Séminaire de Master 2ème année

Sous la direction de Thomas Blossier

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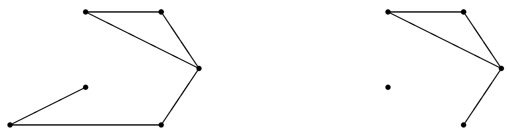
Extending partial isomorphisms of finite graphs

Abstract : This paper aims at proving a theorem given in [4] by *Hrushovski* concerning finite graphs, and gives a generalization (without any proof) of this theorem for a finite structure in a finite relational language. *Hrushovski*'s therorem is the following : any finite graph G embeds in a finite supergraph H so that any local isomorphism of G extends to an automorphism of H.

Introduction

In this paper, we call *finite graph* (G, R) any finite structure G with one binary symmetric reflexive relation R (that is $\forall x \in G\ xRx$ and $\forall x, y\ xRy \Longrightarrow yRx$). We call *vertex* of such a graph any point of G, and *edge*, any couple (x, y) such that xRy. Geometrically, a finite graph (G, R) is simply a finite set of points, some of them being linked by edges (see *picture 1*). A *subgraph* (F, R') of (G, R) is any subset F of G along with the binary relation R' induced by R on F.

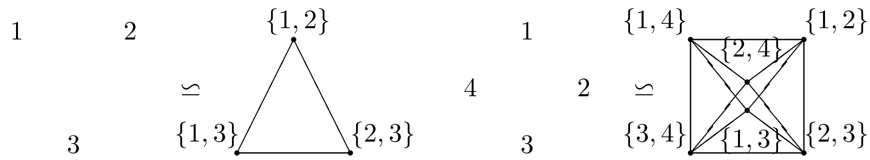


Picture 1 — *A graph* (G, R) *and a subgraph* (F, R') *of* (G, R).

We call *isomorphism* between two graphs (G, R) and (G', R') any bijection that preserves the binary relations, that is, any bijection σ that sends an edge on an edge along with σ^{-1}. If $(G, R) = (G', R')$, then such a σ is called an *automorphism* of (G, R). A *local isomorphism* of (G, R) is an isomorphism between two subgraphs of (G, R).

Let us give another example of graph :

Definition — Let X be a finite set and n a positive integer. We denote by $G(X, n)$ the graph the vertices of which are the $n-$elements subsets of X, with the binary relation R defined by xRy if and only if $x \cap y \neq \emptyset$.

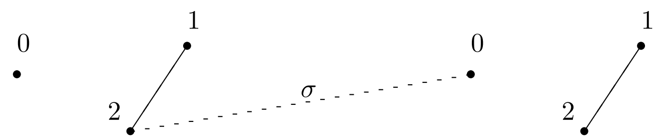


Picture 2 — *The graphs* $G(\{1, 2, 3\}, 2)$ *and* $G(\{1, 2, 3, 4\}, 2)$.

These graphs are quite interesting for two reasons : their vertices are kind of symmetric as each of them has the same number of neighbours. Moreover, they have plenty of automorphisms : any permutation α of X induces a natural automorphism on $G(X, n)$ that we will denote α^*.

1 Extending local isomorphisms to the graph

Let us consider a finite graph (G, R) and a local isomorphism σ of this graph. It is natural to wonder whether there is a way to extend σ to an automorphism of the whole graph G. The answer is clearly negative : just consider the graph $(\{0,1,2\}, R = (1,2))$, and the application σ that sends $\{2\}$ on $\{0\}$ (see *picture 3*). As $\{0\}$ is part of no edge, there is no way to define $\sigma(\{1\})$ so as to maintain an edge between the images of $(1,2)$.



Picture 3 — *The graph* $(\{0,1,2\}, R = (1,2))$ *and a local isomorphism* σ.

So, generally speaking, there is no way to extend a local isomorphism to an automorphism of the graph. However, are there some special local isomorphisms that could be easily extended to an automorphism? It would be convenient for G to have a large amount of automorphisms, so as to have better chances to extend σ.

We are now going to try to find a family $\mathcal{F}$ of subgraphs of $G(X,n)$ where any isomorphism between two elements of $\mathcal{F}$ could be extended to an automorphism of $G(X,n)$, the advantage of $G(X,n)$ being that it has plenty of automorphisms : namely, any permutation α of X induces a natural automorphism α^*.

Definition — A subgraph G_0 of $G(X,n)$ is said to be poor if any couple $x \neq y$ in G_0 has one element in common in X at most, and any $x \in X$ belongs to two different elements of G_0 at most.

This definition has been built to have the following Proposition :

Proposition 1 — Any isomorphism between two poor subgraphs F_1 and F_2 of $G(X,n)$ extends to an automorphism of $G(X,n)$.

Proof — Let $\sigma : F_1 \to F_2$ be an isomorphism between F_1 and F_2. We build a permutation α of X such that α^* extends σ. Let $x \in X$. There are three cases :

(*i*) Either x belongs to two elements f and f' in F_1. Then there is nothing but one choice for $\alpha_1(x)$: it has to be the unique element of $\sigma(f) \cap \sigma(f')$.

(*ii*) Or x belongs to just one element f in F_1. Then let α_f be a bijection between those x in f and the y being just in $\sigma(f)$.

