5).*
Then, under $(\mathcal{H})$ *and as* n *tends to infinity, the joint empirical processes converge weakly in* $\{\ell^\infty([0,1]^d)\}^{2^d}$ *as following*

$$\Big(D_{\emptyset,n}, \{D_{\mathcal{A},n}\}_{\mathcal{A} \in \mathcal{P}_d^\star} \Big) \underset{n\to\infty}{\overset{w}{\rightsquigarrow}} \Big(\mathbb{W}_C - \mathbf{M}'_\emptyset(C; \mathbb{W}_C), \{\mathbf{M}'_\mathcal{A}(C; \mathbb{W}_C)\}_{\mathcal{A} \in \mathcal{P}_d^\star} \Big) . \tag{5.6}$$

The last lines of ([112], Section 2) list carefully the conditions ensuring the required convergence of the empirical copula process for the following list of well-known empirical copulas

- the non-parametric estimators

$$\tilde{C}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} \prod_{i=1}^{d} \mathbf{1}_{\{\hat{F}_{i,n}(X_{ji}) \le x_i\}} \qquad \text{and} \qquad \hat{C}_n(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} \prod_{i=1}^{d} \mathbf{1}_{\{R_{ji,n} \le n\, x_i\}} ,$$

where $\hat{F}_{i,n}$ is the ith marginal empirical cdf and $R_{ji,n}$ is the rank of X_{ji} among $X_{1i}, \ldots, X_{ni}$,
- the empirical checkerboard copula

$$C_n^{\#}(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} \prod_{i=1}^{d} \min\{\max\{n\, x_i - R_{ji,n} + 1, 0\}, 1\} ,$$

- and the empirical beta copula,

$$C_n^{\beta}(\mathbf{x}) = \frac{1}{n} \sum_{j=1}^{n} \prod_{i=1}^{d} \mathrm{F}_{n,R_{ji,n}}(x_i) ,$$

where $\mathrm{F}_{n,r}$ stands for the pdf of the Beta distribution $\mathcal{B}(r, n+1-r)$.

Before applying our results with another copula estimator than those listed above, one should first check that the associated empirical copula process satisfies the required convergence. For instance, in the case of the check-min-erboard estimators of C studied in [125] and [40], only part of the question is answered by [112].

The limiting covariance structures in (5.6) of the processes $\mathbb{D}_\emptyset := \mathbb{W}_C - \mathbf{M}'_\emptyset(C; \mathbb{W}_C)$ and $\mathbb{D}_\mathcal{A} := \mathbf{M}'_\mathcal{A}(C; \mathbb{W}_C)$ depend on the unknown copula C. For this reason it is not directly applicable for statistical testing. Multiplier bootstrap (see [154], [146], [20], [155], [24], [111], [15]) and subsampling (see [142], [14], [11], [12], [112]) have been introduced in the literature to reproduce independently the asymptotic behavior of such processes. They are extended to the present setting in ([M12], Theorem 3.2) for subsampling, ([M12], Theorem 3.3) for weighted subsampling and ([M12], Theorem 3.4) for bootstrap based on multipliers. For other references on empirical copula processes, see also [59], [72], [76], [60], [17], [26], [22], [25], [70], [157], [105] and [23].

5.3.2 New test statistics

Set $\hat{\mathbf{U}}_{j\cdot,n} = (R_{j1,n}, \ldots, R_{jd,n})/n$. The Cramér-von Mises statistics associated with subset $\mathcal{A}$ and weight function q is defined by

$$S_{\mathcal{A},n,q} = \frac{1}{n} \sum_{j=1}^{n} \left\{ \frac{D_{\mathcal{A},n}(\hat{\mathbf{U}}_{j\cdot,n})}{q(\hat{\mathbf{U}}_{j\cdot,n})} \right\}^2$$

and its associated limit as

$$\mathbb{S}_{\mathcal{A},q} = \int_{[0,1]^d} \left\{ \frac{\mathbb{D}_{\mathcal{A}}(\mathbf{x})}{q(\mathbf{x})} \right\}^2 dC(\mathbf{x}) \, .$$

