Lecture # 1
THE DIRECT METHOD: A FEW EXAMPLES

(a) Basic examples

In what follows, $\Omega \subset \mathbb{R}^N$ is a "smooth" bounded open set.
In items $\boxed{\text{A}}$, $\boxed{\text{B}}$, $\boxed{\text{C}}$, $a \in C(\overline{\Omega})$, $a \geq 0$, and $f \in C(\overline{\Omega})$.

$\boxed{\text{A}}$ The problem

$$\begin{cases} -\Delta u + a(x)u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique weak solution $u \in H_0^1(\Omega)$.

Useful reference: [4, Corollary 3.23].

$\boxed{\text{B}}$ Same for the problem

$$\begin{cases} -\Delta u + a(x)|u|^{q-1} \operatorname{sgn} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

with $1 < q < \infty$.

Useful results:

Exercise. Let $1 < q < \infty$. Then

$$L^q(X, \mathscr{T}, \mu) \ni u \mapsto G(u) := |u|^{q-1} \operatorname{sgn} u \in L^{q/(q-1)}(X, \mathscr{T}, \mu)$$

is continuous.

Lemma. Let $1 < q < \infty$. Then

$$L^q(X, \mathscr{T}, \mu) \ni u \mapsto F(u) := \int_X |u|^q \, d\mu$$

is C^1, and

$$F'(u)(\varphi) = q \int_\Omega |u|^{q-1} (\operatorname{sgn} u) \, \varphi, \ \forall \, u, \varphi \in L^q(X, \mathscr{T}, \mu).$$

$\boxed{\text{C}}$ The problem

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + a(x)|u|^{q-1}\operatorname{sgn} u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

with $1 < p, q < \infty$, has a unique weak solution $u \in W_0^{1,p}(\Omega)$.

Useful result:

Exercise. Let $1 < p < \infty$. Then

$$L^p(X, \mathscr{T}, \mu; \mathbb{R}^d) \ni f \mapsto F(f) := \int_X |f|^p \, d\mu$$

is C^1, and

$$F'(f)(g) = p \int_\Omega |f|^{p-2} f \cdot g, \ \forall f, g \in L^p(X, \mathscr{T}, \mu; \mathbb{R}^d).$$

$\boxed{\text{D}}$ **Definition.** A *Carathéodory function* is a function $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^d$ such that

(i) $x \mapsto f(x, u, \xi)$ is (Lebesgue) measurable, $\forall (u, \xi) \in \mathbb{R}^m \times \mathbb{R}^d$.

(ii) $(u, \xi) \mapsto f(x, u, \xi)$ is continuous, for a.e. $x \in \Omega$.

Theorem. (Tonelli, Mac Shane, Morrey, ...) Let $1 \le p, q \le \infty$. Let f be a Carathéodory function such that:

a) $f(x, u, \xi) \ge a(x) \cdot u + b(x) \cdot \xi$, $\forall u, \xi$, for a.e. x, for some $a \in L^{q'}(\Omega; \mathbb{R}^m)$, $b \in L^{p'}(\Omega; \mathbb{R}^d)$.

b) $\xi \mapsto f(x, u, \xi)$ is convex for a.e. $x \in \Omega$.

Set

$$L^q(\Omega; \mathbb{R}^m) \times L^p(\Omega; \mathbb{R}^d) \ni (u, \xi) \mapsto L(u, \xi) := \int_\Omega f(x, u(x), \xi(x)) \, dx \in \mathbb{R} \cup \{\infty\}.$$

Then

$$[u_j \to u \text{ in } L^q(\Omega; \mathbb{R}^m),\ \xi_j \rightharpoonup \xi \text{ in } L^p(\Omega; \mathbb{R}^d)] \implies \varliminf L(u_j, \xi_j) \ge L(u, \xi).$$

(When $p = \infty$, we may replace $\rightharpoonup$ by $\overset{*}{\rightharpoonup}$.)

Useful results:

Exercise. If f is a Carathéodory function and $(u, \xi) : \Omega \to \mathbb{R}^m \times \mathbb{R}^d$ is measurable, prove that $\Omega \ni x \mapsto f(x, u(x), \xi(x))$ is measurable.

