Matrix analytic methods

Lecture 3: Branching processes

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Matrix analytic methods

- are used to define a new class of branching processes called Markovian binary trees,
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- inspire us to construct numerical procedures with probabilistic interpretations;
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- are used to define a new class of branching processes called Markovian binary trees,
- inspire us to construct numerical procedures with probabilistic interpretations;
- branching processes can be modeled by structured Markov chains and solved by adapting tools developed for the latter.
The Markovian arrival process

A Markovian arrival process is a two-dimensional process \( \{(M(t), \varphi(t)), t \in \mathbb{R}^+\} \) where

- \( M(t) \in \mathbb{N} \): the number of events until time \( t \)
- \( \varphi(t) \in \{1, 2, \ldots, n\} \): the phase at time \( t \).

Its generator has the following structure

\[
\begin{bmatrix}
D_0 & D_1 & 0 & 0 & \cdots \\
0 & D_0 & D_1 & 0 & \cdots \\
0 & 0 & D_0 & D_1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]

This process generalizes the Poisson process of arrivals.
The transient Markovian arrival process \((\alpha, D_0, D_1, d)\)

- \(\alpha\): the initial probability vector
- \(D_0\): the matrix of phase transition rates between two events
- \(D_1\): the matrix of phase transition rates at an event epoch
- \(d\): the vector of transition rates to absorption
The individual’s lifetime in an MBT

\[
\begin{array}{ccc}
\varphi_0 & \xrightarrow{i \to j} & \xrightarrow{i \to k} & \xrightarrow{i \to 0} \\
\end{array}
\]

\[
(D_0)_{ij} \quad j \quad \ldots \quad d_i \\
\]

\[
B_{i,jk}
\]

- \( n \) transient phases, 1 absorbing phase 0;
- \( \varphi_0 \in \{1, \ldots, n\} \): the initial phase, with distribution \( \alpha \);
- \( D_0 \): the matrix of phase transition rates between two events;
- \( B \): the Birth rate matrix; for instance \( B = (\alpha \otimes D_1) \);
- \( d \): the death rate vector;

with \( D_0 \mathbf{1} + B \mathbf{1} + \mathbf{d} = \mathbf{0} \).
- An MBT models the evolution of a family or population over time.
- We assume that the individuals behave independently of each other.
Example: female families

- $n = 22$ age classes:

  $0, 1-4, 5-9, 10-14, \ldots, 90-94, 95-99, \geq 100$.

\[ \alpha = [1, 0, \ldots, 0], \quad D_0 = \begin{bmatrix}
  * & 1 & 1/4 & 1/5 \\
  * & 1/4 & * & 1/5 \\
  & \ddots & & \ddots \\
  & & & * & 1/5 \\
  & & & & * \\
\end{bmatrix}, \]

- We use real data from WHO and UN.

- $\gamma$: age-specific fertility rates, $\rightarrow B = \alpha \otimes \text{diag}(\gamma)$,

- $d$: age-specific death rates.

- We assume constant birth and death rates, no migration phenomena, and no dependences between women.
Number of branches at time $t$

$$Z_i(t) = \# \text{ living branches in phase } i \text{ at time } t, \ Z(t) = [Z_i(t)].$$
Number of branches at time $t$

$Z_i(t) = \# \text{ living branches in phase } i \text{ at time } t$, $Z(t) = [Z_i(t)]$.

$F(s, t) = \mathbb{E}[s^{Z(t)} | \varphi_0]$ p.g.f. of the population size at time $t$. 

\[
F(s, t) = e^{D_0 t} s + \int_0^t e^{D_0 u} du + \int_0^t e^{D_0 u} B(F(s, t - u) \otimes F(s, t - u)) du
\]

We obtain

\[
\frac{\partial}{\partial t} F(s, t) = d + D_0 F(s, t) + B(F(s, t) \otimes F(s, t)),
\]

with $F(s, 0) = s$. 

\[
\frac{8}{32}
\]
Number of branches at time $t$

$$Z_i(t) = \# \text{ living branches in phase } i \text{ at time } t, \ Z(t) = [Z_i(t)].$$

$$F(s, t) = \mathbb{E}[s^{Z(t)} | \varphi_0] \text{ p.g.f. of the population size at time } t.$$ 

We condition on the first observable event in the lifetime of the first individual:

$$F(s, t) = e^{D_0 t} s + \int_0^t e^{D_0 u} d u$$

$$+ \int_0^t e^{D_0 u} B \left( F(s, t - u) \otimes F(s, t - u) \right) d u$$
Number of branches at time $t$

$Z_i(t) = \# \text{ living branches in phase } i \text{ at time } t$, $\mathbf{Z}(t) = [Z_i(t)]$.

