Riemann problem for traffic flow on a roundabout

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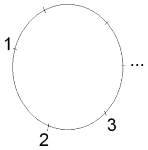
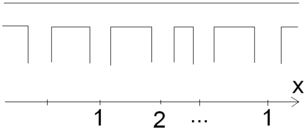
Modelization.





Model

- LWR model on each open segment: let v be a given (decreasing) speed law, then the total density r verifies

$$\partial_t r + \partial_x (rv(r)) = 0;$$

- Special boundary conditions:
  - bounds on the flows of exiting and entering vehicles,
  - conservation of the flow of the vehicles staying on the road.

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The one-T road.

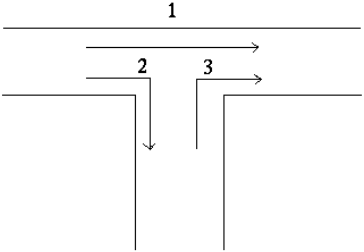


Figure: Road with an entry and an exit in $x = 0$.

ρ_1 : density of vehicles that go straight,
ρ_2 : density of vehicles that are to turn right,
ρ_3 : density of vehicles that are entering the road.

Equations:

We write the mass conservation of the vehicles with the same speed law v (multi-class extension of the LWR model):

$$\begin{cases} \partial_t \rho_1 + \partial_x (\rho_1 v(\rho_1 + \rho_2)) = 0 \\ \partial_t \rho_2 + \partial_x (\rho_2 v(\rho_1 + \rho_2)) = 0 \end{cases} \text{for } x < 0, \qquad \begin{cases} \partial_t \rho_1 + \partial_x (\rho_1 v(\rho_1 + \rho_3)) = 0 \\ \partial_t \rho_3 + \partial_x (\rho_3 v(\rho_1 + \rho_3)) = 0 \end{cases} \text{for } x > 0. \tag{1}$$

We also add piecewise constant initial data (Riemann problem):

$$\begin{aligned} \rho_1(0,x) &= \rho_1^- && \text{for } x < 0 \\ \rho_1(0,x) &= \rho_1^+ && \text{for } x > 0 \\ \rho_2(0,x) &= \rho_2^- && \text{for } x < 0 \\ \rho_3(0,x) &= \rho_3^- && \text{for } x > 0, \end{aligned} \tag{2}$$

Special boundary conditions.

The boundary conditions give bounds on the flows of the vehicles:

$$\begin{array}{lcll} \rho_1 v(\rho_1+\rho_2)(t,0-) & = & \rho_1 v(\rho_1+\rho_3)(t,0+) & \max, \\ \rho_2 v(\rho_1+\rho_2)(t,0-) & \leq & o(t) & \max, \\ \rho_3 v(\rho_1+\rho_3)(t,0+) & \leq & i(t) & \max, \end{array} \quad (3)$$

"max" means here that the flows of ρ_1, ρ_2 and ρ_3 are maximised.
We also add a priority rule :

- we maximise first the flows of ρ_1 and ρ_2
- then the flow of ρ_3.

We can obtain similar result with the other rule.

Result

Theorem

Under the hypotheses

- ***(V)*** *: the speed law* v *is* $\mathcal{C}^{0,1}$*, decreasing and vanishes in 1.*
- ***(R)*** *: the flow* $q(r) = rv(r)$ *is strictly concave and attains its maximum in* r_c*,*

the Riemann problem for the one T road admits a unique weak entropy solution.

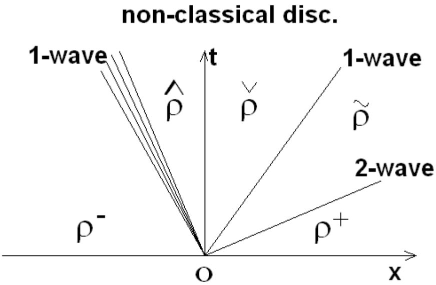
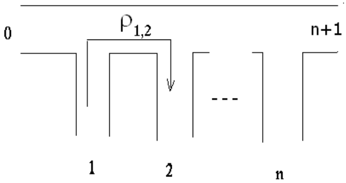


Figure: Solution.

The n-T road.



$\rho_{i,j}$: density of the vehicles which enter in i and exit in j.
(P) we know the numbers $p_{i,j}$ that represent the proportion of vehicles entering in i that are to exit in j.
(T) the vehicles are not authorized to do more than one turn !
This means: $\rho_{i,k}(x_k^+) = 0$, for $i \neq k$, $\rho_{k,j}(x_k^-) = 0$, for $j \neq k$.

Equations

We have:

$$\forall i \in [\![0, n]\!],\ \forall j \in [\![1, n+1]\!], \quad \partial_t \rho_{i,j} + \partial_x \left(\rho_{i,j} v\left(\sum_{l,m} \rho_{l,m}\right) \right) = 0 \qquad (4)$$

with boundary conditions in x_k:

$$\begin{array}{rcll} \text{for } i,j \neq k\,,\ \rho_{i,j} v\left(\sum_{l,m} \rho_{l,m}\right)(t, x_k^-) & = & \rho_{i,j} v\left(\sum_{l,m} \rho_{l,m}\right)(t, x_k^+) & \max, \\ \sum_{0 \leq i \leq n} \rho_{i,k} v\left(\sum_{l,m} \rho_{l,m}\right)(t, x_k^-) & \leq & o_k(t) & \max, \\ \sum_{1 \leq j \leq n+1} \rho_{k,j} v\left(\sum_{l,m} \rho_{l,m}\right)(t, x_k^+) & \leq & i_k(t) & \max, \end{array}$$

the flows being maximized first in x_k^- and then in x_k^+ because of the priority rule.