Let $w = \{w^{\mathcal{A}}\}_{\mathcal{A}\in\mathcal{P}_d}$ be a vector of positive weights. The latter reflects the importance we put in the test $(\mathcal{H})\, C = \mathbf{M}_{\emptyset}(C)$ through $w^{\emptyset}$, or in the test $(\mathcal{H}^{\mathcal{A}})$ $\mathbf{M}_{\mathcal{A}}(C) = 0$ through $w^{\mathcal{A}}$. We introduce the combined Cramér-von Mises statistics and associated limits by

$$S_{w,n,q} = \sum_{\mathcal{A}\in\mathcal{P}_d} w^{\mathcal{A}} S_{\mathcal{A},n,q} \quad \text{and} \quad \mathbb{S}_{w,q} = \sum_{\mathcal{A}\in\mathcal{P}_d} w^{\mathcal{A}} \mathbb{S}_{\mathcal{A},q} \, . \tag{5.7}$$

Corollary 13 *([M12], Corollary 4.1 and 4.2) Let $I_{1,n}$ and $I_{2,n}$ be two integers representing symbolically a version (from bootstrap or multipliers) of the original sample.*
(i) Under the assumptions of ([M12], Theorem 3.1), the combined Cramér-von Mises statistics $S_{w,n,1}$ converges in distribution to $\mathbb{S}_{w,1}$.
(ii) The assumptions of ([M12], Theorem 3.2) allow to define $S^{[I_{1,n}]}_{w,n,1}$ and $S^{[I_{2,n}]}_{w,n,1}$ such that the random vector $\left(S_{w,n,1}, S^{[I_{1,n}]}_{w,n,1}, S^{[I_{2,n}]}_{w,n,1}\right)$ converges in distribution to $(\mathbb{S}_{w,1}, \mathbb{S}^{[1]}_{w,1}, \mathbb{S}^{[2]}_{w,1})$ where the latter is the concatenation of independent copies.
(iii) The assumptions of ([M12], Theorem 3.3) allow to define $S^{[I_{1,n}]}_{w,n,q}$ and $S^{[I_{2,n}]}_{w,n,q}$ such that the random vector $\left(S_{w,n,q}, S^{[I_{1,n}]}_{w,n,q}, S^{[I_{2,n}]}_{w,n,q}\right)$ converges in distribution to $(\mathbb{S}_{w,q}, \mathbb{S}^{[1]}_{w,q}, \mathbb{S}^{[2]}_{w,q})$ where the latter is the concatenation of independent copies.
(iv) The assumptions of ([M12], Theorem 3.4) allow to define $\hat{S}^{(I_{1,n})}_{w,n,1}$ and $\hat{S}^{(I_{2,n})}_{w,n,1}$ such that the random vector $\left(\hat{S}_{w,n,1}, \hat{S}^{(I_{1,n})}_{w,n,1}, \hat{S}^{(I_{2,n})}_{w,n,1}\right)$ converges in distribution to $(\mathbb{S}_{w,1}, \mathbb{S}^{[1]}_{w,1}, \mathbb{S}^{[2]}_{w,1})$ where the latter is the concatenation of independent copies.
Similar results hold true for more standard Cramér-von Mises combined statistics and Kolmogorov combined statistics.

5.3.3 Illustration

Consider a 12-variate random vector $\{X_1, \ldots, X_4\}$, $\{X_5, \ldots, X_8\}$ and $\{X_9, \ldots, X_{12}\}$ that comes from the Normal copula. Let us focus on the test of $(\mathcal{H})$ the block are independent against its negation $(\mathcal{K})$. The correlation matrix controls the departure from the null hypothesis $(\mathcal{H})$ through the value ρ_{inter}. The latter is zero under $(\mathcal{H})$. The statistics W_n, T_n and I_n are those presented in [110] and available in the routine `multIndepTest` of the R-package *copula* [93]. The statistics $S_{w,n,1}$, defined by (5.7), are evaluated for several weights w:

- $w_1 = (1, 0, 0, 0, 0, 0, 0, 0)$ that only measures the left hand term of the decomposition,
- $w_2 = (1, 0, 0, 0, 1, 1, 1, 1)$ that combines the left hand term of the decomposition with the right hand terms of order 2 and 3 (note that the right hand terms associated with singletons all vanish),
- $w_3 = (0, 0, 0, 0, 1, 1, 1, 1)$ that only combines the non-null right hand terms of the decomposition,