$F(s, t) = \mathbb{E}[s^{Z(t)} | \varphi_0]$ p.g.f. of the population size at time $t$.

We condition on the first observable event in the lifetime of the first individual:

$$
F(s, t) = e^{D_0 t} s + \int_0^t e^{D_0 u} d u
$$

$$
+ \int_0^t e^{D_0 u} \mathcal{B}(F(s, t - u) \otimes F(s, t - u)) \text{ } d u
$$

We obtain

$$
\frac{\partial}{\partial t} F(s, t) = d + D_0 F(s, t) + \mathcal{B}(F(s, t) \otimes F(s, t)),
$$

with $F(s, 0) = s$. 
Number of branches at time $t$

$$\frac{\partial}{\partial t} F(s, t) = d + D_0 F(s, t) + B (F(s, t) \otimes F(s, t))$$
Number of branches at time $t$

\[
\frac{\partial}{\partial t} \mathbf{F}(s, t) = \mathbf{d} + D_0 \mathbf{F}(s, t) + B (\mathbf{F}(s, t) \otimes \mathbf{F}(s, t))
\]

$M(t) = (M_{ij}(t)) = \text{Mean number of living individuals at time } t \text{ with}$

\[
M_{ij}(t) = \mathbb{E}[Z_j(t) | \varphi_0 = i] = \frac{\partial F_i(s, t)}{\partial s_j} \bigg|_{s=1}
\]
Number of branches at time $t$

$$\frac{\partial}{\partial t} F(s, t) = d + D_0 F(s, t) + B (F(s, t) \otimes F(s, t))$$

$M(t) = (M_{ij}(t))$ = Mean number of living individuals at time $t$ with

$$M_{ij}(t) = \mathbb{E}[Z_j(t) | \varphi_0 = i] = \frac{\partial F_i(s, t)}{\partial s_j} |_{s=1}$$

We get $\frac{\partial}{\partial t} M(t) = D_0 M(t) + B(1 \otimes M(t) + M(t) \otimes 1)$
Number of branches at time $t$

$$\frac{\partial}{\partial t} \mathbf{F}(s, t) = \mathbf{d} + D_0 \mathbf{F}(s, t) + B \left( \mathbf{F}(s, t) \otimes \mathbf{F}(s, t) \right)$$

$$M(t) = (M_{ij}(t)) = \text{Mean number of living individuals at time } t \text{ with}$$

$$M_{ij}(t) = \mathbb{E}[Z_j(t) | \varphi_0 = i] = \frac{\partial F_i(s, t)}{\partial s_j} \bigg|_{s=1}$$

We get $$\frac{\partial}{\partial t} M(t) = D_0 M(t) + B(1 \otimes M(t) + M(t) \otimes 1)$$

$$\rightarrow M(t) = e^{\Omega t}$$

where $\Omega = D_0 + B (1 \oplus 1)$.
Illustrative example in demography

Mean family size at time $t$
Time until extinction

\[ F(t) = P[\text{Extinction occurs before time } t \mid \varphi_0] : \text{distribution function of the time until extinction} \]
Time until extinction

\[ F(t) = P[\text{Extinction occurs before time } t \mid \varphi_0] : \text{distribution function of the time until extinction} \]

\[ F(t) = F(s, t)|_{s=0} \text{ satisfies} \]

\[ \frac{d}{dt} F(t) = d + D_0 F(t) + B (F(t) \otimes F(t)), \]

with \( F(0) = 0. \)
Illustrative example in demography

Distribution of the time until extinction of a female family

\[ F_1(t) \]

- Brazil
- Congo
- Japan
- Morocco
- SA
- USA
Extinction probability

\[
\frac{\partial}{\partial t} F(s, t) = d + D_0 F(s, t) + B (F(s, t) \otimes F(s, t))
\]
Extinction probability

\[ \frac{\partial}{\partial t} F(s, t) = d + D_0 F(s, t) + B (F(s, t) \otimes F(s, t)) \]

Extinction probability :

\[ q = P[\text{the MBT becomes extinct | } \varphi_0] = \lim_{t \to \infty} F(0, t). \]
Extinction probability

\[ \frac{\partial}{\partial t} F(s, t) = d + D_0 F(s, t) + B (F(s, t) \otimes F(s, t)) \]