Exercise.

1. Let f be a Carathéodory function. Prove that, for each $\varepsilon, M > 0$, there exist: some $\delta = \delta(\varepsilon, M) > 0$ and some compact set $K = K(\varepsilon, M) \subset \Omega$ such that:

i. $|\Omega \setminus K| < \varepsilon$.

ii. $[x \in K, u, v \in \mathbb{R}^m, \xi, \eta \in \mathbb{R}^d, |u| \le M, |\xi| \le M, |u - v| \le \delta, |\xi - \eta| \le \delta] \Rightarrow |f(x, u, \xi) - f(x, v, \eta)| \le \varepsilon$.

(Hint: consider only u, v, ξ, η with rational coordinates.)

2. Prove the *Scorza-Dragoni theorem*: f is a Carathéodory function iff for each $\varepsilon > 0$ there exists some compact set $L_\varepsilon \subset \Omega$ such that:

   i. $|\Omega \setminus L_\varepsilon| < \varepsilon$.

   ii. f is continuous on $L_\varepsilon \times \mathbb{R}^m \times \mathbb{R}^d$.

   (Hint: use Lusin's theorem to find a large set $L \subset \Omega$ such that $L \ni x \mapsto f(x, u, \xi)$ is continuous when u, ξ have rational coordinates.)

Useful references: [6, Theorem 3.4, Section 3.3.1], [4, Corollary 3.9], [3, Theorem 2.2.10].

(b) Notions of convexity

$\boxed{\text{A}}$ **Definition.** A continuous function $f : \mathbb{R}^{Nm} \to \mathbb{R}$ is *quasi-convex* if

$$|U|\, f(\xi) \le \int_U f(\xi + D\varphi(x))\, dx, \forall\, U \subset \mathbb{R}^N \text{ bounded open set}, \quad \forall \xi \in \mathbb{R}^{Nm}, \forall \varphi \in C_c^\infty(U; \mathbb{R}^m). \tag{1}$$

Exercise. Prove that the f is quasi-convex iff (1) is satisfied for *one* non empty U.

Exercise. Assume that U is bounded and convex.

1. Prove that $W^{1,\infty}(U) = \mathrm{Lip}\,(U)$.
2. Prove that (1) still holds when $\varphi \in W_c^{1,\infty}(U, \mathbb{R}^m)$.
3. Prove that, with $(\varphi_j) \subset W^{1,\infty}(U; \mathbb{R}^m)$, $\varphi_j \overset{*}{\rightharpoonup} 0$ iff (φ_j) has uniformly bounded Lipschitz constants and $\varphi_j \to 0$ uniformly on U.

Lemma. (Morrey) If f is quasi-convex and $Q \subset \mathbb{R}^N$ is a cube, then

$$[(\varphi_j) \subset W^{1,\infty}(Q; \mathbb{R}^m),\ \varphi_j \overset{*}{\rightharpoonup} 0] \implies \underline{\lim} \int_Q f(\xi + D\varphi_j(x))\, dx \ge |Q|\, f(\xi), \quad \forall \xi \in \mathbb{R}^{Nm}.$$

Useful reference: [8, Lemma 2.2].

Exercise. Prove a version of Morrey's lemma with Q replaced by a finite volume open set.

$\boxed{\text{B}}$ **Theorem.** (Morrey, ..., Acerbi-Fusco) Let f be a Carathéodory function on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm}$ such that:

a) for a.e. $x \in \Omega$ and each $u \in \mathbb{R}^m$, $\mathbb{R}^{Nm} \ni \xi \mapsto f(x, u, \xi)$ is quasi-convex.

b) $0 \le f(x, u, \xi) \le a(x) + b(u, \xi)$, with $a \in L^1(\Omega)$, $b \in L^\infty_{loc}(\mathbb{R}^m \times \mathbb{R}^{Nm})$.