		ρ_{inter}						
		0.00	*0.025*	*0.05*	*0.075*	*0.10*	*0.125*	*0.15*
Test statistics	W_n	0.060	0.146	0.332	0.540	0.770	0.901	0.982
	T_n	0.047	0.110	0.234	0.404	0.584	0.766	0.932
	I_n	0.054	0.128	0.287	0.503	0.711	0.854	0.971
	$S_{w_1,n,1}$	0.049	0.176	0.376	0.673	0.875	0.964	0.994
	$S_{w_2,n,1}$	0.046	**0.181**	0.388	**0.691**	**0.883**	**0.971**	**0.995**
	$S_{w_3,n,1}$	0.041	0.124	0.311	0.572	0.819	0.953	0.992
	$S_{w_4,n,1}$	0.046	0.173	**0.400**	0.683	0.878	0.969	0.994

Table 5.1: Percentage of rejection of $(\mathcal{H})$ when $n = 200$.

- $w_4 = (8.253373, 0, 0, 0, 2.373714, 2.344580, 2.35670, 1.989524)$ that combines left and right hand terms proportionally to some variances. It is an empirical choice where each $w^{\mathcal{A}}$ is proportional to the estimate of $\text{var}(S_{\mathcal{A},n,1})$ obtained by block bootstrapping.

The numerical illustrations are based on the non-parametric estimator $\hat{C}_n$ for the weight $q = 1$. The results of Table 5.1 show that the combined statistics $S_{w,n,1}$ improve the power of the test procedure. It could be interesting to compare also the performance of the combined statistics $S_{w,n,q}$ for other weights q and other estimates of C.

The next study needs some preliminary remarks. Goodness-of-fit tests arise when C is unknown but assumed to belong to a particular class $(\widetilde{\mathcal{H}})\, C \in \{C_\theta, \theta \in \Theta\}$ where Θ is an open subset of $\mathbb{R}^D$ for some integer $D \geq 1$. Natural tests consist in measuring a "distance" between the empirical copula and an estimate of C obtained under $(\widetilde{\mathcal{H}})$. In the theoretical development, the details for the inference on the parameter have been skept and the testing part of the procedure only has been analyzed. The null hypothesis becomes $(\mathcal{H})\, C = C_{\theta_0}$ where the reader should think of C_{θ_0} as $C_{\hat{\theta}_n}$, where $\hat{\theta}_n$ estimates θ in Θ. Of course, convergence of the practical procedure with respect to $(\widetilde{\mathcal{H}})$ needs appropriate regularity conditions on both the parametric family and the sequence of estimators $\hat{\theta}_n$. We refer to [75] for a review and discussion on combining both testing steps, and more recently to [29].
We thus turn to the Goodness-of-Fit tests $(\mathcal{H})\, C = C_\varphi$ for φ being the Clayton or the Gumbel generator in a 3-dimensional setting. These classes will both be used as the sampling distribution or as the family being tested. As already mentioned, even if the theoretical results have been obtained for a fixed φ, the numerical illustration here goes further since it estimates first $\varphi_{\hat{\theta}}$. To generate the original samples, three values of Kendall's τ are chosen: $\tau = .1$, $\tau = .2$ and $\tau = .3$. Test statistics $S_{w,n,1}$ are given by (5.7). The results are provided in Table 5.2 for $n = 100$. The first lines are dedicated to the test $(\mathcal{H}_3)$ when φ is the Clayton copula. Similarly, Gumbel copula is tested in the last lines of the table. The parameter associated with the generator φ is estimated at each step as the mean of empirical Kendall's τ. The parametric bootstrap described in ([M12], Algorithm 2) with $n_{\text{boot}} = 200$ is used to compute the p-value. The rejection rates are estimated through $n_{\text{rep}} = 500$ repetitions of each experiment. Two characteristics are of interest: the empirical level might be close to the nominal level, arbitrarily fixed at 0.05, and the empirical power. We also add another procedure in Table 5.2. The line `gofCopula` corresponds to the results associated with the command `gofCopula(CopulTest, rCopula(n, CopulSimu), estim.method = "itau")`.

Even if the goal of [M12] is not really to improve a methodology but to transform already known tools in particular cases of more general statements, one can take advantage in analyzing the functional decomposition associated with the null hypothesis in order to take into account subhypotheses and derive powerful weighted test statistics, as illustrated in Tables 5.1 and 5.2.

