Lecture # 4
REGULARITY THEORY

### (a) Exercises for the November 17 session

**Exercise 1.** Let $N \geq 2$ and $u \in C^1(\mathbb{R}^N \setminus \{0\})$ be such that $\partial_1 u \in L^1_{loc}(\mathbb{R}^N)$. Prove that $u \in L^1_{loc}(\mathbb{R}^N)$ and that $\partial_1 u$ is the distributional derivative of $u$. What if $N = 1$?

**Exercise 2.** Let $\omega_N$ be the area of $\mathbb{S}^{N-1}$. Let $E$ be "the" fundamental solution of $-\Delta$ in $\mathbb{R}^N$,

$$E(x) := \begin{cases} -(1/\omega_2)\,\ln|x|, & \text{if } N = 2 \\ (1/[(N-2)\omega_N])\,|x|^{2-N}, & \text{if } N \geq 3 \end{cases}.$$

1. Prove that, in the distributions sense,
$$\partial_j E = g_j, \text{ where } g_j(x) := -\frac{1}{\omega_N}\frac{x_j}{|x|^N}.$$
2. If $1 \leq p \leq \infty$ and $f \in L^p_c(\mathbb{R}^N)$, then, in the distributions sense,
$$\partial_j(f * E) = h_j, \text{where } h_j(x) := \int_{\mathbb{R}^N} f(y)\, g_j(x-y)\, dy.$$

**Exercise 3.** Let $K \in \mathscr{D}'(\mathbb{R}^N) \cap L^1_{loc}(\mathbb{R}^N \setminus \{0\})$. Let $f \in C^\infty_c(\mathbb{R}^N)$ and set $L := \operatorname{supp} f$. Then:

$$(K * f)(x) = \int_{\mathbb{R}^N} f(y)K(x-y)\,dy = \int_L f(y)K(x-y)\,dy,\ \forall\, x \notin \operatorname{supp} L. \tag{1}$$

**Exercise 4.** Let $(X, \mathscr{T}, \mu)$ be a measured space. (Warning: $\mu$ is not supposed $\sigma$-finite.) If $f : X \to \mathbb{R}$ is measurable and $1 \leq p < \infty$, then

$$\|f\|_p^p = p\int_0^\infty t^{p-1} \underbrace{\mu([|f| > t])}_{:=F_f(t)}\, dt.$$

### (b) Exercises for the November 24 session

**Exercise 1.** Let $u \in H^1_0(\Omega)$ be an eigenfunction of $-\Delta$. Prove that $u \in C^\infty(\Omega)$.

In what follows, $\omega$ is an open set in $\mathbb{R}^N$.

**Exercise 2.** Let $\Phi : \omega \to \Omega$ be a bi-Lipschitz $C^1$-diffeomorphism. Prove that, with constants $0 < C_{1,p} < C_{2,p} < \infty$ depending only on $1 \leq p < \infty$ and on $\Phi$, we have

$$C_1\|f \circ \Phi\|_{L^p(\omega)} \leq \|f\|_{L^p(\Omega)} \leq C_2\|f \circ \Phi\|_{L^p(\omega)}, \forall \text{ measurable function } f : \Omega \to \mathbb{R}.$$

Slightly more difficult questions: (i) prove that the above constants depend only on $p$ and on the Lipschitz constants of $\Phi$ and $\Phi^{-1}$; (ii) prove that the above still holds under the weaker assumption that $\Phi$ is a bi-Lipschitz homeomorphism.

**Exercise 3.** Let $\Phi : \omega \to \Omega$ be a $C^1$-diffeomorphism. If $f \in W^{1,1}_{loc}(\Omega)$, prove that $f \circ \Phi \in W^{1,1}_{loc}(\omega)$ and that the chain rule holds, i.e.,

$$\partial_i (f \circ \Phi) = \sum_{j=1}^{N} [(\partial_j f) \circ \Phi] [\partial_i \Phi_j] \text{ in } \mathscr{D}'(\omega), \ \forall\, 1 \le i \le N.$$

