Final exam
December 13, 2023 – 3 hours

Part I. Solving the Laplace equation with divergence form datum. The purpose of this part is to partially prove the following

Theorem. Let Ω be a bounded domain in $\mathbb{R}^N$, of class C^2, and let $1 < p < \infty$. For $F \in L^p(\Omega; \mathbb{R}^N)$, the equation

$$-\Delta u = \operatorname{div} F \text{ in } \mathscr{D}'(\Omega) \tag{1}$$

has a unique solution $u \in W_0^{1,p}(\Omega; \mathbb{R})$. In addition, with some finite constant C independent of F (but possibly depending on p and Ω), we have the estimate $\|u\|_{W^{1,p}} \leq C\|F\|_p$.

Preliminaries. a) The following identity may be useful. If $\omega, \Omega \subset \mathbb{R}^N$ are open sets and $\Phi \in C^1(\omega; \Omega)$, then

$$\nabla(u \circ \Phi) = {}^t J\Phi[(\nabla u) \circ \Phi],\ \forall\, u \in C^1(\Omega; \mathbb{R}).$$

b) We set $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0, \infty)$, $B_+ = \{x \in \mathbb{R}^N_+;\ |x| < 1\}$, and $B_0 = \{(x', 0);\ x' \in \mathbb{R}^{N-1}, |x'| \leq 1\}$.

c) In what follows, C denotes a constant depending possibly on p and Ω, but not on F, u, or the other scalar functions, matrix-valued functions, or vector fields appearing in the equations. This constant may change from a line to another.

d) We always suppose that $1 < p < \infty$. We take for granted the L^p-regularity theory for the equation $-\Delta u = f \in W^{k,p}(\Omega; \mathbb{R})$ and the following variant of the crucial lemma of the L^p-regularity theory.

Crucial lemma. There exist some $\varepsilon_0 > 0$ and $C < \infty$, possibly depending on $1 < p < \infty$ and on N, but not on B, H, h, or w below, such that, for: (a) $w \in W^{2,p}(B_+; \mathbb{R})$ satisfying: (i) there exists some $0 < R < 1$ such that $w(x) = 0$ if $|x| > R$; (ii) $\operatorname{tr}_{|B_0} w = 0$; (b) $H \in L^p(B_+; \mathbb{R}^N)$; (c) $h \in L^p(B_+; \mathbb{R})$; (d) $B \in L^\infty(B_+; M_N(\mathbb{R}))$, satisfying the equation

$$-\Delta w = \operatorname{div}(B\nabla w) + \operatorname{div} H + h \text{ in } \mathscr{D}'(B_+)$$

and the smallness condition $\|B\|_\infty \leq \varepsilon_0$, we have

$$\|w\|_{W^{1,p}} \leq C\|H\|_p + C\|h\|_p.$$

Exercise A. If p, q are conjugated exponents, prove that

$$[u \in W_0^{1,p}(\Omega; \mathbb{R}), -\Delta u = 0, v \in W^{2,q}(\Omega; \mathbb{R}) \cap W_0^{1,q}(\Omega; \mathbb{R})] \implies \int_\Omega u(-\Delta v) = 0,$$

and derive the uniqueness, in $W_0^{1,p}(\Omega; \mathbb{R})$, of a solution of (1).

Exercise B. Assume that the following *a priori* estimate holds.

$$[F \in C_c^\infty(\Omega; \mathbb{R}^N), u \in W_0^{1,p}(\Omega; \mathbb{R}) \text{ solves (1)}] \implies \|u\|_{W^{1,p}} \leq C\|F\|_p + C\|u\|_p. \tag{2}$$

1. Prove that the estimate (2) implies the validity of the following *a priori* estimate.

$$[F \in C_c^\infty(\Omega; \mathbb{R}^N), u \in W_0^{1,p}(\Omega; \mathbb{R}) \text{ solves (1)}] \implies \|u\|_{W^{1,p}} \leq C\|F\|_p.$$