Extinction probability:

\[ q = P[\text{the MBT becomes extinct } | \varphi_0] = \lim_{t \to \infty} F(0, t). \]

Extinction equation:

\[ 0 = d + D_0 q + B (q \otimes q). \]
Extinction probability of an MBT

\[ 0 = d + D_0 q + B (q \otimes q) \iff q = (-D_0)^{-1} d + (-D_0)^{-1} B (q \otimes q) \]

We define

\[ \theta = (-D_0)^{-1} d : \text{the death probability of a branch starting in a given phase} ; \]
\[ \psi = (-D_0)^{-1} B : \text{the birth probability of a branch starting in a given phase}. \]

\[ \Rightarrow \text{Extinction equation : } \]
\[ s = \theta + \psi (s \otimes s). \]
Recall that $M(t) = \exp(\Omega t)$, with $\Omega = D_0 + B (1 \oplus 1)$. Let $\mu$ be the eigenvalue of maximal real part of $\Omega$.

**Theorem**

*Depending on $\mu$, we are in one of the three following cases*

1. *if $\mu < 0$, then the MBT is subcritical and $\mathbf{q} = 1$,*
2. *if $\mu = 0$, then the MBT is critical and $\mathbf{q} = 1$,*
3. *if $\mu > 0$, then the MBT is supercritical and $\mathbf{q} \leq 1$, $\mathbf{q} \neq 1$.*

*In all cases, $\mathbf{q}$ is the minimal nonnegative solution of the fixed point equation*

$$
\mathbf{s} = \theta + \Psi (\mathbf{s} \otimes \mathbf{s}).
$$
Linear algorithms to compute the extinction probability

\[ s = \theta + \Psi (s \otimes s) \]

\[ \equiv \]

\[ s = [I - \Psi (I \otimes s)]^{-1} \theta \]

\[ \equiv \]

\[ s = [I - \Psi (s \otimes I)]^{-1} \theta \]

1. The **Depth** and the **Order** algorithms (Bean, Kontoleon and Taylor, 2005)

2. The **Thicknesses** algorithm (Self, 2006)
Probabilistic interpretation of the Order algorithm

\[ s_0 = \theta \]
\[ s_k = [I - \psi(s_{k-1} \otimes I)]^{-1} \theta, \quad k \geq 1 \]

Order of an MBT = total number of children generations

For \( k \geq 0 \),

- \( \mathcal{M}_k \) = the set of extinct MBTs with an order \( \leq k \) (constraint on the shape of the tree)

- \( \mathcal{M}_k \subseteq \mathcal{M}_{k+1} \subseteq \cdots \subseteq \mathcal{M} \) = the set of all extinct MBTs.

- \( s_k = \mathbb{P}[\mathcal{M}_k \mid \varphi_0] \xrightarrow{k \to \infty} q. \)
Probabilistic interpretation of the Order algorithm

The set of extinct trees with at most 0 children generation:
\[ M_0 = \bot \]

The set of extinct trees with at most \( k \) children generations:
\[ M_k = \ldots \quad \text{for} \quad k \geq 1. \]
Probabilistic interpretation of the Thicknesses algorithm

\[ s_0 = \theta \]
\[ s_{2k-1} = [I - \Psi (s_{2k-2} \otimes I)]^{-1} \theta, \quad k \geq 1 \]
\[ s_{2k} = [I - \Psi (I \otimes s_{2k-1})]^{-1} \theta, \quad k \geq 1 \]

Thickness of an MBT = another quantity about the tree

For \( k \geq 0 \),

- \( \mathcal{M}_k = \) the set of extinct MBTs with a thickness \( \leq k \) (constraint on the shape of the tree)
- \( \mathcal{M}_k \subseteq \mathcal{M}_{k+1} \subseteq \cdots \subseteq \mathcal{M} = \) the set of all extinct MBTs.
- \( s_k = \mathbb{P}[\mathcal{M}_k \mid \varphi_0] \xrightarrow{k \to \infty} q. \)
Probabilistic interpretation of the Thicknesses algorithm

The set of extinct tree with a thickness \( \leq 0 \):

\[
\mathcal{M}^*_0 = \bot
\]

The set of extinct trees with a left thickness \( \leq 2k - 1 \):

\[
\mathcal{M}^*_{2k-1} = \ldots \xrightarrow{\mathcal{M}^*_{2k-2}} \mathcal{M}^*_{2k-2} \xrightarrow{\mathcal{M}^*_{2k-2}} \mathcal{M}^*_{2k-2}
\]

