Final exam
Correction keys

Part I. Solving the Laplace equation with divergence form datum. The purpose of this part is to partially prove the following

Theorem. Let Ω be a bounded domain in $\mathbb{R}^N$, of class C^2, and let $1 < p < \infty$. For $F \in L^p(\Omega; \mathbb{R}^N)$, the equation

$$-\Delta u = \operatorname{div} F \text{ in } \mathscr{D}'(\Omega) \tag{1}$$

has a unique solution $u \in W_0^{1,p}(\Omega; \mathbb{R})$. In addition, with some finite constant C independent of F (but possibly depending on p and Ω), we have the estimate $\|u\|_{W^{1,p}} \le C\|F\|_p$.

Preliminaries. a) The following identity may be useful. If $\omega, \Omega \subset \mathbb{R}^N$ are open sets and $\Phi \in C^1(\omega; \Omega)$, then

$$\nabla(u \circ \Phi) = {}^tJ\Phi[(\nabla u) \circ \Phi], \ \forall\, u \in C^1(\Omega; \mathbb{R}).$$

b) We set $\mathbb{R}^N_+ = \mathbb{R}^{N-1} \times (0, \infty)$, $B_+ = \{x \in \mathbb{R}^N_+;\ |x| < 1\}$, and $B_0 = \{(x', 0);\ x' \in \mathbb{R}^{N-1}, |x'| \le 1\}$.

c) In what follows, C denotes a constant depending possibly on p and Ω, but not on F, u, or the other scalar functions, matrix-valued functions, or vector fields appearing in the equations. This constant may change from a line to another.

d) We always suppose that $1 < p < \infty$. We take for granted the L^p-regularity theory for the equation $-\Delta u = f \in W^{k,p}(\Omega; \mathbb{R})$ and the following variant of the crucial lemma of the L^p-regularity theory.

Crucial lemma. There exist some $\varepsilon_0 > 0$ and $C < \infty$, possibly depending on $1 < p < \infty$ and on N, but not on B, H, h, or w below, such that, for: (a) $w \in W^{2,p}(B_+; \mathbb{R})$ satisfying: (i) there exists some $0 < R < 1$ such that $w(x) = 0$ if $|x| > R$; (ii) $\operatorname{tr}_{|B_0} w = 0$; (b) $H \in L^p(B_+; \mathbb{R}^N)$; (c) $h \in L^p(B_+; \mathbb{R})$; (d) $B \in L^\infty(B_+; M_N(\mathbb{R}))$, satisfying the equation

$$-\Delta w = \operatorname{div}(B\nabla w) + \operatorname{div} H + h \text{ in } \mathscr{D}'(B_+)$$

and the smallness condition $\|B\|_\infty \le \varepsilon_0$, we have

$$\|w\|_{W^{1,p}} \le C\|H\|_p + C\|h\|_p.$$

Exercise A. If p, q are conjugated exponents, prove that

$$[u \in W_0^{1,p}(\Omega; \mathbb{R}), -\Delta u = 0, v \in W^{2,q}(\Omega; \mathbb{R}) \cap W_0^{1,q}(\Omega; \mathbb{R})] \implies \int_\Omega u(-\Delta v) = 0,$$

and derive the uniqueness, in $W_0^{1,p}(\Omega; \mathbb{R})$, of a solution of (1).

Hints. Since $-\Delta u = 0$ and thus $u \in C^\infty(\Omega)$ (Weyl's lemma), we have

$$\int_\Omega \nabla u \cdot \nabla \varphi = 0, \ \forall\, \varphi \in C_c^\infty(\Omega). \tag{2}$$

By passing to the limits and using the fact that $\nabla u \in L^p(\Omega)$, (2) still holds for $\varphi \in W_0^{1,q}(\Omega)$. On the other hand, since $-\Delta v \in L^q(\Omega)$ and $\nabla v \in L^q(\Omega)$, we have

$$\int_\Omega \psi(-\Delta v) = \int_\Omega \nabla \psi \cdot \nabla v, \ \forall\, \psi \in C_c^\infty(\Omega). \tag{3}$$

By passing to the limits and using the L^q integrability of $-\Delta v$ and ∇v, we find that (3) still holds for $\psi \in W_0^{1,p}(\Omega)$.

