SMALL SKEW FIELDS

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Abstract. A division ring of positive characteristic with countably many pure types is a field.

Wedderburn showed in 1905 that finite fields are commutative. As for infinite fields, we know that superstable [1, Cherlin, Shelah] and supersimple [4, Pillay, Scanlon, Wagner] ones are commutative. In their proof, Cherlin and Shelah use the fact that a superstable field is algebraically closed. Wagner showed that a small field is algebraically closed [5], and asked whether a small field should be commutative. We shall answer this question positively in non-zero characteristic.

1. Preliminaries

Definition 1.1. A theory is *small* if it has countably many n-types without parameters for all integer n. A structure is *small* if its theory is so.

We shall denote $dcl(A)$ the definable closure of a set A. Note that if K is a field and A a subset of K, then $dcl(A)$ is a field too. Smallness is clearly preserved under interpretation and addition of finitely many parameters.

Let D, D_1, D_2 be A-definable sets in some structure M with $A \subset M$. We define *the Cantor-Bendixson rank* $CB_A(D)$ and *degree* $dCB_A(D)$ of D over A.

Definition 1.2. By induction, we define

$CB_A(D) \geq 0$ if D is not empty
$CB_A(D) \geq \alpha+1$ if there is an infinite family of disjoint A-definable subsets D_i of D, such that $CB_A(D_i) \geq \alpha$ for all $i < \omega$.
$CB_A(D) \geq \beta$ for a limit ordinal β if $CB_A(D) \geq \alpha$ for all $\alpha < \beta$.

Definition 1.3. $dCB_A(D)$ is the greatest integer d such that D can be divided into d disjoint A-definable sets, with same rank over A as D.

Proposition 1.4. *If M is small and A is a finite set,*

(i) *The rank $CB_A(M)$ is ordinal.*
(ii) *The degree dCB_A is well defined.*
(iii) *If $D_1 \subset D_2$, then $CB_A(D_1) \leq CB_A(D_2)$.*
(iv) $CB_A(D_1 \cup D_2) = max\{CB_A(D_1), CB_A(D_2)\}$.
(v) *CB_A and dCB_A are preserved under A-definable bijections.*

If A is empty, we shall write CB and dCB rather than $CB_\emptyset$ or $dCB_\emptyset$.

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Remark 1.5. Let $H < G$ be A-definable small groups with $H \cap dcl(A) < G \cap dcl(A)$. Then, either $CB_A(H) < CB_A(G)$, or $CB_A(H) = CB_A(G)$ and $dCB_A(H) < dCB_A(G)$.

Corollary 1.6. *A small integral domain with unity is a field.*

Proof. Let R be this ring. If r is not invertible, then $1 \notin rR$ hence $rR \cap dcl(r) \lneq R \cap dcl(r)$. Apply Remark 1.5, but R and rR have same rank and degree over r. $\square$

Note that R need not have a unity (see Corollary 1.10). More generally, if φ is a definable bijection between two definable groups $A \leq B$ in a small structure, then A equals B.

Proposition 1.7. (Descending Chain Condition) *Let G be a small group and g a finite tuple in G. Let H be a subgroup of $dcl(g)$. In H, there is no strictly decreasing infinite chain of subgroups of the form $G_0 \cap H > G_1 \cap H > G_2 \cap H > \cdots$, where the G_i are H-definable subgroups of G.*

Proof. By Remark 5, either the rank or the degree decreases at each step. $\square$

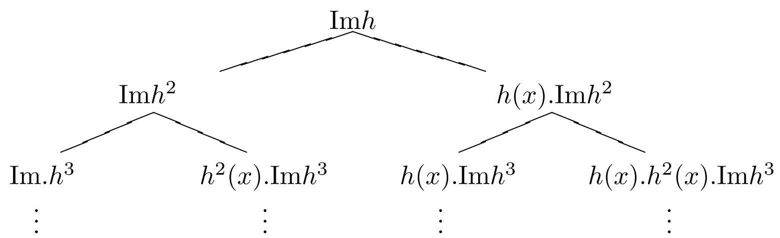
Corollary 1.8. *Let G be a small group, g a finite tuple, H subgroup of $dcl(g)$, and G_i a family of g-definable subgroups of G indexed by I. There is a finite subset I_0 of I such that*

$$\bigcap_{i \in I} G_i \cap H = \bigcap_{i \in I_0} G_i \cap H$$

Another chain condition on images of endomorphisms :

Proposition 1.9. *Let G be a small group and h a group homomorphism of G. There exists some integer n such that* $\mathrm{Im}h^n$ *equals* $\mathrm{Im}h^{n+1}$.

Proof. Suppose that the chain $(\mathrm{Im}h^n)_{n \geq 1}$ be strictly decreasing. Consider the following tree $G(x)$



Consider the partial type $\{x \notin h^{-n}\mathrm{Im}h^{n+1},\ n \geq 1\}$. We call it $\Phi(x)$ The sequence $(h^{-n}\mathrm{Im}h^{n+1})_{n \geq 1}$ is increasing, and each set $G \setminus h^{-n}\mathrm{Im}h^{n+1}$ is non-empty, so Φ is finitely consistent. Let b be a realization of Φ in a saturated model. The graph $G(b)$ has 2^ω consistent branches, whence $S_1(b)$ has cardinal 2^ω, a contradiction with G being small. $\square$

Corollary 1.10. *Let G be a small group and h a group homomorphism of G. There exists some integer n such that G equals* $\mathrm{Ker}h^n \cdot \mathrm{Im}h^n$.

