A muggle's approach to weakly converging sequences in L^p

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Captatio benevolentiæ. If $1 < p < \infty$, a bounded sequence in L^p contains a weakly convergent subsequence. Most often, this result is proved using the theory of reflexive spaces. The argument can be simplified when L^q is separable (which, in general, is not the case). In that case, the conclusion can be obtained via a diagonal process, encoded in Lemma 1 below. We propose a rather direct approach valid in the general case, relying solely on uniform convexity and Lemma 1. There is no claim of originality in the proof. Its only interest is that it allows to get the desired result without appealing to reflexivity.

More specifically, the setting is the following. Let $1 < p < \infty$ and let q be the conjugate of p. Let $(X, \mathscr{T}, \mu)$ be a measure space. We want to prove, using relatively low technology, the following classical fact.

Theorem. For each bounded sequence $(f_j) \subset L^p = L^p(X, \mathscr{T}, \mu)$, there exist a subsequence (f_{j_k}) and $f \in L^p$ such that $\int f_{j_k} g \to \int f g$, $\forall\, g \in L^q$.

The final ingredient of the proof is the following application of the diagonal process.

Lemma 1. Let E be a separable normed space. For each bounded sequence $(\varphi_j) \subset E^*$, there exist a subsequence (φ_{j_k}) and $\varphi \in E^*$ such that $\varphi_{j_k}(x) \to \varphi(x)$, $\forall\, x \in E$.

In order to derive the theorem, we start with four simple observations, Lemmas 2–5.

Lemma 2. Let $(F_n)_{n \geq 1}$ be a non-decreasing sequence of subspaces of a normed space E and let F be the closure of $\cup_n F_n$. If $\varphi \in F^*$, then

$$\|\varphi\|_{F^*} = \lim_n \|\varphi\|_{(F_n)^*}. \tag{1}$$

(With a slight abuse of notation, $\|\varphi\|_{(F_n)^*}$ stands for $\|\varphi_{|F_n}\|_{(F_n)^*}$.)

Proof. "$\geq$" is clear. For the opposite inequality, we may assume that $\|\varphi\|_{F^*} > 0$. Let $\varepsilon > 0$. Let $x \in F$ be such that $\|x\| = 1$ and $\varphi(x) \geq (1-\varepsilon)\|\varphi\|_{F^*}$. Let n_0 and $y \in F_{n_0}$ be such that $\|y\| \leq 1$ and $\|x - y\| \leq \varepsilon$. For each $n \geq n_0$, we have

$$\|\varphi\|_{(F_n)^*} \geq \varphi(y) = \varphi(x) - \varphi(x-y) \geq (1-\varepsilon)\|\varphi\|_{F^*} - \varepsilon\|\varphi\|_{F^*} = (1-2\varepsilon)\|\varphi\|_{F^*},$$

whence the conclusion. □

Lemma 3. Let E be a uniformly convex Banach space. If $(x_n) \subset E$ is a sequence such that $(\|x_n\|)$ converges and $\|x_n + x_m\| \geq 2\|x_n\|$, $\forall\, m \geq n$, then (x_n) converges.

Proof. We may assume that $\lim_n \|x_n\| > 0$ and $x_n \neq 0$, $\forall\, n$. Write $x_n = \|x_n\| y_n$. It suffices to prove that (y_n) converges. Let $\varepsilon > 0$ and let $\delta > 0$ correspond to ε in the definition of the uniform convexity. Then

$$\left\|\frac{y_n + y_m}{2}\right\| = \left\|\frac{x_n + x_m}{2\|x_n\|} + \frac{x_m}{2}\left(\frac{1}{\|x_m\|} - \frac{1}{\|x_n\|}\right)\right\| \geq \frac{\|x_n + x_m\|}{2\|x_n\|} - \frac{\|x_m\|}{2}\left|\frac{1}{\|x_m\|} - \frac{1}{\|x_n\|}\right| \geq 1 - \delta,$$

the latter inequality being valid for large n and m. For such n and m, we have $\|y_n - y_m\| < \varepsilon$, whence the conclusion. □