By combining (2) (with $\varphi = v$) with (3) (with $\psi = u$), we find that $\displaystyle\int_\Omega u(-\Delta v) = 0$.

For any $g \in C_c^\infty(\Omega)$, we have $g \in L^q(\Omega)$, and thus there exists $v \in W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega)$ such that $-\Delta v = g$. We find that $\displaystyle\int_\Omega ug = 0, \forall\, g \in C_c^\infty(\Omega)$, and thus (by the localization principle) $u = 0$ a.e. By linearity, this implies the uniqueness, in $W_0^{1,p}(\Omega;\mathbb{R})$, of a solution of (1). □

Exercise B. Assume that the following *a priori* estimate holds.

$$[F \in C_c^\infty(\Omega;\mathbb{R}^N), u \in W_0^{1,p}(\Omega;\mathbb{R}) \text{ solves (1)}] \implies \|u\|_{W^{1,p}} \le C\|F\|_p + C\|u\|_p. \tag{4}$$

1. Prove that the estimate (4) implies the validity of the following *a priori* estimate.

$$[F \in C_c^\infty(\Omega;\mathbb{R}^N), u \in W_0^{1,p}(\Omega;\mathbb{R}) \text{ solves (1)}] \implies \|u\|_{W^{1,p}} \le C\|F\|_p. \tag{5}$$

2. Prove that the estimate (4) (and possibly other ingredients, to be specified) implies the theorem.

Hints. 1. Argue by contradiction. Then there exist sequences $(F_j) \subset C_c^\infty(\Omega;\mathbb{R}^N)$, $(u_j) \subset W_0^{1,p}(\Omega)$ such that, for each j, $-\Delta u_j = \operatorname{div} F_j$, $\|u_j\|_{W^{1,p}} = 1$, while $\|F_j\|_p \to 0$. Moreover, by (4), there exists some $C > 0$ such that $\|u_j\|_p \ge C$. Using the Rellich-Kondrachov theorem and passing, up to a subsequence, to weak limits, we obtain the existence of some $u \in W_0^{1,p}(\Omega)$ such that $-\Delta u = 0$ and $\|u\|_p \ge C$. This contradicts the conclusion of Exercise A.

2. If $F \in C_c^\infty(\Omega;\mathbb{R}^N)$, then $\operatorname{div} F \in L^p(\Omega)$ and thus, by the L^p-regularity theory, (1) has a solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. By the assumption (4), u satisfies (5). By density of $C_c^\infty(\Omega;\mathbb{R}^N)$ in $L^p(\Omega;\mathbb{R}^N)$ and the *a priori* estimate (5), for every $F \in L^p(\Omega;\mathbb{R}^N)$ there exists some $u \in W_0^{1,p}(\Omega)$ satisfying (1). In addition, we have $\|u\|_{W^{1,p}} \le C\|F\|_p$. Uniqueness follows from Exercise A. □

Exercise C. Let $u \in W_{loc}^{1,1}(\Omega;\mathbb{R})$ and $F \in L_{loc}^1(\Omega;\mathbb{R}^N)$ satisfy (1). Let $\Phi : \omega \to \Omega$ be a C^1-diffeomorphism. Set $v = u \circ \Phi$. Find (explicitly) a matrix-valued function $A \in C(\omega; M_N(\mathbb{R}))$ and a vector field $G \in L_{loc}^1(\omega;\mathbb{R}^N)$ such that

$$-\operatorname{div}(A\nabla v) = \operatorname{div} G \text{ in } \mathscr{D}'(\omega), \tag{6}$$

and carefully justify and give a precise meaning to (6).