Proof. Take n as in Proposition 1.9, and write f for h^n. We have $\mathrm{Im} f^2 = \mathrm{Im} f$, so for all $g \in G$ there exists some element g' such that $f(g) = f^2(g')$. Hence $f(gf(g')^{-1}) = 1$ and $gf(g')^{-1} \in \mathrm{Ker} f$, that is $g \in \mathrm{Ker} f \cdot \mathrm{Im} f$. □

It was shown in [6] that a definable injective homomorphism of a small group is surjective. Note that this follows again from Corollary 1.10.

2. Small skew fields

Recall a result proved in [5] :

Fact 2.1. *An infinite small field is algebraically closed.*

From now on, consider an infinite small skew field D. We begin by analysing elements of finite order.

Lemma 2.2. *Let $a \in D$ an element of order $n < \omega$. Then a is central in D.*

Proof. Either D has zero characteristic, so $Z(D)$ is infinite, hence algebraically closed. But $Z(D)(a)$ is an extension of $Z(D)$ of degree $d \leq n$, whence $a \in Z(D)$.

Or D has positive characteristic. Suppose that a is not central, then [3, Lemma 3.1.1] there exists x in D such that $xax^{-1} = a^i \neq a$. If x has finite order, then all elements in the multiplicative group $\langle x, a\rangle$ have finite order. Hence $\langle x, a\rangle$ is commutative [3, Lemma 3.1.3], contradicting $xax^{-1} \neq a$. So x has infinite order. Conjugating m times by x, we get $x^m a x^{-m} = a^{i^m}$. But a and a^i have same order n, with $gcd(n, i) = 1$. Put $m = \phi(n)$. By Fermat's Theorem, $i^m \equiv 1[n]$, so x^m and a commute. Then $L = Z(C_D(a, x^m))$ is a definable infinite commutative subfield of D which contains a. Let L^x be $\{l \in L,\ x^{-1}lx = l\}$. This is a proper subfield of L. Moreover $1 < [L^x(a) : L^x] \leq n$. But L^x is infinite as it contains x. By Fact 2.1, it is algebraically closed and cannot have a proper extension of finite degree. □

Proposition 2.3. *Every element of D has a n^{th} root for each integer n.*

Proof. Let $a \in D$. If a has infinite order, $Z(C_D(a))$ is an infinite commutative definable subfield of D. Hence it is algebraically closed, and a has an n^{th} root in $Z(C_D(a))$. Otherwise a has finite order. According to Lemma 2.2 it is central in D. Let $x \in D$ have infinite order. Then $a \in Z(C_D(a, x))$, a commutative, infinite, definable, and thus algebraically closed field. □

Remark 2.4. Note that since $D^\times$ is divisible, it has elements of arbitrary large finite order, which are central by Lemma 2.2. Taking D omega-saturated, we can suppose $Z(D)$ infinite.

Let us now show that a small skew field is *connected*, that is to say, has no definable proper subgroup of finite index.

Proposition 2.5. *D is connected.*

Proof. Multiplicatively : By Proposition 2.3, $D^\times$ is divisible so has no subgroup of finite index. Additively : Let H be a definable subgroup of D^+ of finite index n. In zero characteristic, D^+ is divisible, so $n = 1$. In general, let k be an infinite finitely generated subfield of D. Consider a finite intersection $G = \bigcap_{i \in I} d_i H$ of left translates of H by elements in k such that $G \cap k$ is minimal ; this exists by the chain condition. By minimality, $G \cap k = \bigcap_{d \in k} dH \cap k$, so $G \cap k$ is a left ideal of k. Furthermore, G is a finite intersection of subgroups of finite index in D^+ ; it has therefore finite index in D. Thus $G \cap k$ has finite index in $D \cap k = k$, and cannot be trivial, so $G \cap k = k = H \cap k$. This holds for every infinite finitely generated k, whence $H = D$. □

Now we look at elements of infinite order.

Lemma 2.6. *$a \in D$ have infinite order. Then $C_D(a) = C_D(a^n)$ for all $n > 0$.*

Proof. Clearly $C_D(a) \leq C_D(a^n)$. Consider $L = Z(C_D(a^n))$. It is algebraically closed by Fact 2.1, but $L(a)$ is a finite commutative extension of L, whence $a \in L$ and $C_D(a^n) \leq C_D(a)$. □

Now suppose that D is not commutative. We shall look for a commutative centralizer C and show that the dimension $[D : C]$ is finite. This will yield a contradiction.