**Exercise 4.** Let $\Phi : \omega \to \Omega$ be a $C^1$-diffeomorphism. Let $u \in W^{1,1}_{loc}(\Omega)$ satisfy $-\Delta u = f \in L^1_{loc}(\Omega)$ in the distributions sense. Set $v := u \circ \Phi \in W^{1,1}_{loc}(\omega)$. Then, in the distributions sense, we have, in $\omega$,

$$-\operatorname{div}(A \nabla v) = g \in L^1_{loc}(\omega), \tag{2}$$

where

$$A = A(x) := |J\,\Phi| \, [(J\Phi)^{-1}] \, [{}^t[(J\Phi)^{-1}]], \ g := |J\,\Phi| \, f \circ \Phi.$$

**Exercise 5.** Set $\mathbb{R}^N_+ := \{x \in \mathbb{R}^N ;\, x_N > 0\}$. Let $u \in W^{1,1}(\mathbb{R}^N_+)$. Let $h \in \mathbb{R}^{N-1} \times \{0\}$. Give a meaning to and prove the equality $\operatorname{tr}\, u(\cdot + h) = (\operatorname{tr}\, u)(\cdot + h)$.

**Exercise 6.** Let $p$, $q$ be conjugated exponents, $g \in L^q(\mathbb{R}^N_+)$, $w \in W^{1,p}(\mathbb{R}^N_+)$, $h \in \mathbb{R}^{N-1} \times \{0\}$. Then

$$\left| \int_{\mathbb{R}^N_+} (g(x+h) - g(x))\, w(x)\, dx \right| \le |h| \, \|g\|_q \, \|\nabla w\|_p.$$

**Exercise 7.** Let $f \in L^1_{loc}(\mathbb{R}^N_+)$. Then

$$\lim_{t \to 0} \frac{f(\cdot + t e_j) - f}{t} = \partial_j f \text{ in } \mathscr{D}'(\mathbb{R}^N_+), \ \forall\, 1 \le j \le N-1.$$

**Exercise 8.** Let $u \in W^{2,1}(\mathbb{R}^N_+)$.

1. Let $\Sigma := \mathbb{R}^{N-1} \times \{0\}$, that we identify with $\mathbb{R}^{N-1}$. When $\varphi \in C^2_c(\overline{\mathbb{R}^N_+})$, prove the generalized (second) Green formula

$$\int_{\mathbb{R}^N_+} (-\Delta u)\, \varphi = \int_{\mathbb{R}^{N-1}} [\operatorname{tr}_{|\Sigma}\, \partial_N u]\, \varphi - \int_{\mathbb{R}^{N-1}} [\operatorname{tr}_{|\Sigma}\, u]\, \partial_N \varphi + \int_{\mathbb{R}^N_+} u\, (-\Delta \varphi).$$

2. If $F : \mathbb{R}^N_+ \to \mathbb{R}$, set

$$F^*(x) = F^*(x_1, \ldots, x_N) := \begin{cases} F(x), & \text{if } x_N > 0 \\ -F(x_1, \ldots, x_{N-1}, -x_N), & \text{if } x_N < 0 \end{cases}.$$

Let $u \in W^{2,1}(\mathbb{R}^N_+)$ satisfy $\operatorname{tr}_{|\Sigma}\, u = 0$. Prove that $-\Delta(u^*) = (-\Delta u)^*$.

### (c) Warnings

A **Exercise (Weierstrass' counterexample to Dirichlet's principle)** Let $0 < \alpha < 1$ and set

$$v(x, y) := (x^2 - y^2)\,(-\ln(x^2 + y^2))^\alpha,\ \forall\,(x, y) \in \mathbb{D} := \{(x, y) \in \mathbb{R}^2;\ x^2 + y^2 < 1\}.$$

Prove that:

(a) $v \notin C^2(\mathbb{D})$.

(b) The distributional Laplacian $f := \Delta v$ is continuous on $\mathbb{D}$.

(c) The equation $\Delta u = f$ has no classical (i.e., $C^2$) solution near the origin.