Hints. If $u \in C^\infty(\Omega)$ and $\varphi \in C_c^\infty(\Omega)$, then the chain rule $\nabla(u \circ \Phi) = {}^t J\Phi[(\nabla u) \circ \Phi]$ and a change of variables yield

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla \varphi &= \int_\omega [(\nabla u) \circ \Phi] \cdot [(\nabla \varphi) \circ \Phi]\, |J\Phi| \\ &= \int_\omega [({}^t J\Phi)^{-1} \nabla(u \circ \Phi)] \cdot [({}^t J\Phi)^{-1} \nabla(\varphi \circ \Phi)]\, |J\Phi| \\ &= \int_\omega [A\nabla(u \circ \Phi)] \cdot [\nabla(\varphi \circ \Phi)], \text{ with } A = |J\Phi|\, [(J\Phi)^{-1}]\, [({}^t J\Phi)^{-1}].\end{aligned} \tag{7}$$

The first equality (and thus the full chain of identities) in (7) still holds when $u \in W^{1,1}_{loc}(\Omega)$ and $\varphi \in C_c^1(\Omega)$. To see this, consider an open smooth bounded set V such that $\operatorname{supp} \varphi \subset V \Subset \Omega$, and set $U := \Phi^{-1}(V)$. Then $\Phi : U \to V$ is a bi-Lipschitz C^1-diffeomorphism, and thus (known exercise) $W^{1,1}(V) \ni u \mapsto (\nabla u) \circ \Phi \in L^1(U)$ is continuous, and the same holds for $C^1(\overline{V}) \ni \varphi \mapsto (\nabla \varphi) \circ \Phi \in L^\infty(U)$. We conclude *via* Hölder's inequality and approximation, using the density of $C^\infty(V) \cap W^{1,1}(V)$ in $W^{1,1}(V)$, and of $C_c^\infty(V)$ in $C_c^1(V)$.

On the other hand, if (1) holds, then

$$\int_\Omega \nabla u \cdot \nabla \varphi = -\int_\Omega F \cdot \nabla \varphi,\ \forall\, \varphi \in C_c^\infty(\Omega). \tag{8}$$

Arguing as above on the right-hand side of (8), we may rewrite (8) as

$$\int_\omega [A\nabla(u \circ \Phi)] \cdot [\nabla(\varphi \circ \Phi)] = -\int_\omega [|J\Phi|\, (J\Phi)^{-1}(F \circ \Phi)] \cdot [\nabla(\varphi \circ \Phi)], \quad \forall\, \varphi \in C_c^1(\Omega). \tag{9}$$

Set $G = |J\Phi|\, (J\Phi)^{-1}(F \circ \Phi)$. If $\zeta \in C_c^1(\omega)$, then (7) and (9) with $\varphi := \zeta \circ \Phi^{-1}$ yield

$$\int_\omega (A\nabla v) \cdot \nabla \zeta = -\int_\omega G \cdot \nabla \zeta,$$

so that v satisfies $-\operatorname{div}(A\nabla v) = \operatorname{div} G$ in $\mathscr{D}'(\omega)$. □

Exercise D. Let $v \in W^{1,p}(B_+; \mathbb{R})$, $A \in C(B_+; M_N(\mathbb{R}))$, and $G \in L^p(B_+; \mathbb{R})$ satisfy (6) with $\omega = B_+$. Assume, moreover, that A is symmetric. (Is this requirement restrictive?)