			Simulated copulas					
			Clayton			Gumbel		
			$\tau = .1$	$\tau = .2$	$\tau = .3$	$\tau = .1$	$\tau = .2$	$\tau = .3$
Tested copulas	Clayton	$S_{w_1,n,1}$	0.056	0.044	0.048	**0.364**	**0.690**	**0.934**
		$S_{w_2,n,1}$	0.052	0.044	0.038	0.294	0.674	0.924
		$S_{w_3,n,1}$	0.044	0.060	0.046	0.160	0.494	0.852
		`gofCopula`	`0.050`	`0.044`	`0.044`	`0.214`	`0.670`	`0.924`
	Gumbel	$S_{w_1,n,1}$	0.282	0.720	0.922	0.070	0.048	0.056
		$S_{w_2,n,1}$	**0.306**	**0.766**	0.948	0.036	0.042	0.052
		$S_{w_3,n,1}$	0.254	0.710	**0.960**	0.024	0.042	0.048
		`gofCopula`	`0.288`	`0.716`	`0.930`	`0.030`	`0.062`	`0.054`

Table 5.2: Rejection rates of the null hypothesis based on sample size $n = 100$, parametric bootstrap size $n_{\text{boot}} = 200$ and number of repetitions of the experiment $n_{\text{rep}} = 500$.

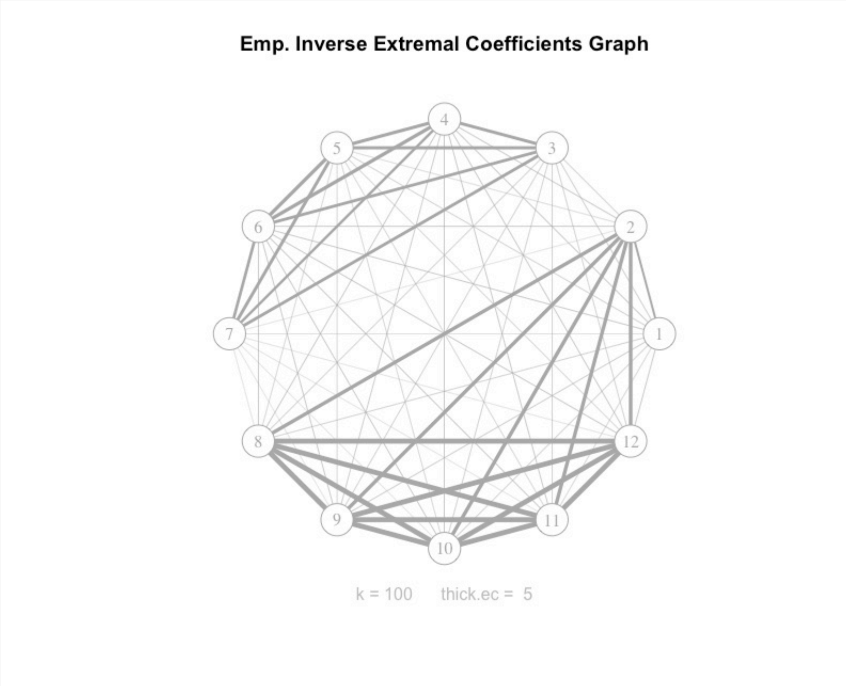
Concluding remarks

Identifying and modeling dependencies with copulas remain an important topic, which has become very popular over the last decades since it has been applied in almost every discipline. [M12] unifies the treatment of several copula-based tests of the structure of dependence, as [43], [74], [73], [110], [111] among others. The solution to dip them in the functional decomposition context of [117] (and its recent version [M15]) in order to reveal a common pattern is successful both theoretically and empirically. In a tail dependence context, the empirical study of the stdf has been handled in [M5] in order to cancel the asymptotic bias and in [M14] to provide a finite overview of its support. Once again, their implementation on data lead to really satisfactory results. As with the previous two chapters, this one ends with an illustration from the *satdad* R-package [M11]. Using simulations in the domain of attraction (with an Archimax sampling) and in the attractor (with a multivariate Fréchet sampling), the tail dependograph is estimated and plotted.