B Useful reference for items B and C: [10]

**Exercise** Let $\alpha \in \mathbb{R} \setminus \{-1, 1 - N\}$ and set

$$u(x) := x_1\,|x|^\alpha,\ \forall\,x \in \mathbb{R}^N \setminus \{0\},\ \beta := -\frac{\alpha(\alpha + N)}{(\alpha + 1)(\alpha + N - 1)}.$$

Then

$$\sum_{1 \le i \le N} \partial_i \left( \sum_{1 \le j \le N} (\delta_{ij} + \beta\, x_i x_j\, |x|^{-2})\partial_j u \right) = 0 \text{ in } \mathbb{R}^N \setminus \{0\}.$$

C **Theorem** (Serrin) A homogeneous uniformly elliptic equation in divergence form may have locally unbounded $W^{1,1}_{loc}(\Omega)$ weak solutions.

More specifically, if $N \ge 2$ and $0 < \varepsilon < 1$, and we set

$$u(x) := \frac{x_1}{|x|^{N-1+\varepsilon}},\ x \in \mathbb{R}^N,$$

and

$$A(x) := \mathrm{Id}_N + \frac{b}{|x|^2}(x_i\,x_j)_{1 \le i,j \le N},\ x \in \mathbb{R}^N,\ \text{with } b := \frac{N - 1}{\varepsilon(\varepsilon + N - 2)} - 1,$$

then $u \in W^{1,1}_{loc}(\mathbb{R}^N) \setminus L^\infty_{loc}(\mathbb{R}^N)$, $A$ is uniformly elliptic in $\mathbb{R}^N$, and

$$\operatorname{div}(A\nabla u) = 0 \text{ in } \mathscr{D}'(\mathbb{R}^N).$$

### (d) Singular integrals

General reference: [8, Section 3]

A **Proposition.** With the above notation, let $K := \partial_k g_j = \partial_k \partial_j E$ (in the distributions sense). Then:

(a) $K \in \mathscr{D}'(\mathbb{R}^N) \cap C^\infty(\mathbb{R}^N \setminus \{0\})$, and in particular (1) holds.

(b) $K \in \mathscr{S}'$ and, in the distributions sense,

$$\widehat{K}(\xi) = \ell_{j,k}, \text{ where } \ell_{j,k}(\xi) := -\frac{\xi_j \xi_k}{|\xi|^2}.$$

(c) For some finite $C$, we have $|\nabla K(x)| \le C/|x|^{N+1}$, $\forall\, x \in \mathbb{R}^N \setminus \{0\}$.

Useful reference: [6, Theorem 2.3.4]

B **Marcinkiewicz interpolation theorem** (special case) Let $(X, \mathscr{T}, \mu)$ be a measured space. Let $1 < r < \infty$ and let $T$ be a linear operator on $L^1 \cap L^r(X)$ such that, for every $f \in L^1 \cap L^r(X)$, $Tf$ is a measurable function on $X$ and, for some $K_1, K_r < \infty$, we have

$$\mu([|Tf| > t]) \le K_1 \frac{\|f\|_1}{t}, \ \forall\, f \in L^1 \cap L^r(X), \ \forall\, t > 0,$$
$$\mu([|Tf| > t]) \le K_r \frac{\|f\|_r^r}{t^r}, \ \forall\, f \in L^1 \cap L^r(X), \ \forall\, t > 0.$$

Then, for every $1 < p < r$ and some $C_p < \infty$,

$$\|Tf\|_p \le C_p \|f\|_p, \ \forall\, f \in L^1 \cap L^r(X),$$

and in particular $T$ admits a unique linear continuous extension from $L^p(X)$ into $L^p(X)$.

In the special case where $\mu$ is a Radon measure in $\mathbb{R}^N$, the same holds if $T$ is initially defined on $C_c(\mathbb{R}^N)$ or even $C_c^\infty(\mathbb{R}^N)$.

C **Calderón-Zygmund decomposition, second form** Let $f \in C_c(\mathbb{R}^N)$ and $t > 0$. Then, with finite constants independent of $f$ and $t$ there exist: a family of disjoint cubes $C_n \subset \mathbb{R}^N$ and functions $g, h_n \in L_c^\infty(\mathbb{R}^N)$ (depending on $f$ and $t$) such that

(a) $g = f$ in $\mathbb{R}^N \setminus \cup_n C_n$.

(b) $|g| \le Ct$.