1. Let $\zeta \in C^1(B_+; \mathbb{R})$ and set $w = \zeta v$. Carefully justify the equality

$$-\operatorname{div}(A\nabla w) = \operatorname{div}(\zeta G) - G \cdot \nabla \zeta - 2\operatorname{div}(vA\nabla \zeta) + v \operatorname{div}(A\nabla \zeta) \text{ in } \mathscr{D}'(B_+). \tag{10}$$

2. Assume furthermore that: (i) for some $0 < R < 1$, we have $\zeta(x) = 0$ if $|x| \geq R$; (ii) $\zeta \in C^2(\overline{B_+})$; (iii) $v \in W^{2,p}(B_+)$; (iv) $\operatorname{tr}_{|B_0} v = 0$. Write $A = \mathrm{I}_N + B$. Under an appropriate smallness condition on B, prove the *a priori* estimate

$$\|w\|_{W^{1,p}} \leq C\|G\|_p + C\|v\|_p. \tag{11}$$

3. Sketch the strategy for deriving (4) from (11) (and possibly other ingredients).

Hints. The assumption on A is not restrictive, since the A found in the previous exercise is indeed symmetric. (On the other hand, even without the symmetry assumption, a formula in the spirit of (10) holds, and can be obtained following the same lines as below.)

1. We rely on the following identities, valid (by approximation, possibly based on the use of Hölder's inequality) under the assumptions $A \in L^\infty_{loc}(B_+; M_N(\mathbb{R}))$, A symmetric, $\zeta \in C^1(B_+;\mathbb{R})$, $v \in W^{1,1}_{loc}(B_+;\mathbb{R})$:

$$\operatorname{div}(A\nabla w) = \operatorname{div}(vA\nabla\zeta + \zeta A\nabla v), \tag{12}$$

$$\operatorname{div}(vA\nabla\zeta) = v\operatorname{div}(A\nabla\zeta) + (A\nabla\zeta)\cdot\nabla v, \tag{13}$$

$$\begin{aligned}\operatorname{div}(\zeta A\nabla v) =&\zeta\operatorname{div}(A\nabla v) + (A\nabla v)\cdot\nabla\zeta = \zeta\operatorname{div}(A\nabla v) + (A\nabla\zeta)\cdot\nabla v\\ =&\zeta\operatorname{div}(A\nabla v) - v\operatorname{div}(A\nabla\zeta) + \operatorname{div}(vA\nabla\zeta),\end{aligned} \tag{14}$$

$$\begin{aligned}-\operatorname{div}(A\nabla w) =&\zeta\operatorname{div}G - 2\operatorname{div}(vA\nabla\zeta) + v\operatorname{div}(A\nabla\zeta)\\ =&\operatorname{div}(\zeta G) - G\cdot\nabla\zeta - 2\operatorname{div}(vA\nabla\zeta) + v\operatorname{div}(A\nabla\zeta),\end{aligned} \tag{15}$$

where the first term on the right-hand side of (14) and the first term on the right-hand side of (15) are respectively defined by

$$[\zeta\operatorname{div}(A\nabla v)](\psi) = -\int_{B_+}(A\nabla v)\cdot[\nabla(\zeta\psi)],\ \forall\,\psi\in C^\infty_c(B_+;\mathbb{R}),$$

$$(\zeta\operatorname{div}G)(\psi) = -\int_{B_+}G\cdot[\nabla(\zeta\psi)],\ \forall\,\psi\in C^\infty_c(B_+;\mathbb{R}).$$

The second identity in (15) is the desired one.

2. By known properties of traces and products, for ζ and w as in the statement, w satisfies the assumptions of the crucial lemma. With ε_0 as in the crucial lemma and taking (15) into account, we have

$$\begin{aligned}\|w\|_{W^{1,p}} \le& C\|\zeta G - 2vA\nabla\zeta\|_p + C\|-G\cdot\nabla\zeta + v\operatorname{div}(A\nabla\zeta)\|_p\\ \le& C\|G\|_p + C\|v\|_p.\end{aligned}$$