The set of extinct trees with a right thickness \( \leq 2k \):

\[
\mathcal{M}^*_{2k} = \ldots \xrightarrow{\mathcal{M}^*_{2k-1}} \mathcal{M}^*_{2k-1} \xrightarrow{\mathcal{M}^*_{2k-1}} \mathcal{M}^*_{2k-1}
\] for \( k \geq 1 \).
Quadratically convergent algorithm: Newton

\[ \mathcal{F}(s) = s - \theta - \psi(s \otimes s) = 0 \]

\[ \Rightarrow \text{Newton's iteration method:} \]

\[ x_{k+1} = x_k - (\mathcal{F}_{x_k})^{-1} \mathcal{F}(x_k), \quad k \geq 0, \]

which leads to the Newton algorithm:

\[ x_0 = \theta, \]
\[ x_{k+1} = [I - \psi(x_k \oplus x_k)]^{-1}[\theta - \psi(x_k \otimes x_k)], \quad k \geq 1. \]
Illustrative example in demography

Extinction probability of a female family
QBD approach

The MBT can be modeled by a level-dependent QBD \((X(t), \varphi(t))\) where

- \(X(t) = Z(t) \mathbf{1} \in \mathbb{N}\) is the total MBT size at time \(t = \) the level,
- \(\varphi(t) = (Z_1(t), Z_2(t), \ldots, Z_n(t)) \in \mathbb{N}^n\) is the phase.

\[
Q = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Q_{10} & Q_{11} & Q_{12} & 0 & 0 & \cdots \\
0 & Q_{21} & Q_{22} & Q_{23} & 0 & \cdots \\
0 & 0 & Q_{32} & Q_{33} & Q_{34} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]
QBD approach (2)

We can then obtain

- the distribution of the **maximum population size** given extinction as

\[
P[\text{Max} \leq m|\varphi_0, \text{Ext}] = \text{Diag}(q)^{-1} G_1(m)
\]

where \( G_1(m) \) is the \( m \)th iterate of a linear algorithm for QBDs.
QBD approach (2)

We can then obtain

- the distribution of the maximum population size given extinction as

\[ P[\text{Max} \leq m | \varphi_0, \text{Ext}] = \text{Diag}(q)^{-1} G_1(m) \]

where \( G_1(m) \) is the \( m \)th iterate of a linear algorithm for QBDs.

- the mean time until extinction given extinction, which is equivalent to the mean first passage time to level 0,
QBD approach (2)

We can then obtain

- the distribution of the maximum population size given extinction as

  $$P[\text{Max} \leq m | \varphi_0, \text{Ext}] = \text{Diag}(\mathbf{q})^{-1} G_1(m)$$

  where $G_1(m)$ is the $m$th iterate of a linear algorithm for QBDs.

- the mean time until extinction given extinction, which is equivalent to the mean first passage time to level 0,

- the mean time until the population reaches $k$ individuals, which is equivalent to the mean first passage time to level $k$. 
MBT with catastrophes
Assume that

- the catastrophes occur following a Poisson process with parameter $\beta$ (or more generally following a MAP),
- they arrive independently of the evolution of the MBT,
- an individual in phase $i$ is killed with probability $\varepsilon_i$. 
MBT with catastrophes

Assume that

- the catastrophes occur following a Poisson process with parameter $\beta$ (or more generally following a MAP),
- they arrive independently of the evolution of the MBT,
- an individual in phase $i$ is killed with probability $\varepsilon_i$.

$\Rightarrow$ No more independence in the evolution of the individuals:

$$q \neq \theta + \psi(q \otimes q).$$
Extinction probability

\[ \hat{Z}(t) = [\hat{Z}_1(t), \hat{Z}_2(t), \ldots, \hat{Z}_n(t)]^T : \text{population size at time } t. \]
Extinction probability

- $\hat{Z}(t) = [\hat{Z}_1(t), \hat{Z}_2(t), \ldots, \hat{Z}_n(t)]^T$: population size at time $t$.

- $\hat{F}(s, t) = \mathbb{E}[s\hat{Z}(t) | \varphi_0]$ satisfies

$$
\frac{\partial}{\partial t} \hat{F}(s, t) - \frac{\partial}{\partial s} \hat{F}(s, t) \cdot a(s) = \beta [\hat{F}(\Delta s + \epsilon, t) - \hat{F}(s, t)]
$$

$$
\hat{F}(s, 0) = s,
$$

with $a(s) = d + D_0 s + B (s \otimes s)$ and $\Delta = \text{diag}(1 - \epsilon)$. 
Extinction probability

\( \hat{Z}(t) = [\hat{Z}_1(t), \hat{Z}_2(t), \ldots, \hat{Z}_n(t)]^T \) : population size at time \( t \).