Lemma 4. Let $(A_n) \subset \mathscr{T}$. For each n, there exist $k = k(n) \geq 1$ and $B_j = B_j^n \in \mathscr{T}$, $1 \leq j \leq k$, mutually disjoint and such that

$$\operatorname{Vect}\left(\{\chi_{A_1}, \ldots, \chi_{A_n}\}\right) \subset F_n := \operatorname{Vect}\left(\left\{\chi_{B_1^n}, \ldots, \chi_{B_{k(n)}^n}\right\}\right) \subset \operatorname{Vect}\left(\left\{\chi_{B_1^{n+1}}, \ldots, \chi_{B_{k(n+1)}^{n+1}}\right\}\right). \tag{2}$$

If, in addition, each A_j is of finite measure, then so is each B_ℓ.

Proof. Proof by induction on n, the case where $n = 1$ being obvious. With $B_1 := B_1^n, \ldots$, $B_k := B_k^n$, we have

$$A_{n+1} = (B_1 \cap A_{n+1}) \sqcup (B_2 \cap A_{n+1}) \sqcup \ldots \sqcup (B_k \cap A_{n+1}) \sqcup \left(A_{n+1} \setminus \cup_{1 \leq j \leq k} B_j\right),$$

and thus, for $n + 1$, the conclusion of the lemma holds for the disjoint list of sets

$$B_1 \setminus A_{n+1}, B_1 \cap A_{n+1}, \ldots, B_k \setminus A_{n+1}, B_k \cap A_{n+1}, A_{n+1} \setminus \cup_{1 \leq j \leq k} B_j.$$ □

Lemma 5. Let

$$G := \operatorname{Vect}\left(\{\chi_{B_1}, \ldots, \chi_{B_k}\}\right), \text{ with } B_1, \ldots, B_k \in \mathscr{T} \text{ mutually disjoint and of finite measure.} \tag{3}$$

For each $g \in G$ such that $\|g\|_q > 0$, there exists some $f \in G$ such that

$$\int f g = \|g\|_q \text{ and } \|f\|_p = 1. \tag{4}$$

Proof. Set $f := C|g|^{q-1} \operatorname{sgn} g$, where $C := 1/\|g\|_q^{1/(p-1)}$. Clearly, $f \in G$ and (4) holds. □

Now comes the argument at the heart of the proof of the theorem. It is a sort of "infinite" version of Lemma 5, and is reminiscent of the equality $(\ell^p)^* = \ell^q$.

Lemma 6. Let (F_n) be a non-decreasing sequence of finite dimensional spaces, each one of the form (3). Set $F := \cup_n F_n$ and denote by P, respectively Q, the closure in F in L^p, respectively L^q.

Then P^* can be isometrically identified with Q, in the following sense. (i) If $\varphi \in P^*$, there exists $g \in Q$ such that $\|g\|_q = \|\varphi\|_{P^*}$ and

$$\varphi(f) = \int f g,\ \forall\, f \in P. \tag{5}$$

(ii) For each $g \in Q$, the functional φ as in (5) belongs to P^* and satisfies $\|\varphi\|_{P^*} = \|g\|_q$.

Similar conclusions if we reverse the roles of P and Q.

Proof. *Step 1.* Approximation of g. This is done by a Galerkin method. Since φ is linear on the finite dimensional space F_n, there exists some $g_n \in F_n$ such that

$$\varphi(f) = \int f g_n, \ \forall f \in F_n. \tag{6}$$

Moreover, by (6), Hölder's inequality, and Lemma 5, we have

$$\|g_n\|_q = \|\varphi\|_{(F_n)^*}. \tag{7}$$

Step 2. The sequence $(g_n)_{n\geq 1}$ is a Cauchy sequence in L^q. This is the key step, and relies on the uniform convexity of L^q (and thus of Q). We may assume that, for $n \geq n_0$, we have $\|g_n\|_q > 0$ (for, otherwise, $g_n \equiv 0$ and we are done). By Lemma 5, for $n \geq n_0$ there exists some $f_n \in F_n$ such that $\|f_n\|_p = 1$ and $\|g_n\|_q = \varphi(f_n) = \int f_n g_n$. If $m \geq n \geq n_0$, then