If $(u_j) \subset W^{1,\infty}(\Omega; \mathbb{R}^m)$ and $u_j \overset{*}{\rightharpoonup} u$, then

$$\underline{\lim} \int_\Omega f(x, u_j(x), Du_j(x))\, dx \geq \int_\Omega f(x, u(x), Du(x))\, dx.$$

Useful result:

Exercise. (Easy version of Lebesgue's differentiation theorem) Let $Q := (0,1)^N$ and let $g \in L^1(Q)$. Let $\ell \geq 1$ be an integer and

$$g_\ell(x) := \fint_C g(y)\, dy \text{ if } x \text{ belongs to the dyadic cube } C \text{ of size } 2^{-\ell}.$$

Then, up to a subsequence $\ell_n \to \infty$, $g_\ell \to g$ a.e.

Useful references: [1, Theorem II.1], [10, Corollary, p. 13].

For the record [1, Theorem II.4]:

Theorem. (Acerbi-Fusco) Let $1 \leq p < \infty$. Let f be a Carathéodory function on $\Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm}$ such that:

a) for a.e. $x \in \Omega$ and each $u \in \mathbb{R}^m$, $\mathbb{R}^{Nm} \ni \xi \mapsto f(x, u, \xi)$ is quasi-convex.
b) $0 \leq f(x, u, \xi) \leq a(x) + C(|u|^p + |\xi|^p)$, with $a \in L^1(\Omega)$ and C finite.

If $(u_j) \subset W^{1,p}(\Omega; \mathbb{R}^m)$ and $u_j \rightharpoonup u$, then

$$\underline{\lim} \int_\Omega f(x, u_j, Du_j(x))\, dx \geq \int_\Omega f(x, u, Du(x))\, dx.$$

$\boxed{\text{C}}$ **Theorem.** (Morrey) Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm}$ be continuous. If, for every open set $U \subset \Omega$,

$$[u_j \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(U)] \implies$$
$$\underline{\lim} \int_U f(x, u_j(x), Du_j(x))\, dx \geq \int_U f(x, u(x), Du(x))\, dx,$$

then, for each $x \in \Omega$ and $u \in \mathbb{R}^m$, $\mathbb{R}^{Nm} \ni \xi \mapsto f(x, u, \xi)$ is quasi-convex.

Useful result:

Lemma. Let $Q := (0,1)^N$ and let $\zeta \in C_c^\infty(Q; \mathbb{R}^m)$, extended as a smooth 1-periodic function to $\mathbb{R}^m$. Let $U \subset \Omega$ be relatively compact. Let $u_0 \in C(\Omega; \mathbb{R}^m)$, $\xi_0 \in C(\Omega; \mathbb{R}^{Nm})$. Set $\zeta_j(x) := 2^{-j}\zeta(2^j x)$, $\forall\, j \geq 1$, $\forall\, x \in \mathbb{R}^N$. Then

$$\begin{aligned}\lim \int_U & f(x, u_0(x), \xi_0(x) + D\zeta_j(x))\, dx \\ &= \int_U \int_Q f(x, u_0(x), \xi_0(x) + D\zeta(y))\, dy dx\end{aligned}$$

and

$$\lim \int_U f(x, u_0(x) + \zeta_j(x), \xi_0(x) + D\zeta_j(x))\, dx$$
$$= \int_U \int_Q f(x, u_0(x), \xi_0(x) + D\zeta(y))\, dy dx.$$

For the record [1, Theorem II.2]:

Theorem. (Acerbi-Fusco) Let $f : \Omega \times \mathbb{R}^m \times \mathbb{R}^{Nm}$ be a Carathéodory function such that $0 \leq f(x, u, \xi) \leq a(x) + b(u, \xi), \forall\, x \in \Omega, \forall\, (u, \xi) \in \mathbb{R}^m \times \mathbb{R}^{Nm}$, where $a \in L^1(\Omega)$ and $b \in L^\infty_{loc}(\mathbb{R}^m \times \mathbb{R}^{Nm})$. If, for every open set $U \subset \Omega$,

$$[u_j \overset{*}{\rightharpoonup} u \text{ in } W^{1,\infty}(U)] \implies$$
$$\underline{\lim} \int_U f(x, u_j(x), Du_j(x))\, dx \geq \int_U f(x, u(x), Du(x))\, dx,$$

then, for a.e. $x \in \Omega$ and each $u \in \mathbb{R}^m$, $\mathbb{R}^{Nm} \ni \xi \mapsto f(x, u, \xi)$ is quasi-convex.

