Lecture # 4
EXISTENCE METHODS

(a) Concentration-compactness

Useful general reference: [6, Section I.4]

A **Exercise.** Let $F_m : [0, \infty) \to [0, 1]$, $m \geq 0$, be *non decreasing* functions. Prove that, up to a subsequence, F_m converges simply a.e.

First concentration-compactness lemma (Lions) Let (μ_m) be a sequence of Borel probability measures on $\mathbb{R}^N$. Then, up to a subsequence, one of the following holds:

(a) (Compactness) There exists a sequence $(x_m) \subset \mathbb{R}^N$ such that, for every $\varepsilon > 0$, there exists some $R = R(\varepsilon)$ satisfying $\mu_m(B_R(x_m)) > 1 - \varepsilon, \forall\, m$.

(b) (Vanishing) For every $R > 0$,

$$\sup_{x \in \mathbb{R}^N} \mu_m(B_R(x)) \to 0 \text{ as } m \to \infty.$$

(c) (Dichotomy) There exists some $0 < \lambda < 1$ and sequences $(x_m) \subset \mathbb{R}^N$, $R_j \to \infty$ such that

$$\sup_m |\mu_m(B_{R_j}(x_m)) - \lambda| \to 0,$$
$$\sup_m |\mu_m(\mathbb{R}^N \setminus \overline{B}_{2R_j}(x_m)) - (1 - \lambda)| \to 0 \text{ as } j \to \infty.$$

B **Brezis-Lieb lemma** Let $(X, \mathscr{T}, \mu)$ be a measured space and $0 < p < \infty$. Let $f_j, f : X \to \mathbb{C}$ be measurable functions such that:

(i) $f_j \to f$ a.e.

(ii) For some finite C, $\displaystyle\int_X |f_j|^p \leq C, \forall\, j.$

Then

$$\int_X ||f_j|^p - |f|^p - |f_j - f|^p| \to 0,$$

In particular, if $p \geq 1$, $X = \mathbb{R}^N$ with the Lebesgue measure, and we set

$$\mu_j := (|f_j|^p - |f|^p - |f_j - f|^p)\, dx,$$

then $\mu_j \overset{*}{\rightharpoonup} 0$ in the sense of measures.

Useful references: [1, Theorem 1.9], [5, Exercice de synthèse #10]

C **Theorem** (Lions) Let $a = a(x) \in C(\mathbb{R}^N, (0,\infty))$ be such that

$$\lim_{|x|\to\infty} a(x) = a_\infty > 0.$$

Let $1 < p < \dfrac{N+2}{N-2}$ and set

$$I := \inf\left\{\int_{\mathbb{R}^N} (|\nabla u|^2 + au^2);\ u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{p+1} = 1\right\},$$
$$I_\infty := \inf\left\{\int_{\mathbb{R}^N} (|\nabla u|^2 + a_\infty u^2);\ u \in H^1(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{p+1} = 1\right\}.$$

If $I < I_\infty$, then the inf in I is attained. Up to a multiplicative constant, a minimizer is a non trivial solution $u \in H^1(\mathbb{R}^N)$ of

$$-\Delta u + au = |u|^{p-1}u \text{ in } \mathbb{R}^N.$$

D **Exercise.** Let μ be a finite *diffuse* Borel measure in $\mathbb{R}^N$. Prove that

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^N} \mu(B_r(x)) = 0.$$

Exercise. Let ω, λ be finite Borel measures in $\mathbb{R}^N$ and $1 \le p < q < \infty$. Assume that, for some $0 < S < \infty$, we have

$$S\left(\int_{\mathbb{R}^N} |f|^q\, d\omega\right)^{p/q} \le \int_{\mathbb{R}^N} |f|^p\, d\lambda,\ \forall \text{ Borel function } f : \mathbb{R}^N \to \mathbb{R}. \tag{1}$$

Prove that:

(a) ω is a purely atomic measure, i.e., there exist $\alpha_j > 0$, $x_j \in \mathbb{R}^N$ such that $\omega = \sum_j \alpha_j \delta_{x_j}$.