\( \hat{F}(s, t) = E[s\hat{Z}(t) | \varphi_0] \) satisfies

\[
\frac{\partial}{\partial t} \hat{F}(s, t) - \frac{\partial}{\partial s} \hat{F}(s, t) \cdot a(s) = \beta [\hat{F}(\Delta s + \epsilon, t) - \hat{F}(s, t)]
\]

\( \hat{F}(s, 0) = s \),

with \( a(s) = d + D_0 s + B (s \otimes s) \) and \( \Delta = \text{diag}(1 - \epsilon) \).

*Idea*: write the backward Kolmogorov equation for \( P[\hat{Z}(t) = z] \).
Extinction probability

\( \hat{Z}(t) = [\hat{Z}_1(t), \hat{Z}_2(t), \ldots, \hat{Z}_n(t)]^T \): population size at time \( t \).

\( \hat{F}(s, t) = E[s \hat{Z}(t) | \varphi_0] \) satisfies

\[
\frac{\partial}{\partial t} \hat{F}(s, t) - \frac{\partial}{\partial s} \hat{F}(s, t) \cdot a(s) = \beta [\hat{F}(\Delta s + \epsilon, t) - \hat{F}(s, t)]
\]

\[\hat{F}(s, 0) = s,\]

with \( a(s) = d + D_0 s + B (s \otimes s) \) and \( \Delta = \text{diag}(1 - \epsilon) \).

**Idea**: write the backward Kolmogorov equation for \( P[\hat{Z}(t) = z] \).

\( \triangleright \) Extinction probability \( \hat{q} \) of the MBT, given its initial phase:

\[ \hat{q} = \lim_{t \to \infty} \hat{F}(0, t). \]
Method I: Numerical solution of PDE\(^1\)

When \( n = 1 \):

\[
\frac{\partial}{\partial t} \hat{F}(s, t) - \frac{\partial}{\partial s} \hat{F}(s, t) a(s) = \beta [\hat{F}(\delta s + \epsilon, t) - \hat{F}(s, t)]
\]

\(^1\)joint work with Pauline Lafitte, Université de Lille 1
Method I: Numerical solution of PDE$^1$

When $n = 1$:

$$\frac{\partial}{\partial t} \hat{F}(s, t) - \frac{\partial}{\partial s} \hat{F}(s, t) a(s) = \beta [\hat{F}(\delta s + \epsilon, t) - \hat{F}(s, t)]$$

- Grid of points $(s_k, t^n)$, $s_k = s_{k-1} + \Delta s$, $-1 \leq s_k \leq 1$, $t^n = t^{n-1} + \Delta t$, $t^n \geq 0$

---

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Method I : Numerical solution of PDE

When \( n = 1 \):

\[
\frac{\partial}{\partial t} \hat{F}(s, t) - \frac{\partial}{\partial s} \hat{F}(s, t) a(s) = \beta [\hat{F}(\delta s + \epsilon, t) - \hat{F}(s, t)]
\]

▶ Grid of points \((s_k, t^n)\), \(s_k = s_{k-1} + \Delta s, -1 \leq s_k \leq 1, t^n = t^{n-1} + \Delta t, t^n \geq 0\)

▶ Fractional step method :

1. \(\frac{\partial}{\partial t} \hat{F}(s, t) - \frac{\partial}{\partial s} \hat{F}(s, t) a(s) = 0\)

2. \(\frac{\partial}{\partial t} \hat{F}(s, t) = \beta [\hat{F}(\delta s + \epsilon, t) - \hat{F}(s, t)]\)

---

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Method I : Numerical solution of PDE

When \( n = 1 \):

\[
\frac{\partial}{\partial t} \hat{F}(s, t) - \frac{\partial}{\partial s} \hat{F}(s, t) a(s) = \beta [\hat{F}(\delta s + \epsilon, t) - \hat{F}(s, t)]
\]

- Grid of points \((s_k, t^n)\), \( s_k = s_{k-1} + \Delta s, -1 \leq s_k \leq 1, \)
  \( t^n = t^{n-1} + \Delta t, t^n \geq 0 \)