(c) $\operatorname{supp} h_n \subset C_n$, $\forall\, n$.

(d) $\displaystyle\int h_n = 0, \forall\, n.$

(e) $\displaystyle⨍ |h_n| \le Ct, \forall\, n.$

(f) $\displaystyle f = g + \sum_n h_n$ (pointwise).

(g) $\displaystyle\sum_n |C_n| \le C \frac{\|f\|_1}{t}.$

(h) $\displaystyle\|g\|_1 + \sum_n \|h_n\|_1 \le C\|f\|_1.$

(i) $\displaystyle\left\| \sum_n h_n \right\|_2^2 = \sum_n \|h_n\|_2^2 \le \|f\|_2^2.$

D **Calderón-Zygmund theorem adapted to the Laplace equation** Let $K \in \mathscr{S}'(\mathbb{R}^N) \cap C^1(\mathbb{R}^N \setminus \{0\})$ satisfy

(i) $\widehat{K}$ is a bounded *real* function.

(ii) $|\nabla K(x)| \le C/|x|^{N+1}, \forall\, x \in \mathbb{R}^N \setminus \{0\}$, for some finite $C$.

Let $Tf := K * f, \forall\, f \in C_c^\infty(\mathbb{R}^N)$. Then

$$\|Tf\|_p \le C_{p,N} \|f\|_p,\ \forall\, 1 < p < \infty,\ \forall\, f \in C_c^\infty(\mathbb{R}^N).$$

In particular, for $1 < p < \infty$, $T$ admits a unique linear continuous extension from $L^p(\mathbb{R}^N)$ into itself.

**Corollary.** Let $1 < p < \infty$ and $f \in L^p_c(\mathbb{R}^N)$, and set $u := E * f$. Then

$$\|\partial_j \partial_k u\|_p \le C_{p,N} \|f\|_p,\ \forall\, 1 \le j, k \le N.$$

E **A standard "elliptic estimate"** Let $1 < p < \infty$, $K \subset \Omega \subset \mathbb{R}^N$, with $K$ compact and $\Omega$ open. If $-\Delta u = f \in L^p(\Omega)$, then $u \in W^{2,p}_{loc}(\Omega)$ and, for some finite $C = C_{p,N,\Omega,K}$,

$$\|u\|_{W^{2,p}(K)} \le C(\|f\|_{L^p(\Omega)} + \|u\|_{L^1(\Omega)}).$$

**(e) $L^p$ regularity theory**

Useful references: [4, Chapter 9] for the regularity theory, [8, Section 1.5] for trace theory

A **Main regularity theorem** (**Calderón**, **Zygmund**, **Koselev**, Greco, Agmon, Douglis, Nirenberg, ...) Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$-domain. Let $1 < p < \infty$ and $f \in L^p(\Omega)$. Then the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3}$$

has a unique (generalized) solution $u \in W^{2,p}(\Omega)$. In addition, for some finite $C$ independent of $f$, $\|u\|_{W^{2,p}(\Omega)} \le C\|f\|_p$.

**Exercise.** The above $u$ is not only a distributional solution, but also a *strong solution*, in the sense that for a.e. $x \in \Omega$ we have

$$-\sum_{j=1}^N \partial_{jj} u(x) = f(x).$$

B **Toolbox** In what follows, $\omega, \Omega \subset \mathbb{R}^N$ are bounded open sets.

For the record.

**Rademacher's theorem** A Lipschitz function $f : \Omega \to \mathbb{R}$ is differentiable a.e., and its distributional gradient and point gradient coincide.

Useful reference: [3, Section 3.1.2]

**Exercise.** Let $\Phi : \overline{\omega} \to \overline{\Omega}$ be a $C^{1,1}$-diffeomorphism and $1 \le p \le \infty$. Prove that $f \mapsto \|f \circ \Phi\|_{W^{2,p}(\omega)}$ is equivalent to the usual norm on $W^{2,p}(\Omega)$.