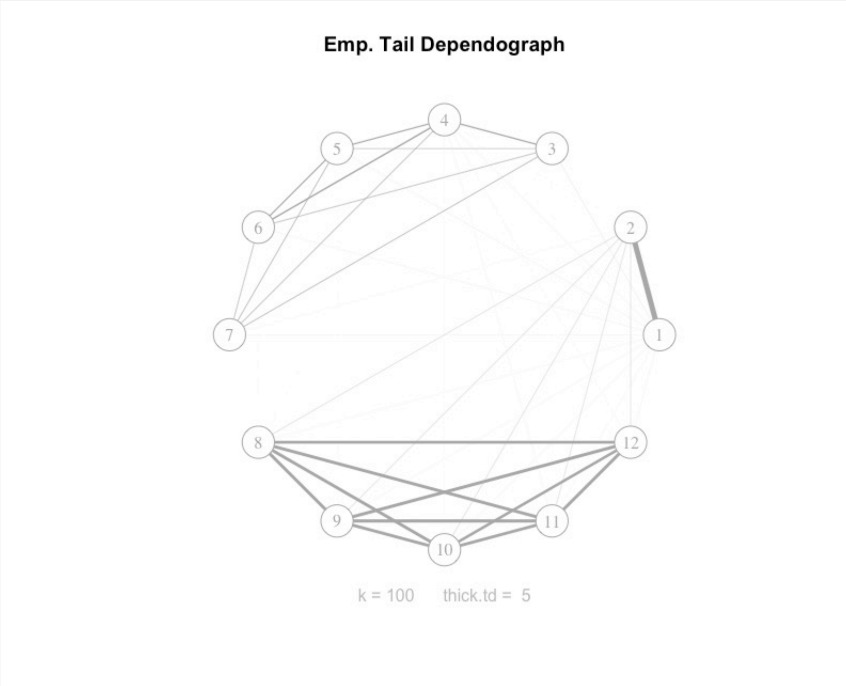
Third illustration of [M11]

Consider the 12-dimensional dependence structure generated at the end of Chapter 3 as well as the two samples of multivariate extreme value random vectors and Archimax random vectors. Recall the theoretical Inverse extremal coefficient graph and the theoretical tail dependograph obtained at the end of Chapter 4. We now provide their empirical counteparts on both samples.

```
> graphsEmp(sample = X_fr12, k = 100, which = "iecgraph")
```

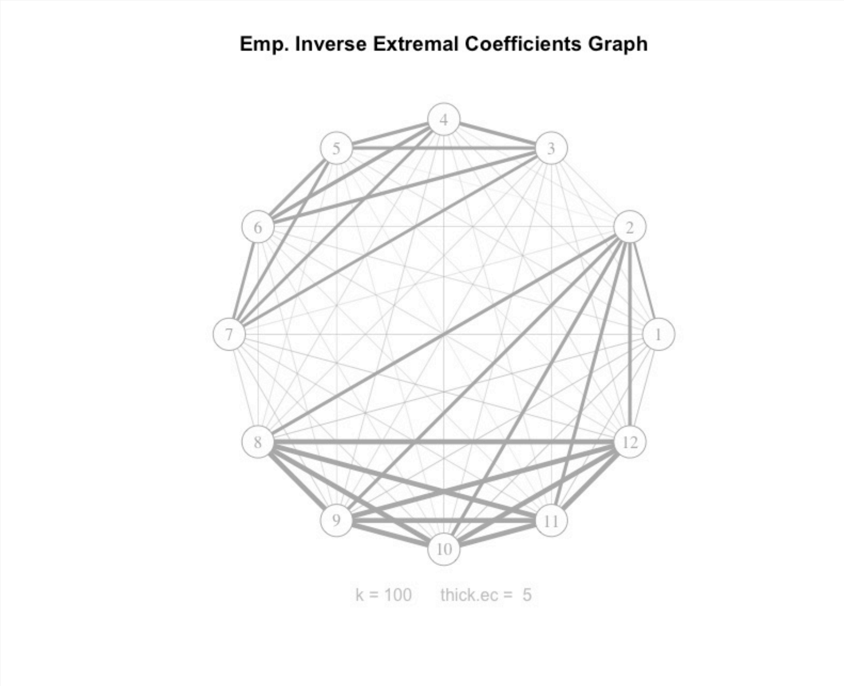



```
> graphsEmp(sample = X_fr12, k = 100)
```