$\boxed{\text{D}}$ **Proposition.** Assume that $N = 1$ and let $f : \mathbb{R}^m \to \mathbb{R}$ be continuous. Then f is quasi-convex iff f is convex.

For the record:

Theorem. Assume that $m = 1$ (i.e., we work with scalar functions u) and let $f : \mathbb{R}^N \to \mathbb{R}$. Then f is quasi-convex iff f is convex.

Useful reference: [6, Theorem 3.1, Section 3.3.1].

$\boxed{\text{E}}$ We identify $\mathbb{R}^{Nm}$ with $M_{m,N}(\mathbb{R})$. Let $A \in M_{m,N}(\mathbb{R})$. Given $1 \leq \ell \leq K := \min(m, N)$, and $I = \{i_1 < i_2 < \ldots < i_\ell\} \subset \{1, \ldots, m\}$, $J = \{j_1 < j_2 < \ldots < j_\ell\} \subset \{1, \ldots, N\}$, let $A_{I,J}$ denote the minor of order ℓ of A formed with the rows $i_1, \ldots, i_\ell$, respectively the columns $j_1, \ldots, j_\ell$. Let M be the number of all possible minors. We order the minors as $A^1, \ldots, A^M$.

Definition. (Morrey, Ball) A function $f : \mathbb{R}^{Nm} \to \mathbb{R}$ is *polyconvex* if there exists some convex function $g : \mathbb{R}^M \to \mathbb{R}$ such that $f(A) = g(A^1, \ldots, A^M)$.

Proposition. (Morrey, Ball) A polyconvex function is quasi-convex.

A useful result:

Lemma.

1. If $\Omega \subset \mathbb{R}^k$ is smooth bounded and $u, v \in C^\infty(\overline{\Omega}; \mathbb{R}^k)$ are such that $u = v$ near $\partial\Omega$, then

$$\int_\Omega \det\,(\nabla u)(x)\, dx = \int_\Omega \det\,(\nabla v)(x)\, dx.$$

2. Let $U \subset \mathbb{R}^N$ be open bounded. If I, J are as above, $A \in M_{m,N}(\mathbb{R})$ and $\zeta \in C_c^\infty(U; \mathbb{R}^m)$, then

$$\int_U (A + D\varphi(x))_{I,J}\, dx = |U|\, A_{I,J}.$$

Useful references: [2, Section 4], [6, Section 4.1]

(c) Passing to the weak limits in nonlinear quantities

$\boxed{\text{A}}$ **Theorem** (Reshetnyak) If $u^j, u \in W^{1,N}(\Omega, \mathbb{R}^N)$ and $u^j \rightharpoonup u$ in $W^{1,N}$, then

$$\det(\nabla u^j) \to \det(\nabla u) \text{ in } \mathscr{D}'(\Omega).$$

Useful reference: [9]

$\boxed{\text{B}}$ **Definition.** (Ball) Let $u = (u_1, \dots, u_N) \in W^{1,N^2/(N+1)}(\Omega, \mathbb{R}^N)$. Then

$$\operatorname{Det}(\nabla u) := *d(u_1 du_2 \wedge \cdots \wedge du_N) \in \mathscr{D}'(\Omega).$$

Exercise. Using the Sobolev embeddings, prove that the above definition makes sense.

Exercise. If $u \in W^{1,N}(\Omega, \mathbb{R}^N)$, prove that $\operatorname{Det}(\nabla u) = \det(\nabla u)$.