**Lemma.** Let $A \in \mathrm{Lip}_{loc}(\omega)$, $v \in W^{1,1}_{loc}(\omega)$, $\zeta \in \mathrm{Lip}_{loc}(\omega)$, and $g \in L^1_{loc}(\omega)$. If (2) holds, then

$$- \operatorname{div} (A\nabla(\zeta v)) = \zeta\, g - v \operatorname{div} (A\nabla\zeta) - (A\nabla v) \cdot \nabla\zeta - (A\nabla\zeta) \cdot \nabla v.$$

**Adapted covering lemma.** Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$-domain. Given a number $\varepsilon > 0$, there exist:

(i) An integer $m$,

(ii) $m$ balls $B_j = B_{\rho_j}(0)$, $1 \le j \le m$,

(iii) $C^{1,1}$ diffeomorphisms $\Phi_j : B_j \to \mathbb{R}^N$, $1 \le j \le m$,

(iv) $\psi \in C^\infty_c(\Omega)$ and $\zeta_j \in C^\infty_c(B_j)$, $1 \le j \le m$,

such that:

(a) $\Omega \subset \cup_j \Phi_j(B_j)$,

(b) $B_j \cap (\Phi_j)^{-1}(\Omega) = B_j^+ := \{x = (x_1, \dots, x_N) \in B_j;\ x_N > 0\}$,

(c) $B_j \cap (\Phi_j)^{-1}(\partial\Omega) = B_j^0 := \{x = (x_1, \dots, x_N) \in B_j;\ x_N = 0\}$,

(d) $B_j \cap (\Phi_j)^{-1}((\mathbb{R}^N \setminus \overline{\Omega})) = B_j^- := \{x = (x_1, \dots, x_N) \in B_j;\ x_N < 0\}$,

(e) $\rho_j \le 1$, $1 \le j \le m$,

(f) Given $u : \Omega \to \mathbb{R}$, if we set $u_0 := \psi u : \Omega \to \mathbb{R}$, $v_j := u_{|\Phi_j(B_j^+)} \circ \Phi_j : B_j^+ \to \mathbb{R}$, and $w_j := \zeta_j v_j$, $1 \le j \le m$, then

$$\|u\|_{W^{2,p}(\Omega)} \sim \|u_0\|_{W^{2,p}(\Omega)} + \sum_{1 \le j \le m} \|w_j\|_{W^{2,p}(B_j^+)},\ \forall\, 1 \le p \le \infty,$$

(g) If we set

$$A(x) = A_j(x) := |J\,\Phi_j(x)|\,[(J\Phi_j)^{-1}(x)]\,[{}^t[(J\Phi_j)^{-1}(x)]],\ 1 \le j \le m,\ x \in B_j,$$

then $x \mapsto A(x)$ is Lipschitz and $A(x) = \mathrm{Id} + B(x)$, where $\|B(x)\|_\infty \le \varepsilon$, $1 \le j \le m$, $x \in B_j$,

(h) If $u \in W^{1,1}_{loc}(\Omega)$ satisfies, in $\mathscr{D}'(\Omega)$, $-\Delta u = f \in L^1_{loc}(\Omega)$, then $u_0$ and $w_j$ satisfy

$$\begin{aligned} -\Delta u_0 &= \psi\, f - 2\nabla\psi \cdot \nabla u - u\,\Delta\psi \ \text{ in } \mathscr{D}'(\mathbb{R}^N), \\ -\operatorname{div}(A_j\nabla w_j) &= \zeta_j\,(f_{|\Phi_j(B_j^+)} \circ \Phi_j)|J\,\Phi_j| - v_j \operatorname{div}(A_j\nabla\zeta_j) \\ &\quad - (A_j\nabla v_j)\cdot\nabla\zeta_j - (A_j\nabla\zeta_j)\cdot\nabla v_j \text{ in } \mathscr{D}'(B_j^+),\ 1 \le j \le m. \end{aligned}$$

**Exercise.** Let $u \in W^{1,1}(\Omega)$ and $\varphi \in C^1(\overline{\Omega})$. Prove that $\operatorname{tr}(\varphi u) = \varphi_{|\partial\Omega} \operatorname{tr} u$.