$$\|g_n\|_q = \varphi(f_n) = \int f_n g_n = \int f_n g_m = \int f_n \frac{g_n + g_m}{2} \leq \left\|\frac{g_n + g_m}{2}\right\|_q. \tag{8}$$

Step 2 follows from (8) and Lemma 3.

Step 3. Proof of (i). Let $g = \lim_n g_n$, so that $g \in Q$. By (7) and Lemma 2, we have $\|g\|_q = \|\varphi\|_{P^*}$. On the other hand, if $f \in F$, then $f \in F_m$ for some m, and therefore, for $n \geq m$, we have $\varphi(f) = \int f g_n$. Passing to the limits over n, we find that $\varphi(f) = \int f g$. Next, if $f \in P$, consider a sequence $(f_j) \subset F$ such that $f_j \to f$ in L^p. By the above, we have $\varphi(f) = \lim_j \varphi(f_j) = \lim_j \int f_j g = \int f g$. The equality $\|g\|_q = \|\varphi\|_{P^*}$ follows from Lemma 2.

Step 4. We may assume that $\|g\|_q > 0$. Clearly, $\varphi \in P^*$ and $\|\varphi\|_{P^*} \leq \|g\|_q$. To prove the opposite inequality, let $0 < \varepsilon < \|g\|_q$. Let n and $h_n \in F_n$ such that $\|g - h_n\|_q < \varepsilon$. Let $f \in F_n$ satisfy $\|f\|_p = 1$ and $\int h_n f = \|h_n\|_q$ (see Lemma 5). Then

$$\|\varphi\|_{P^*} \geq \int f g = \int f h_n - \int f(g - h_n) \geq \|h_n\|_q - \|f\|_p \|g - h_n\|_q \geq \|g\|_q - 2\varepsilon,$$

whence the conclusion. □

Proof of the theorem. *Step 1.* Reduction to simple functions. Consider, for each j, a simple function $\widetilde{f}_j$ such that $\|\widetilde{f}_j - f_j\|_p < 2^{-j}$, $\forall j$. If we find some subsequence $(\widetilde{f}_{j_k})$ and some $f \in L^p$ such that $\int \widetilde{f}_{j_k} g \to \int f g$, $\forall g \in L^q$, then for the same f, we have $\int f_{j_k} g \to \int f g$, $\forall g \in L^q$, and we are done. Therefore, it suffices to prove the theorem for simple functions.

Step 2. Use of the diagonal process. Since each f_j is a simple function in L^p, there exists of sequence $(A_n) \subset \mathscr{T}$ of sets of finite measure such that each f_j is a finite linear combination of the χ_{A_n}'s. Let F_n be as in Lemma 4. With the notation in Lemma 6, we have $(f_j) \subset P = Q^*$. Viewed as a sequence of linear continuous functionals on Q, (f_j) is bounded. On the other hand, Q is clearly separable, since, with the notation in Lemma 4, it is generated by the countable family $\left\{\chi_{B_j^n}; n \geq 1, 1 \leq j \leq k(n)\right\}$. By Lemma 1 and Lemma 6, there exist a subsequence (f_{j_k}) and some $f \in P$ such that $\int f_{j_k} h \to \int f h$, $\forall h \in Q$.

Step 3. Conclusion. Let $g \in L^q$. Set $\varphi(u) := \int u g$, $\forall u \in L^p$. By Lemma 6, there exists some $h \in Q$ such that $\varphi(u) = \int u h$, $\forall u \in P$. By the previous step, we thus have $\int f_{j_k} g \to \int f g$, $\forall g \in L^q$. □