(b) $\sum_j (\alpha_j)^{p/q} < \infty$.

(c) $\lambda \ge S \sum_j (\alpha_j)^{p/q} \delta_{x_j}$.

(d) (1) holds if and only if it holds for $f \in C_c^\infty(\mathbb{R}^N)$.

Hint. *Step 1.* Assume first that λ is diffuse. Using the previous exercise, prove that, for every cube $C \subset \mathbb{R}^N$, $\omega(C) = 0$, and thus $\omega = 0$.

Step 2. Apply Step 1 to ω_0 and λ_0, where ω_0, respectively λ_0, is the diffuse part of ω, respectively λ.

Exercise. Let $1 \le p < \infty$ and $k \ge 1$ be such that $kp < N$. Let $\dfrac{1}{q} := \dfrac{1}{p} - \dfrac{k}{N}$.

Set

$$\dot{W}^{k,p} := \{u \in \mathscr{D}'(\mathbb{R}^N);\ D^k u \in L^p, u \in L^q\}.$$

Prove that, if we endow $\dot{W}^{k,p}$ with the norm $u \mapsto \left\|D^k u\right\|_p$, then $C_c^\infty(\mathbb{R}^N)$ is dense in $\dot{W}^{k,p}$. In particular, prove that we have the Sobolev inequality

$$S\|u\|_q^p \le \left\|D^k u\right\|_p^p,\ \forall\, u \in \dot{W}^{k,p}, \tag{2}$$

for some (optimal Sobolev constant) $0 < S < \infty$.

Useful reference for $k = 1$: [4, Lemma 14]. Hint for $k \ge 2$: prove the following result:

Exercise. Let k, p, and q be as above. For $R > 0$, set $A_R := \{x \in \mathbb{R}^N;\ R \le |x| \le 2R\}$. If $v \in C^\infty(A_R)$, then for every $\varepsilon > 0$ there exists some finite $C(\varepsilon)$ (independent of R and v) such that

$$\sum_{\ell=0}^{k-1} R^{-(k-\ell)} \left\|D^\ell v\right\|_{L^p(A_R)} \le \varepsilon \left\|D^k v\right\|_{L^p(A_R)} + C(\varepsilon)\|v\|_{L^q(A_R)}.$$

Second concentration-compactness lemma (Lions) Let $1 < p < \infty$, k, q, and S be as above. Let $(u_m) \subset \dot{W}^{k,p}$ and $u \in \dot{W}^{k,p}$ be such that:

(i) $u_m \rightharpoonup u$ in $\dot{W}^{k,p}$ and $u_m \to u$ a.e.

(ii) $|u_m|^q\, dx \overset{*}{\rightharpoonup} |u|^q dx + \omega$ in the sense of measures, for some (non-negative) Borel measure ω.

(iii) $|D^k u_m|^p dx \overset{*}{\rightharpoonup} |D^k u|^p dx + \mu$ in the sense of measures, for some (non-negative) Borel measure μ.

Then:

(a) ω is a purely atomic measure: $\omega = \sum_j \alpha_j \delta_{x_j}$, with $\alpha_j > 0$, $x_j \in \mathbb{R}^N$.

(b) We have $\sum_j (\alpha_j)^{p/q} < \infty$.

(c) We have $\mu \ge S \sum_j (\alpha_j)^{p/q} \delta_{x_j}$.

E **Theorem** (Aubin, Talenti, Lions) Let $k \ge 1$ and $1 < p < \infty$ be such that $kp < N$. Then there exists some $u \in \dot{W}^{k,p} \setminus \{0\}$ such that equality holds in (2).

(b) Mountain pass solutions. A glimpse of other topological methods

Useful references: [2, Chapter 4], [7, Chapter 1]

A **Ekeland's Variational Principle** Let (M, δ) be a complete metric space, $\Phi : M \to \mathbb{R}$, and $\varepsilon > 0$. Assume that:

(i) Φ is l.s.c. and $m := \inf \Phi > -\infty$.