Equivalently, prove that, if $u \in W^{1,N}(\Omega, \mathbb{R}^N)$, then

$$\int_\Omega \det(\nabla u)\, \varphi = -\int_\Omega \det(\varphi, u_2, \dots, u_N)\, u_1, \forall\, \varphi \in C_c^\infty(\Omega, \mathbb{R}).$$

$\boxed{\text{C}}$ **Theorem.** (Reshetnyak, Ball, Brezis-Nguyen) Let $N^2/(N+1) < p \leq N$. Let $u^j, u \in W^{1,p}(\Omega, \mathbb{R}^N)$ be such that $u^j \rightharpoonup u$ in $W^{1,p}$. Then

$$\operatorname{Det}(\nabla u^j) \to \operatorname{Det}(\nabla u) \text{ in } \mathscr{D}'(\Omega).$$

Useful result:

Lemma. Let $p \geq N - 1$ and let $q \geq 1$ satisfy $(N-1)/p + 1/q = 1$. If $u, v \in C^\infty(\overline{\Omega}, \mathbb{R}^N)$, then

$$\left| \int_\Omega [\det(\nabla v) - \det(\nabla u)]\, \varphi \right| \leq C_{N,\Omega} \|v - u\|_q \left(\|\nabla u\|_p + \|\nabla v\|_p\right)^{N-1} \|\nabla \varphi\|_\infty,$$
$$\forall\, \varphi \in C_c^\infty(\Omega, \mathbb{R}).$$

Useful reference: [5, Theorem 1]

Exercise. When $N = 2$, establish the above theorem by proving the following stronger statement: if $p > 4/3$ and $u^j = (u_1^j, u_2^j) \rightharpoonup u = (u_1, u_2)$ in $W^{1,p}(\Omega, \mathbb{R}^2)$, then $u_1^j\, \nabla u_2^j \to u_1\, \nabla u_2$ in $\mathscr{D}'(\Omega)$.

$\boxed{\text{D}}$ For the record:

Theorem. (Edelsen, Ericksen, Ball) Let $f : \mathbb{R}^{Nm} \to \mathbb{R}$ be a continuous function such that, for some $1 \le p < \infty$,

$$[u^j \rightharpoonup u \text{ in } W^{1,p}(\Omega, \mathbb{R}^m)] \implies [f(Du^j) \to f(Du) \text{ in } \mathscr{D}'(\Omega)].$$

Then f is an affine function of the minors of Du.

Similarly when $p = \infty$, for the $\overset{*}{\rightharpoonup}$ convergence.

Useful reference: [6, Theorem 1.5 in Section 4.1.2, and Section 4.2.2].

$\boxed{\text{E}}$ **Gap (or Lavrentiev) phenomen**

Theorem. (Maniá) Let

$$F(x) := \int_0^1 (x^3(t) - t)^2 \, x'^6(t) \, dt, \ \forall \, x \in W^{1,1}((0,1)) \text{ with } x(0) = 0 \text{ and } x(1) = 1.$$

Then we have the following *Lavrentiev phenomenon*

$$\inf\{F(x);\ x \in C^1([0,1])\} > \inf\{F(x);\ x \in W^{1,1}((0,1))\}.$$

Useful reference: [7]

References

[1] Emilio Acerbi and Nicola Fusco. Semicontinuity problems in the calculus of variations. *Arch. Rational Mech. Anal.*, 86(2):125–145, 1984.

[2] John M. Ball. Convexity conditions and existence theorems in nonlinear elasticity. *Arch. Rational Mech. Anal.*, 63(4):337–403, 1976/1977.

[3] Vladimir I. Bogachev. *Measure theory. Vol. I.* Springer-Verlag, Berlin, 2007.

[4] Haim Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[5] Haim Brezis and Hoai-Minh Nguyen. The Jacobian determinant revisited. *Invent. Math.*, 185(1):17–54, 2011.

[6] Bernard Dacorogna. *Direct methods in the calculus of variations*, volume 78 of *Applied Mathematical Sciences*. Springer-Verlag, Berlin, 1989.

[7] Philip D. Loewen. On the Lavrentiev phenomenon. *Canad. Math. Bull.*, 30(1):102–108, 1987.

[8] Charles B. Morrey, Jr. Quasi-convexity and the lower semicontinuity of multiple integrals. *Pacific J. Math.*, 2:25–53, 1952.

[9] Yuriĭ G. Reshetnyak. Mappings with bounded distortion as extremals of integrals of Dirichlet type. *Sibirsk. Mat. Ž.*, 9:652–666, 1968.

[10] Elias M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.