Theorem

Under the hypotheses ***(V)****,* ***(F)*** *and* ***(P)****, there exists* $T > 0$ *such that the Riemann problem for the n-T road admits a unique weak entropy solution for* $t \in [0, T]$*.*
Furthermore, we can give a lower bound for the time of existence: let $L = \min(x_{k+1} - x_k) > 0$*, then* $T \geq \frac{L}{2V}$*.*

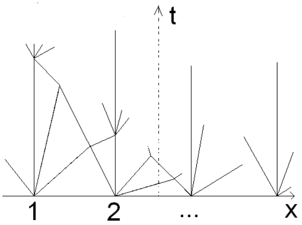


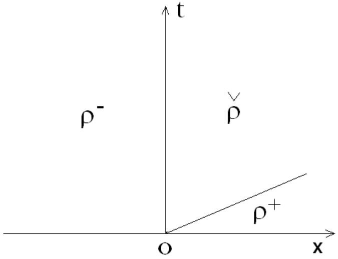
Figure: Solution with n points of entry and exit.

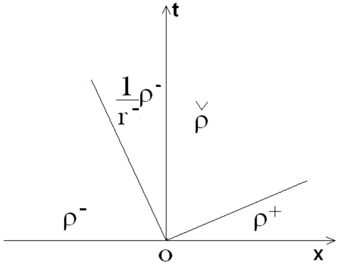
Discontinuity points.

We have some points of discontinuity for the Riemann solver :

- when $o, \rho_2^- \longrightarrow 0$;
- when $r^+ = \rho_1^+ + \rho_3^+ \longrightarrow 1$ and $\rho_1^- \longrightarrow 0$,

For example, in the case: $o = 0$, i large, $\rho_1^- < r_c$, $r^+ \geq r_c$, we obtain





Invariant sets.

We have some invariant sets.

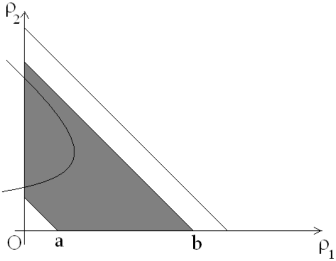


Figure: Invariant set.

On $\mathcal{S}$ the Riemann solver for the considered problem is not continuous. However, it is continuous on some subset: for $o \in [\varepsilon, 1]$ with $\varepsilon > 0$ and on $\rho \in T_{0,b}$, with $b < 1$ and $T_{0,b}$ invariant, the solution is obtained continuously.

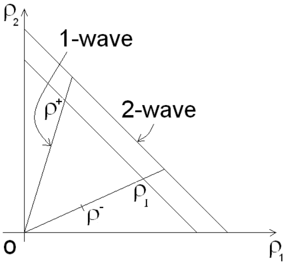


Figure: Waves curves.

We consider here a multi-class extension of the LWR model. The solution of the Riemann problem is obtained by following the Hugoniot loci.

- The 1- waves are shocks or rarefaction waves;
- the 2-waves are contact discontinuities.

We call $N(\rho^-)$ the set of states attainable in $x = 0$ by the left and $P(\rho^+)$ the set of states attainable in $x = 0$ by the right.

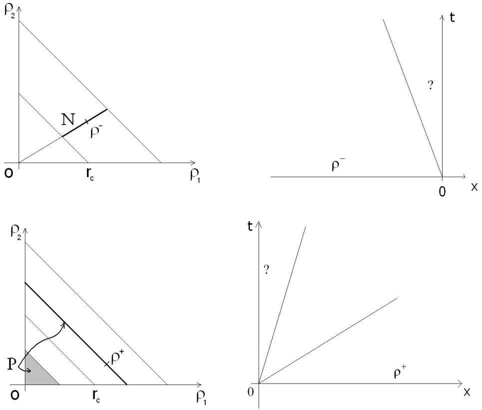


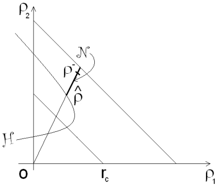
Figure: Left-Riemann problem(top), right-riemann problem (bottom).

Authorized set.

Let d be a function such that $d(r) = q_c$ if $r \leq r_c$ and $d(r) = q(r)$ if $r \geq r_c$. Then, we introduce:

$$\begin{aligned}
\mathcal{N}(\rho_1^-, \rho_2^-) &= N(\rho_1^-, \rho_2^-) \bigcap \{(\rho_1, \rho_2), \rho_2 v(\rho_1 + \rho_2) \leq o\} \\
&\bigcap \{(\rho_1, \rho_2), \rho_1 v(\rho_1 + \rho_2) \leq d(r^+)\}, \\
\mathcal{P}(\rho_1^+, \rho_3^+) &= P(\rho_1^+, \rho_3^+) \cap \{(\rho_1, \rho_3), \rho_1 v(\rho_1 + \rho_3) = M\} \\
&\cap \{(\rho_1, \rho_3), \rho_3 v(\rho_1 + \rho_3) \leq i\}.
\end{aligned}$$

We maximise the flow of ρ_2 on $\mathcal{N}(\rho^-)$ and then the flow of ρ_3 on $\mathcal{P}(\rho^+)$.



Conclusion:

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