3. We want to prove the validity of (4). Let $F\in C^\infty_c(\Omega;\mathbb{R}^N)$ and let $u\in W^{2,p}(\Omega)\cap W^{1,p}_0(\Omega)$ solve (1). Considering an adapted covering of $\overline{\Omega}$ and an adapted partition of unity, we may write $u = \sum\zeta_j u = \sum u_j$, with $\zeta_0\in C^\infty_c(\Omega)$ and, for $j\ge 1$, ζ_j supported in a small ball centered at some point of $\partial\Omega$. By item 2 and known properties on norm equivalences after composition with a diffeomorphism, for $j\ge 1$ we have

$$\|u_j\|_{W^{1,p}} \le C\|F\|_p + \|u\|_p. \tag{16}$$

For $j = 0$, we start from

$$-\Delta(\zeta_0 u) = \operatorname{div}(\zeta_0 F) - F\cdot\nabla\zeta_0 - 2\operatorname{div}(u\nabla\zeta_0) + u\operatorname{div}(A\nabla\zeta_0) = K \tag{17}$$

(which is a special case of (15)) and use the identity $\zeta_0 u = -E * K$ (valid since $\zeta_0 u$ is compactly supported) to derive the estimate

$$\|\zeta_0 u\|_{W^{1,p}} \le C\|F\|_p + C\|u\|_p. \tag{18}$$

Combining (16) and (18), we obtain (4). □

Part II. A uniqueness result. In what follows, B denotes the unit ball in $\mathbb{R}^N$. The purpose of this part is to establish the implication

$$[u \in W_0^{1,1}(B;\mathbb{R}), -\Delta u = 0] \implies u = 0. \tag{19}$$

Preliminaries. a) The following result (see, e.g., [1, Proposition 9.18]) may be useful. Let $\Omega \subset \mathbb{R}^N$ be an open set and let $u \in W_0^{1,1}(\Omega;\mathbb{R})$. Let $\widetilde{u}$ be the extension of u with the value 0 outside Ω. Then $\widetilde{u} \in W^{1,1}(\mathbb{R}^N;\mathbb{R})$ and, in addition, $\nabla\widetilde{u}$ is the extension of ∇u with the value 0 outside Ω.

b) We set, for $r > 0$, $B_r := \{x \in \mathbb{R}^N;\ |x| < r\}$, $S_r := \{x \in \mathbb{R}^N;\ |x| = r\}$.

Exercise A. Let $u \in C(B;\mathbb{R}) \cap W_0^{1,1}(B;\mathbb{R})$. For $0 < r < 1$, prove that

$$\int_{S_r} |u| \le \int_{B\setminus B_r} |\nabla u|. \tag{20}$$

A possible approach consists of arguing by regularization, by carefully justifying the limiting argument.

Hints. Let $0 < r < R$ and $v \in C^1(\mathbb{R}^N)$ be such that $v(x) = 0$ if $|x| \ge R$. Then

$$\begin{aligned}
\int_{S_r} |v| =& r^{N-1}\int_{S_1} |v(ry)|\, d\mathscr{H}^{N-1}(y) = r^{N-1}\int_{S_1} \left|\big[v(ty)\big]_{t=r}^{t=R}\right| d\mathscr{H}^{N-1}(y)\\
=& r^{N-1}\int_{S_1}\left|\int_r^R [\nabla v(ty)]\cdot y\, dt\right| d\mathscr{H}^{N-1}(y)\\
\le& r^{N-1}\int_{S_1}\int_r^R |(\nabla v)(ty)|\, dt\, d\mathscr{H}^{N-1}(y)\\
\le& \int_{S_1}\int_r^R t^{N-1}|(\nabla v)(ty)|\, dt\, d\mathscr{H}^{N-1}(y) = \int_{B_R\setminus B_r} |\nabla v| = \int_{\mathbb{R}^N\setminus B_r} |\nabla v|.
\end{aligned} \tag{21}$$