**Exercise.** Let $\Omega$ be a $C^1$-domain, and let $\Psi : U \to \mathbb{R}^N$ be a $C^1$-diffeomorphism from an open set $U \subset \mathbb{R}^N$ into its image. Set $\Xi := \Psi_{|U} : U \to \Psi(U)$ and $\Phi := \Xi^{-1}$. Set also $\Sigma := \partial\Omega \cap U$ and $\Lambda := \Psi(\Sigma)$. Let $u \in W^{1,1}(U)$ and set $v := u \circ \Phi$. Give a meaning to and prove the equality $\operatorname{tr}_{\Psi(\Sigma)} v = (\operatorname{tr}_\Sigma(u)) \circ [(\Psi)^{-1}_{|\Lambda}]$.

C **Theorem (Higher order regularity)** Let $k \geq 0$, $\Omega \in C^{k+1,1}$, and $1 < p < \infty$. If $f \in W^{k,p}(\Omega)$, then the solution $u$ of (3) satisfies $u \in W^{k+2,p}(\Omega)$ and, for some finite $C$ independent of $f$, $\|u\|_{W^{k+2,p}(\Omega)} \leq C\|f\|_{W^{k,p}(\Omega)}$.

D For the record, we mention some results in lower order regularity theory.

**Theorem.** Let $\Omega \in C^{1,1}$ and $1 < p < \infty$. For $F \in L^p(\Omega; \mathbb{R}^N)$, the equation

$$-\Delta u = \operatorname{div} F \text{ in } \mathscr{D}'(\Omega)$$

has a unique solution $u \in W_0^{1,p}(\Omega)$. In addition, with some finite constant $C$ independent of $F$, we have the estimate $\|\nabla u\|_p \leq C\|F\|_p$.

**Theorem** (Stampacchia) Let $\Omega \in C^{1,1}$. For $f \in L^1(\Omega)$, the equation

$$-\Delta u = f \text{ in } \mathscr{D}'(\Omega)$$

has a unique solution $u \in W_0^{1,1}(\Omega)$. Moreover, this $u$ satisfies $u \in \cap_{1 \leq p < N/(N-1)} W_0^{1,p}(\Omega)$ and, with finite constants $C_p$ independent of $f$,

$$\|\nabla u\|_p \leq C_p\|f\|_1, \; \forall\, 1 \leq p < \frac{N}{N-1}.$$

Useful reference: [9, Section 4.1]

## Big picture of the proofs of Theorems A and C

**I. Preliminaries**

(a) Localization of the problem.
(b) The effects of the change of coordinates: equation, traces.
(c) Equivalence of norms. Reduction to local estimates.
(d) Choice of adapted neighborhoods of points of $\partial\Omega$.

**II. $L^2$ theory**

(a) Interior estimates.
(b) Boundary estimates *via* Nirenberg's quotient method.

**III. $L^p$, $p > 2$, theory**

(a) Interior estimates *via* Calderón-Zygmund theory. Induction principle.

(b) Boundary estimates. Reflection and contraction principles.

**IV.** $L^p, p < 2$**, theory**

(a) Uniqueness *via* duality.

(b) *A priori estimates via* uniqueness.

(c) Existence *via a priori* estimates.

**V. Higher order regularity** by induction

### (f) A glimpse of the $C^\alpha$ regularity theory

Useful reference: [4, Lemma 4.4, Theorem 6.14, Theorem 6.19]. For the record:

**Theorem (**$C^\alpha$ **regularity)** (Kellogg) Let $0 < \alpha < 1$, $k \geq 0$, $\Omega \in C^{k+2,\alpha}$. If $f \in C^{k,\alpha}(\overline{\Omega})$, then the solution of (3) satisfies $u \in C^{k+2,\alpha}(\overline{\Omega})$. In addition, for some finite $C$ independent of $f$, $\|u\|_{C^{k+2,\alpha}(\overline{\Omega})} \leq C\|f\|_{C^{k,\alpha}(\overline{\Omega})}$.

**Lemma (Hölder estimates for the Newtonian potential)** (Korn) Let $0 < \alpha < 1$. If $f \in C^\alpha_c(\mathbb{R}^N)$ and $u := E * f$, then, for some finite $C$ independent of $f$,

$$\left|D^2 u\right|_{C^\alpha(\mathbb{R}^N)} \leq C|f|_{C^\alpha(\mathbb{R}^N)}.$$

### (g) Power growth nonlinearities. Bootstrap

Useful reference: [8, Section 3.3.2]. In what follows, we assume that $N \geq 3$.

