Solutions to selected problems in TD 4 of the functional analysis course at Lyon 1

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Exercise 8.

5. We claim first that for every $v \in C$, we have

$$|f - g| \leq |f - v| \quad \text{a.e.} \tag{1}$$

Indeed, let $x \in X$. If $f(x) \geq 0$, then $g(x) = f(x)$ and so the inequality is satisfied. If $f(x) < 0$, then $g(x) = 0$ and so $|f(x) - g(x)| = |f(x)| = -f(x)$. The right hand side is $|v(x) - f(x)| = v(x) - f(x)$ (for almost every $x < 0$) since $v(x) \geq 0$ and $-f(x) > 0$. The inequality becomes $-f(x) \leq v(x) - f(x)$ which is true because $v(x) \geq 0$. It follows that

$$||f - g||_\infty \leq ||f - v||_\infty$$

and this means that $||f - g||_\infty = d(f, C)$ (distance in L^∞). On the other hand, C is convex and closed in L^∞. Indeed, let $f, g \in C$ and $t \in [0, 1]$, then $f + g \in L^\infty$ and $(1 - t)f + tg \geq 0$ a.e. This proves that C is convex. To prove that C is closed in $L^\infty(X)$, let (f_n) be a sequence of C that converges to f in $L^\infty(X)$. Then, $f_n \geq 0$ a.e. and f_n converges a.e. to f. It follows that $f \geq 0$ a.e. and so $f \in C$.

The argument we used to prove uniqueness breaks down because $||f - g||_\infty = ||f - v||_\infty$ does not imply that $|f - g| = |f - v|$. Here's a counterexample. Equip $\mathbb{R}^2$ with the infinity norm. This is $L^\infty(\{1, 2\})$ where $\{1, 2\}$ is equipped with the counting measure. Let $f = (-1, 0)$ and let $v_0 = (0, 1)$. Then $f^+ = (0, 0)$. However $||f - f^+||_\infty = ||f - v_0||_\infty = 1$.

6. Let us check that $C := \{f \in L^p; |f| \leq h\}$ is convex and closed in $L^p(X)$ for all $p \in [1, \infty]$. Indeed, let $f, g \in C$ and $t \in [0, 1]$, then $f + g \in L^p$ and

$$|(1 - t)f + tg| \leq (1 - t)|f| + t|g| \leq (1 - t)h + th = h \quad \text{a.e.}$$

This proves that C is convex. To prove that C is closed in $L^p(X)$, let (f_n) be a sequence of C that converges to f in $L^p(X)$. Then, $|f_n| \leq h$ a.e. and there is a subsequence f_{n_k} that converges a.e. to f. It follows that $|f| \leq h$ a.e. and so $f \in C$. Now, let

$$g = \begin{cases} f & \text{if } |f| \leq h \\ h & \text{if } f > h \\ -h & \text{if } f < -h. \end{cases}$$

Then $g \in C$ and $||f - g||_p = d(f, C)$ for every $p \in [1, \infty]$. Indeed, we claim first that for every $v \in C$,

$$|f - g| \leq |f - v| \quad \text{a.e.} \tag{2}$$

If $|f| \leq h$, then $g = f$ and the inequality holds. If $f > h$, then $|f - g| = f - h$ and $|f - v| = f - v$. In this case the inequality is equivalent to $f - h \leq f - v$ or which is equivalent to $-h \leq -v$ which is equivalent to $v \leq h$, which is true because $v \in C$. Finally, if $f < -h$, then $g = -h$ and so $|f - g| = f + h$. On the other hand, $|f - v| = v - f$. The inequality becomes $= -h - f \leq v - f$ or $-h \leq v$, which is true because $v \in C$. It follows from this inequality that for every $p \in [1, \infty]$ and every $v \in C$.

$$||f - g||_p \leq ||f - v||_p,$$

and so $||f - g||_p = d(f, C)$. We prove uniqueness for $p < \infty$. Suppose that there is an element $v_0 \in C$ such that $||f - g||_p = ||f - v_0||_p$. This means that

$$\int_X (|f - v_0|^p - |f - g|^p) = 0.$$

But, the integrand is nonnegative. Therefore, it must vanish a.e. and this gives

$$|f - g| = |f - v_0| \quad \text{a.e.}$$

If $|f| \leq h$, then $g = f$ and so $|f - v_0| = 0$, i.e., $v_0 = f$. If $f > h$, then $g = h$ and we get $f - h = f - v_0$. Therefore $v_0 = h$. If $f < -h$, then $g = -h$ and the equality becomes $-h - f = v_0 - f$ and so $v_0 = -h$. This means that $v_0 = g$. Hence uniqueness.