Lemma 2.7. *Let $a \in D$, $t \in D \setminus im(x \mapsto ax - xa)$ and $\varphi : x \mapsto t^{-1} \cdot (ax - xa)$. Then $D = im\varphi \oplus ker\varphi$. Moreover, if $k = dcl(a, t, \overline{x})$, where $\overline{x}$ is a finite tuple, then $k = im\varphi \cap k \oplus ker\varphi \cap k$.*

Proof. Let $K = ker\varphi = C_D(a)$. Put $I = im\varphi$; this is a right $K-$vector space, so $I \cap K = \{0\}$, since $1 \in K \cap I$ is impossible by the choice of t. Consider the morphism

$$\tilde{\varphi} : \begin{array}{ccc} D^+/K & \longrightarrow & D^+/K \\ x & \longmapsto & \varphi(x) \end{array}$$

$\tilde{\varphi}$ is an embedding, and D^+/K is small ; by Corollary 10, $\tilde{\varphi}$ is surjective and $D/K = \tilde{\varphi}(D/K)$, hence $D = I \oplus K$. Now, let $k = dcl(a, t, \overline{x})$ where $\overline{x}$ is a finite tuple of parameters in D. I and K are $k-$definable. For all $\alpha \in k$ there exists a unique couple $(\alpha_1, \alpha_2) \in I \times K$ such that $\alpha = \alpha_1 + \alpha_2$, so α_1 and α_2 belong to $dcl(\alpha, a, t) \leq k$, that is to say $k = I \cap k \oplus K \cap k$. □

Lemma 2.8. *For every $a \notin Z(D)$, the map $\varphi_a : x \mapsto ax - xa$ is onto.*

Proof. Suppose φ_a not surjective. Let $t \notin im\varphi_a$, and $k = dcl(t, a, \overline{x})$ be a non commutative subfield of D for some finite tuple $\overline{x}$. Consider the morphism

$$\varphi : \begin{array}{ccc} D^+ & \longrightarrow & D^+ \\ x & \longmapsto & t^{-1}.(ax - xa) \end{array}$$

Set $H = im\varphi$, and $K = C_D(a) = ker\varphi$. By Lemma 2.7 we get $k = H \cap k \oplus K \cap k$. Let $N = \bigcap_I a_i H$ be a finite intersection of left-translates of H by elements in k, such that $N \cap k$ be minimal. We have

$$N \cap k = \bigcap_{i \in I} a_i H \cap k = \bigcap_{d \in k} dH \cap k,$$

so $N \cap k$ is a left ideal. Moreover, $H \cap k$ is a right $K \cap k$ vector-space of codimension 1. Then $N \cap k$ has codimension at most $n = \mid I \mid$. If $N \cap k = k$, then $H \cap k = k$, whence $K \cap k = \{0\}$, a contradiction. So $N \cap k$ is trivial and, k is a $K \cap k$-vector space of dimension at most n. By [2, Corollary 2 p.49] we get $[k : K \cap k] = [Z(k)(a) : Z(k)]$. But $Z(k) = Z(C_D(k)) \cap k$ with $Z(C_D(k))$ algebraically closed. Note that every element of k commutes with $Z(C_D(k))$, so $a \in Z(k)$, which is absurd if we add $b \notin C_D(a)$ in k. □

Theorem 2.9. *A small field in non-zero characteristic is commutative.*

Proof. Let $a \in D$ be non-central, and let us show that $x \mapsto ax - xa$ is not surjective. Otherwise there exists x such that $ax - xa = a$, hence $axa^{-1} = x + 1$. We would then have $a^p x a^{-p} = x + p = x$, and $x \in C_D(a^p) \setminus C_D(a)$, a contradiction with Lemma 2.2. □

3. Open problems

3.1. **Zero characteristic.** Note that we just use characteristic p in proof of theorem 19 to show that there exist $a \notin Z(D)$ such that $x \mapsto ax - xa$ is not surjective. Thus questions 1 and 2 are equivalent :

Question 1. Is a small skew field D of zero characteristic commutative ?

Question 2. Is every $x \mapsto ax - xa$ surjective onto D for $a \notin Z(D)$?

3.2. **Weakly small fields.** Weakly small structures have been introduced to give a common generalization of small and minimal structures. Minimal fields are known to be commutative.

Definition 3.1. A structure M is *weakly small* if for all finite set of parameters A in M, there are only countably many 1-types over A.

Question 3. Is a weakly small field algebraically closed ?

Question 4. Is a weakly small skew field commutative ?

Note that a positive answer to question 3 implies a positive answer to question 4, as all the proves given still hold. In general, one can prove divisibility and connectivity of an infinite weakly small field.

Proposition 3.2. *Every element in an infinite weakly small field D has a n^{th} root for all $n \in \omega$.*

Proof. Let $a \in D$. In zero characteristic, $Z(C_D(a))$ is an infinite definable commutative subfield of D, hence weakly small. According to [5, Proposition 9], every element in $Z(C_D(a))$ has a n^{th} root. In positive characteristic, we can reason as in the proof of Lemma 12, and find y with infinite order which commutes with a. Apply one more time [5, Proposition 9] to $Z(C_D(a, y))$. □

So $D^\times$ is divisible and the proof of Proposition 2.5 still holds.

Proposition 3.3. *An infinite weakly small field is connected.*

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