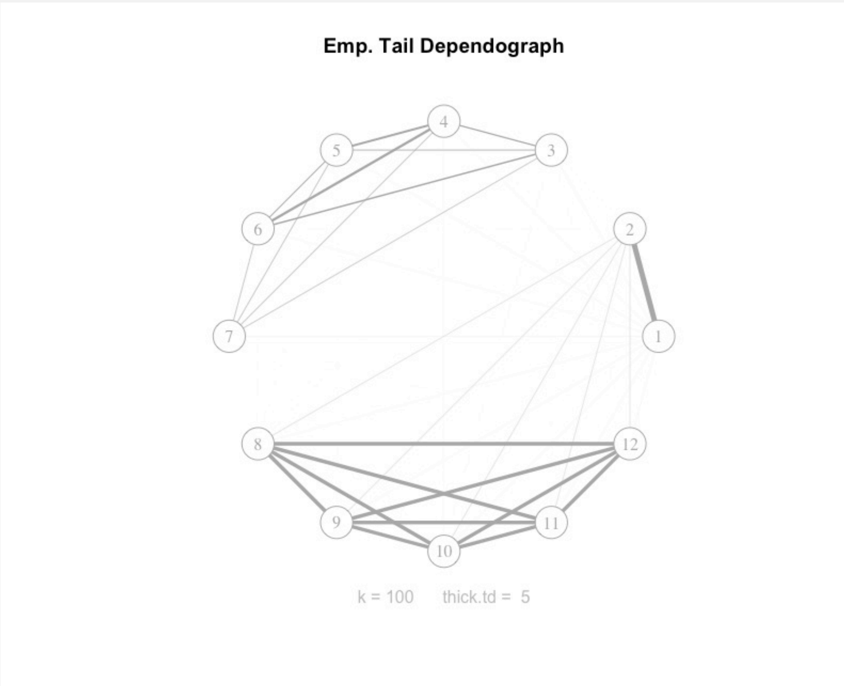


Third illustration of [M11]

```
> graphsEmp(sample = X_archimax12, k = 100, which = "iecgraph")
```




```
> graphsEmp(sample = X_archimax12, k = 100)
```



The empirical tail dependograph still nicely separates the support into two groups. Although these two groups are less dissociated in the empirical inverse extremal coefficients graph, it wouldn't be complicated to obtain an empirical threshold so that values smaller than it are set to 0 and the two groups are reconstructed. One could say that the empirical inverse extremal coefficients graph needs some (easy) post-cleaning.

Summary and perspectives

Summary My research work lies at the interface of extreme value theory, sensitivity analysis, and non-parametric inference. The links between the first two themes was not obvious but their combination has proved fruitful. The aim of extreme value theory is to propose probabilistic models that allow for extrapolation of a phenomenon to rarely observed values. It allows goodness-of-fit tests, statistical evaluations, and comparison of the efficiency of different procedures. Finally, applying it enables us to improve our understanding of various environmental or financial phenomena, for example.

At the heart of my habilitation thesis is the stable tail dependence function, which provides a complete characterization of the asymptotic dependence structure. Its study involves very interesting mathematical concepts, such as multivariate monotonicity, homogeneity, or spectral representation. Furthermore, it becomes additive on asymptotically independent components. This search for additivity is also explored through superset importance indices in global sensitivity analysis. In particular, the Hoeffding-Sobol decomposition has allowed me to introduce new concepts such as tail superset importance coefficients and the tail dependograph. More generally, the analysis of functional decompositions using commutative and idempotent operators yield a better understanding of the similarities and differences between Hoeffding-Sobol and Möbius decompositions. This has led to the emergence of a general framework for analyzing various statistical dependence hypotheses.

Perspectives Regarding the new family of indices resulting from this, namely the tail superset importance coefficients, my goal was to hierarchize the strength of dependence and proceed with dimension reduction. However, the fact that there is no longer a probabilistic interpretation, unlike what extremal coefficients offer, contributes to this difficulty. The only way to proceed is to first reconstruct the stdf. A less difficult project, still related to this collection of coefficients, is the following: An ongoing work, not presented in this text, allows me to produce a new test of asymptotic independence. Its performance could be compared to the test associated with the tail superset importance coefficients.