Let now $u \in W_0^{1,1}(B)$. If ρ is a standard mollifier, then $\operatorname{supp}(\widetilde{u} * \rho_\varepsilon) \subset B_{1+\varepsilon}$. Applying (21) (with $v = \widetilde{u} * \rho_\varepsilon$ and $R = 1 + \varepsilon$), we find that

$$\int_{S_r} |\widetilde{u} * \rho_\varepsilon| \le \int_{\mathbb{R}^N\setminus B_r} |(\nabla\widetilde{u}) * \rho_\varepsilon|. \tag{22}$$

Passing to the limits in (22) (using the fact that $\widetilde{u} * \rho_\varepsilon \to u$ uniformly on S_r as $\varepsilon \to 0$, since u is continuous in B, and that $(\nabla\widetilde{u}) * \rho_\varepsilon \to \nabla\widetilde{u}$ in $L^1(\mathbb{R}^N)$), we find that

$$\int_{S_r} |u| \le \int_{\mathbb{R}^N\setminus B_r} |\nabla\widetilde{u}| = \int_{B\setminus B_r} |\nabla u|. \qquad \square$$

Exercise B. We now prove (19).

1. Let $g \in C_c^\infty(B;\mathbb{R})$ and let $v \in H_0^1(B;\mathbb{R})$ solve $-\Delta v = g$. For $0 < r < 1$, prove that

$$\begin{aligned}\left|\int_{B_r} ug\right| &\leq \|\nabla v\|_{L^\infty(B)} \int_{S_r} |u| + \|v\|_{L^\infty(S_r)} \int_{S_r} |\nabla u| \\ &\leq \|\nabla v\|_{L^\infty(B)} \int_{B\setminus B_r} |\nabla u| + (1-r)\|\nabla v\|_{L^\infty(B)} \int_{S_r} |\nabla u|.\end{aligned} \tag{23}$$

2. Conclude, using an appropriate sequence $r_j \to 1$.

Proof. 1. By Weyl's lemma, $u \in C^\infty(B)$. By the L^p-regularity theory, we also have $v \in C^\infty(B) \cap C^1(\overline{B})$. Green's second formula and the previous exercise yield

$$\begin{aligned}\left|\int_{B_r} ug\right| =& \left|-\int_{S_r} u\frac{\partial v}{\partial \nu} + \int_{S_r} v\frac{\partial u}{\partial \nu}\right| \leq \|\nabla v\|_{L^\infty(B)} \int_{S_r} |u| + \|v\|_{L^\infty(S_r)} \int_{S_r} |\nabla u| \\ \leq& \|\nabla v\|_{L^\infty(B)} \int_{B\setminus B_r} |\nabla u| + \|v\|_{L^\infty(S_r)} \int_{S_r} |\nabla u| \\ \leq& \|\nabla v\|_{L^\infty(B)} \int_{B\setminus B_r} |\nabla u| + (1-r)\|\nabla v\|_{L^\infty(B)} \int_{S_r} |\nabla u|,\end{aligned}$$

where the last inequality follows from the fact that $v = 0$ on ∂B, so that, for $x \in S_r$, we have

$$|v(x)| = |v(x) - v(x/|x|)| \leq |x - x/|x|| \, \|\nabla v\|_{L^\infty(B)} = (1-r) \, \|\nabla v\|_{L^\infty(B)}.$$

2. Recall that, if $f : [0,1] \to \mathbb{R}_+$ is (Lebesgue) integrable, then there exists a sequence $t_j \to 0$ such that $t_j f(t_j) \to 0$. Applying this to $f(t) = \int_{S_{1-t}} |\nabla u|$, we find that there exists a sequence $r_j \to 1$ such that

$$(1 - r_j) \int_{S_{r_j}} |\nabla u| \to 0. \tag{24}$$

Applying (23) with $r = r_j$ and using (24) and dominated convergence, we find that

$$\int_B ug = \lim_{j\to\infty} \int_{B_{r_j}} ug = 0, \; \forall \, g \in C_c^\infty(B). \tag{25}$$

By the localisation principle, $u = 0$ a.e., and thus $u = 0$ (since u is continuous). □

References

[1] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.