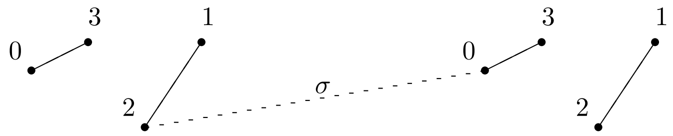
(*iii*)Or x is in none of the elements of F_1. Let α_2 be a bijection between those x and the y in none of the $\sigma(f)$.

Then, the union $\alpha = \bigcup \alpha_i$ defines a permutation of X. □

2 Extending the graph to a supergraph

As we generally fail to extend a local isomorphism to an automorphism of the graph, the next natural question is : can we extend the graph G to a larger graph H so that each local isomorphism of G extends to an automorphism of H?

Let us have a new look at the graph $(\{0,1,2\}, R = (1,2))$ of *picture 3*, and the application $\sigma : \{2\} \mapsto \{0\}$. There is no way to extend σ because $\{2\}$ and $\{0\}$ do not have the same number of *neighbours* (points in relation with them). But there is a way to solve this problem, namely by adding a fourth point $\{3\}$ to the graph so that $\{0\}R\{3\}$ (see *picture 4*). Then σ extends in a natural way.

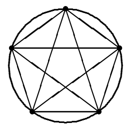


Picture 4 — *Extending the graph* $(\{0,1,2\}, R = (1,2))$.

We have just seen that a necessary condition for H is that every vertex of G should have the same *valency* in H, that is, the same number of neighbours in H:

Proposition 2 — Any finite graph (G,R) is a subgraph of a graph (H,R') with uniform valency.

Proof — Let n be the maximum valency of G. One can suppose n odd (or replace n by $n+1$). Around each vertex g of G, let's add as many new vertices linked to g so that g is surrounded by n neighbours. So, in this new graph, each vertex has valency n or 1. Let's put these new vertices of valency 1 on a circle, and link each of them with the $(n-1)/2$ previous and next ones on the circle (on the picture, $n=5$). Then each new vertex has $1+2(n-1)/2 = n$ neighbours. A problem arises when the $(n-1)/2$ previous and next ones on the circle are not distinct. This happens when there is less than n points on the circle. But one can always add n new vertices of valency 1 linked to another one. $\square$



But, such a graph H with uniform valency embeds in a $G(X,n)$, which is quite interesting as we saw that those $G(X,n)$ do have plenty of automorphisms. Just take X as the set of all the edges of H and n as the valency of each vertex of H (the application that sends a vertex x of H on the set of n edges adjacent to x is a local isomorphism from H to $G(X,n)$). And the image of H in $G(X,n)$ is poor! Therefore :

Proposition 3 — Any graph with uniform valency $n \geqslant 2$ is isomorphic to a poor subgraph of $G(X,n)$.

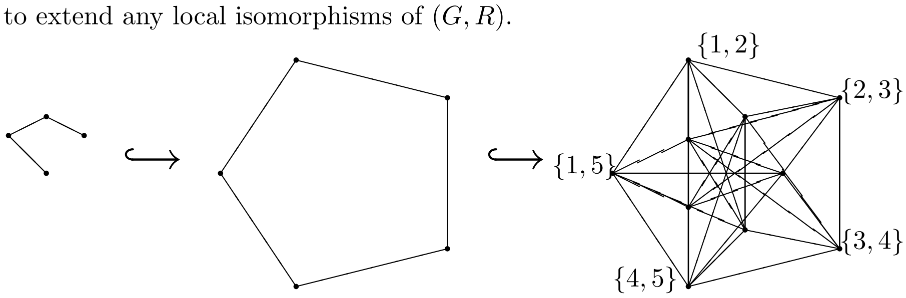
Then any finite graph G embeds in a poor subgraph G_0. Noting that a subgraph of a poor graph is poor, we have answered our second question :

Theorem *(Hrushovski)* — Any finite graph G embeds in a finite supergraph H so that any local isomorphism of G extends to an automorphism of H.

3 Example, and generalization

Let's give a simple example of the preceding construction. Take the following graph (G, R) :

The maximum valency of the vertices is 2, so first build a supergraph (H, R') of (G, R) with uniform valency 2 as shown in *picture 5*. (H, R') has 5 edges so embeds in $G(\{1, 2, 3, 4, 5\}, 2)$, which is the supergraph we're looking for. In fact, as this is a fairly simple example, the graph (H, R') would be big enough to extend any local isomorphisms of (G, R).



Picture 5 — *Extending* (G, R) *to* $G(\{1, 2, 3, 4, 5\}, 2)$.

A graph is nothing but a structure in the language $\mathcal{L} = \{R\}$ in the theory of a symmetric, reflexive relation. Whenever the language $\mathcal{L}$ should have more than one binary relation, we would speak of multi-colored graphs.

In fact, the theorem we have just given a proof of, not only extends to any multi-colored graph (with a finite number of colors), but also to any finite structure with a finite relational language :

Theorem *(Herwig)* — Let $\mathcal{M}$ be a finite structure in a finite relational language $\mathcal{L}$, and σ a local isomorphism of $\mathcal{M}$. Then there exists a finite superstructure $\mathcal{N}$ and an automorphism f of $\mathcal{N}$ extending σ.

References

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[3] B. Herwig, *Extending partial isomorphisms on finite structures*, Combinatorica, 1993.

[4] E. Hrushovski, *Extending partial isomorphisms of graphs*, Combinatorica 12 (p.411-416), 1992