As for the work that analyzes dependence, there is a theoretical passage to be filled in when it comes to the case of goodness-of-fit. Work remains to be done and several choices are possible. Apart from the specific case of gof tests, another natural question that has not really been addressed but only measured on examples is the optimal choice of weights that contribute to the definition of the new test statistics, and reflect the importance we put respectively on the global hypothesis and the sub hypotheses.

Over time, another objective is to improve and enrich my package by adding elements from past collaborations, as well as elements from my future publications.

If I extract myself from the text I presented and think about the path taken to construct the coming years, I remember three things. My participation in research projects has allowed me to study concrete problems. This is undoubtedly the greatest challenge of my work: taking into account all the specificities and complications of real data. It is time-consuming but very informative. Attending seminars and conferences, even (or especially) when the field is not truly our own, can sometimes prove fruitful. Finally, the most important thing in my opinion is to continue ongoing collaborations and start new ones. To conclude, I would like to mention that supervising PhD or master's students involves regular questioning of our research project. This work of transmission and support is at the heart of our missions as teacher-researchers. I hope that the future holds some opportunities for me and that, with this habilitation, I will be better equipped to contribute.

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Non parametric statistics and global sensitivity analysis tools in the study of (tail) dependence

Abstract:
My research work lies at the interface of extreme value theory, sensitivity analysis, and non-parametric inference. The links between the first two themes are not obvious but their combination has proved fruitful. The aim of extreme value theory is to propose probabilistic models that allow for extrapolation of a phenomenon to rarely observed values. It allows goodness-of-fit tests, statistical evaluations, and comparison of the efficiency of different procedures. Finally, applying it enables us to improve our understanding of various environmental or financial phenomena, for example.
At the heart of my habilitation thesis is the stable tail dependence function, which provides a complete characterization of the asymptotic dependence structure. Its study involves very interesting mathematical concepts, such as multivariate monotonicity, homogeneity, or spectral representation. Furthermore, it becomes additive on asymptotically independent components. This search for additivity is also explored through superset importance indices in global sensitivity analysis. In particular, the Hoeffding-Sobol decomposition has allowed me to introduce new concepts such as tail superset importance coefficients and the tail dependograph. More generally, the analysis of functional decompositions using commutative and idempotent operators yield a better understanding of the similarities and differences between Hoeffding-Sobol and Möbius decompositions. This has led to the emergence of a general framework for analyzing various statistical dependence hypotheses.

Statistique non paramétrique et outils de l'analyse de sensibilité globale pour l'étude de la dépendance (asymptotique)

Résumé:
Mes travaux de recherche se situent à l'interface de la théorie des valeurs extrêmes, de l'analyse de sensibilité et de l'inférence non paramétrique. Les liens entre les deux premiers thèmes ne sont pas évidents mais leur combinaison s'est avérée féconde. La théorie des valeurs extrêmes vise à proposer des modèles probabilistes permettant l'extrapolation d'un phénomène vers des valeurs rarement observées. Elle permet des tests d'adéquation, des évaluations statistiques et la comparaison de l'efficacité de différentes procédures. En l'appliquant, elle améliore enfin notre compréhension de divers phénomènes environnementaux ou financiers, par exemple.
Au cœur de mon mémoire d'habilitation à diriger des recherches se trouve la *stable tail dependence function*, qui fournit une caractérisation complète de la structure de dépendance asymptotique. Son étude implique des concepts mathématiques très intéressants, tels que la monotonicité multivariée, l'homogénéité ou la représentation spectrale. De plus, elle devient additive sur les composantes asymptotiquement indépendantes. Cette recherche d'additivité est également explorée à travers les indices d'importance superset en analyse de sensibilité globale. En particulier, la décomposition de Hoeffding-Sobol m'a permis d'introduire de nouveaux concepts tels que les *tail superset importance coefficients* et le *tail dependograph*. Plus généralement, l'analyse de décompositions fonctionnelles à partir d'opérateurs commutatifs et idempotents permet une meilleure compréhension des similitudes et des différences entre les décompositions de Hoeffding-Sobol et de Möbius. Cela a conduit à l'émergence d'un cadre général pour l'analyse de diverses hypothèses statistiques de dépendance.

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