2. Prove that the estimate (2) (and possibly other ingredients, to be specified) implies the theorem.

Exercise C. Let $u \in W^{1,1}_{loc}(\Omega;\mathbb{R})$ and $F \in L^1_{loc}(\Omega;\mathbb{R}^N)$ satisfy (1). Let $\Phi : \omega \to \Omega$ be a C^1-diffeomorphism. Set $v = u \circ \Phi$. Find (explicitly) a matrix-valued function $A \in C(\omega; M_N(\mathbb{R}))$ and a vector field $G \in L^1_{loc}(\omega;\mathbb{R}^N)$ such that

$$-\operatorname{div}(A\nabla v) = \operatorname{div} G \text{ in } \mathscr{D}'(\omega), \tag{3}$$

and carefully justify and give a precise meaning to (3).

Exercise D. Let $v \in W^{1,p}(B_+;\mathbb{R})$, $A \in C(B_+; M_N(\mathbb{R}))$, and $G \in L^p(B_+;\mathbb{R})$ satisfy (3) with $\omega = B_+$. Assume, moreover, that A is symmetric. (Is this requirement restrictive?)

1. Let $\zeta \in C^1(B_+;\mathbb{R})$ and set $w = \zeta v$. Carefully justify the equality

$$-\operatorname{div}(A\nabla w) = \operatorname{div}(\zeta G) - G\cdot\nabla\zeta - 2\operatorname{div}(vA\nabla\zeta) + v\operatorname{div}(A\nabla\zeta) \text{ in } \mathscr{D}'(B_+). \tag{4}$$

2. Assume furthermore that: (i) for some $0 < R < 1$, we have $\zeta(x) = 0$ if $|x| \geq R$; (ii) $\zeta \in C^2(\overline{B_+})$; (iii) $v \in W^{2,p}(B_+)$; (iv) $\operatorname{tr}_{|B_0} v = 0$. Write $A = I_N + B$. Under an appropriate smallness condition on B, prove the *a priori* estimate

$$\|w\|_{W^{1,p}} \leq C\|G\|_p + C\|v\|_p. \tag{5}$$

3. Sketch the strategy for deriving (2) from (5) (and possibly other ingredients).

Part II. A uniqueness result. In what follows, B denotes the unit ball in $\mathbb{R}^N$. The purpose of this part is to establish the implication

$$[u \in W^{1,1}_0(B;\mathbb{R}), -\Delta u = 0] \implies u = 0. \tag{6}$$

Preliminaries. a) The following result (see, e.g., [1, Proposition 9.18]) may be useful. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in W^{1,1}_0(\Omega;\mathbb{R})$. Let $\widetilde{u}$ be the extension of u with the value 0 outside Ω. Then $\widetilde{u} \in W^{1,1}(\mathbb{R}^N;\mathbb{R})$ and, in addition, $\nabla\widetilde{u}$ is the extension of ∇u with the value 0 outside Ω.
b) We set, for $r > 0$, $B_r := \{x \in \mathbb{R}^N;\ |x| < r\}$, $S_r := \{x \in \mathbb{R}^N;\ |x| = r\}$.

Exercise A. Let $u \in C(B;\mathbb{R}) \cap W^{1,1}_0(B;\mathbb{R})$. For $0 < r < 1$, prove that

$$\int_{S_r} |u| \leq \int_{B\setminus B_r} |\nabla u|. \tag{7}$$

A possible approach consists of arguing by smoothing, by carefully justifying the limiting argument.

Exercise B. We now prove (6).

1. Let $g \in C^\infty_c(B;\mathbb{R})$ and let $v \in H^1_0(B;\mathbb{R})$ solve $-\Delta v = g$. For $0 < r < 1$, prove that

$$\begin{aligned} \left|\int_{B_r} ug\right| &\leq \|\nabla v\|_{L^\infty(B)} \int_{S_r} |u| + \|v\|_{L^\infty(S_r)} \int_{S_r} |\nabla u| \\ &\leq \|\nabla v\|_{L^\infty(B)} \int_{B\setminus B_r} |\nabla u| + (1-r)\|\nabla v\|_{L^\infty(B)} \int_{S_r} |\nabla u|. \end{aligned} \tag{8}$$

2. Conclude, using an appropriate sequence $r_j \to 1$.

References

[1] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.