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a measurable function satisfying

$$|f(x,t)| \leq C(1 + |t|^p), \ \forall\, x \in \Omega, \ \forall\, t \in \mathbb{R}.$$

Let $u$ satisfy

$$u \in H^1_{loc}(\Omega), \ x \mapsto f(x, u(x)) \in L^1_{loc}(\Omega)$$
$$-\Delta u = f(x, u(x)) \text{ in } \mathscr{D}'(\Omega).$$

A **Exercise.** Assume that $p < \dfrac{N+2}{N-2}$. Then $u \in W^{2,r}_{loc}(\Omega), \forall\, r < \infty$.

B **Proposition.** The same holds when $p = \dfrac{N+2}{N-2}$.

Moreover, if $u \in H^1_{loc}(\Omega)$ satisfies

$$-\Delta u = a(x)u + b(x), \text{ with } a \in L^{N/2}_{loc}(\Omega), \ b \in L^\infty_{loc}(\Omega),$$

then $u \in L^r_{loc}(\Omega), \forall\, r < \infty$.

C **Exercise.** Let $p > \dfrac{N+2}{N-2}$. Prove that the equation $-\Delta u = |u|^p$ has a locally unbounded solution $u \in H^1 \cap L^p(B_1(0))$, of the form $u(x) = \lambda |x|^{-\alpha}$, for appropriate constants $\lambda, \alpha > 0$.

**(h) A glimpse of the De Giorgi regularity theory**

Useful references: [4, Sections 8.5–8.9], [5, Chapter 4]. For the record:

A **Theorem** (local boundedness; **Stampacchia**, Ladyzhenskaya, Uraltseva, Trudinger,...) Let $A = A(x)$ be uniformly elliptic in $\Omega := B_1(0)$. Let $u \in H^1(\Omega)$ satisfy $-\operatorname{div}(A\nabla u) = f \in L^p(\Omega)$, where $p > \frac{N}{2}$. Then $u \in L^\infty_{loc}(\Omega)$ and, with a finite constant depending only on $0 < R < 1$ and $p$,

$$\|u\|_{L^\infty(B_R(0))} \le C(\|f\|_{L^q(\Omega)} + \|u\|_{L^1(\Omega)}).$$

B **Theorem** (local $C^\alpha$ regularity; **De Giorgi**, Nash, Ladyzhenskaya, Uraltseva, **Moser**,...) There exists some $0 < \alpha < 1$ depending only on $p$ and the ellipticity constants of $A$ such that the above $u$ belongs to $C^\alpha_{loc}(\Omega)$ and satisfies, with a finite constant $C$ depending only on $R$ and $p$

$$|u(x) - u(y)| \le C(\|f\|_{L^q(\Omega)} + \|u\|_{L^1(\Omega)}),\ \forall\, x, y \in B_R(0).$$

**(i) Wente estimates. Compensation phenomena**

Useful references: [1], [2], [7, Section 10.3]

A **Theorem** (Wente) Let $\Omega \in C^{1,1}$ be a bounded domain in $\mathbb{R}^2$, and let $F \in H^1(\Omega; \mathbb{R}^2)$. Then the problem

$$\begin{cases} -\Delta u = \det(JF) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a (unique) weak solution $u \in H^1_0(\Omega)$. In addition, we have $u \in C(\overline{\Omega})$ and, for some finite constant independent of $F$, we have the *Wente estimates*

$$\|u\|_\infty + \|\nabla u\|_2 \le C\|\nabla F\|_2.$$

B For the record:

**Theorem** (Fefferman, Stein, Coifman, Lions, Meyer, Semmes) If $F \in W^{1,N}_c(\mathbb{R}^N; \mathbb{R}^N)$, and we set $u := E * [\det(JF)]$, then $D^2u \in L^1(\mathbb{R}^N)$ and, with a finite constant independent of $F$ and of its support,

$$\left\|D^2 u\right\|_1 \le C\|\nabla F\|_N.$$

# References


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