- Fractional step method :
  1. \( \frac{\partial}{\partial t} \hat{F}(s, t) - \frac{\partial}{\partial s} \hat{F}(s, t) a(s) = 0 \)
  2. \( \frac{\partial}{\partial t} \hat{F}(s, t) = \beta [\hat{F}(\delta s + \epsilon, t) - \hat{F}(s, t)] \)

1. Finite difference methods
   - Semi-Lagrangian method

2. Forward Euler method

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Method II : a recursive integral equation

- Recall that $F(s, t)$ denotes the p.g.f. of the MBT size at time $t$ without catastrophe and may be computed numerically.
Method II: a recursive integral equation

- Recall that $F(s, t)$ denotes the p.g.f. of the MBT size at time $t$ without catastrophe and may be computed numerically.

- $\hat{F}_k(s, t) = E[s^{\hat{X}(t)} 1_{\{N(t) \leq k\}} | \varphi_0]$ : p.g.f of the MBT size at time $t$ on paths with at most $k$ catastrophes until time $t$

\[
\hat{F}_k(s, t) = F(s, t) e^{-\beta t} + \int_0^t \beta e^{-\beta (t-u)} \hat{F}_{k-1}(\Delta F(s, t-u) + \varepsilon, u) \, du,
\]

with $\hat{F}_0(s, t) = F(s, t) e^{-\beta t}$. 

- $\hat{q} = \lim_{t \to \infty} \hat{F}_0(0, t) = \lim_{t \to \infty} \lim_{k \to \infty} \hat{F}_k(0, t)$. (Superlinear convergence)
Method II: a recursive integral equation

Recall that $F(s, t)$ denotes the p.g.f. of the MBT size at time $t$ without catastrophe and may be computed numerically.

\[
\hat{F}_k(s, t) = \mathbb{E}[s^{\hat{X}(t)} 1\{N(t) \leq k\} | \varphi_0]: \text{p.g.f of the MBT size at time } t \text{ on paths with at most } k \text{ catastrophes until time } t.
\]

\[
\hat{F}_k(s, t) = F(s, t) e^{-\beta t} + \int_0^t \beta e^{-\beta (t-u)} \hat{F}_{k-1}(\Delta F(s, t-u) + \epsilon, u) \, du,
\]

with $\hat{F}_0(s, t) = F(s, t) e^{-\beta t}$.

\[
\hat{q} = \lim_{t \to \infty} \hat{F}(0, t) = \lim_{t \to \infty} \lim_{k \to \infty} \hat{F}_k(0, t).
\]

(Superlinear convergence)
Method III: GI/M/1-type Markov chain approach

Two-dimensional Markov process \((\hat{X}(t), \hat{\phi}(t))\) with

- \(\hat{X}(t) = \hat{Z}(t) \mathbf{1}\) the total MBT size at time \(t = \) the level,
- \(\hat{\phi}(t) = (\hat{Z}_1(t), \hat{Z}_2(t), \ldots, \hat{Z}_n(t)) = \) the phase.

\[
Q = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
Q_{10} & Q_{11} & Q_{12} & 0 & 0 & \cdots \\
Q_{20} & Q_{21} & Q_{22} & Q_{23} & 0 & \cdots \\
Q_{30} & Q_{31} & Q_{32} & Q_{33} & Q_{34} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

\(\iff\) Extinction probability \(\hat{q} \equiv \) Probability to go from level 1 to level 0.
Method III: GI/M/1-type Markov chain approach

\( \gamma(i) = \) first passage time to level \( i \).

\( \hat{q} = G_1 = P[\gamma(0) < \infty, \hat{\phi}(\gamma(0)) | \hat{X}(0) = 1, \hat{\phi}(0)]. \)

\( G_1 = 1 - \lim_{M \to \infty} (L_1 L_2 \cdots L_M) \ 1 \) with

\( L_i = P[\gamma(i + 1) < \gamma(0), \hat{\phi}(\gamma(i + 1)) | \hat{X}(0) = i, \hat{\phi}(0)], \)

\( L_1 = (-Q_{11})^{-1} Q_{12}, \)

\( L_i = \left[ I - (-Q_{ii})^{-1} \sum_{j=1}^{i-1} Q_{i(i-j)} \prod_{i-j \leq k \leq i-1} L_k \right]^{-1} (-Q_{ii})^{-1} Q_{i(i+1)}, \)

\( i \geq 2. \)
References


- Hautphenne and Latouche *The Markovian binary tree applied to demography*, Submitted for publication.