(ii) $u \in M$ is such that $\Phi(u) < m + \varepsilon^2$.

Then there exists some $v \in M$ such that:

(a) $\Phi(v) \leq \Phi(u)$.

(b) $\Phi(w) > \Phi(v) - \varepsilon\, \delta(w, v)$, $\forall\, w \in M \setminus \{v\}$.

(c) $\delta(v, u) < \varepsilon$.

In the special case where $(M, \| \, \|)$ is a Banach space and Φ is Gâteaux différentiable at the above v, we have

$$\left| \frac{\partial \Phi}{\partial y}(v) \right| \leq \varepsilon \|y\|, \ \forall\, y \in M.$$

B **Minimax principle** (Shi) Consider:

(i) A Banach space X and $J \in C^1(X, \mathbb{R})$.

(ii) A compact metric space (K, d), a compact subspace $K_0 \subset K$, and a continuous map $\zeta : K_0 \to X$.

Set

$$\begin{aligned}
&M := \{\gamma \in C(K, X);\ \gamma_{|K_0} = \zeta\}, \\
&\Phi(\gamma) := \max_K J \circ \gamma, \ \forall\, \gamma \in M, \\
&c := \inf_M \Phi, \\
&c_0 := \max_{K_0} J \circ \zeta.
\end{aligned}$$

If $c > c_0$, then there exists a sequence $(x_j) \subset X$ such that:

(a) $J(x_j) \to c$.

(b) $J'(x_j) \to 0$ in X'.

Corollary. Assume that, with c as above, J satisfies the *Palais-Smale condition* $(\mathrm{PS})_c$ *at level* c: any sequence $(x_j) \subset X$ satisfying (a) and (b) contains a convergent subsequence. Then J has a critical point x such that $J(x) = c$.

C **Exercise.** Prove the *Ambrosetti-Rabinowitz Mountain pass theorem*. Let X be a Banach space and $J \in C^1(X, \mathbb{R})$. Assume that there exist $R > 0$ and $x_0 \in X$ such that:

(i) $\max\{J(0),\ J(x_0)\} < \inf\{J(x);\ \|x\| = R\}$.

(ii) $\|x_0\| > R$.

Set

$$c := \inf \left\{ \max_{t\in[0,1]} J(\gamma(t));\ \gamma \in C([0,1],X),\ \gamma(0)=0,\ \gamma(1)=x_0, \right\}.$$

If J satisfies the $(\mathrm{PS})_c$ condition, then J has a critical point x such that $J(x) = c$.

Theorem. Let $\Omega \subset \mathbb{R}^N$ be a bounded $C^{1,1}$-domain and $1 < p < \dfrac{N+2}{N-2}$.

Prove that the problem

$$\begin{cases} -\Delta u = \lambda u + u^p & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \\ u > 0 & \text{in } \Omega \end{cases}$$

has a classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ if and only if $\lambda < \lambda_1(\Omega)$.

Useful reference: [7, Theorem 1.19]

D **Exercise.** Prove the *Rabinowitz saddle point theorem*. Let X be a Banach space and $J \in C^1(X,\mathbb{R})$. Let $X = X^- \oplus X^+$, with X^- finite dimensional and X^+ closed.

For fixed $R > 0$, let

$$K := \{x \in X^-;\ \|x\| \le R\} \text{ and } K_0 := \{x \in X^-;\ \|x\| = R\}.$$

Assume that:

(i) $\max\limits_{K_0} J < \inf\limits_{X^+} J.$

(ii) J satisfies the $(\mathrm{PS})_c$ condition, where

$$c := \inf\{\max J \circ g;\ g \in C(K,X), g(x) = x \text{ if } x \in K_0\}.$$

Then J has a critical point x such that $J(x) = c$.

Useful references: [2, Theorem 4.7], [3]

Theorem. Let $N \ge 3$ and $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz domain. Let $a = a(x) \in L^{(2N)/(N+2)}(\Omega)$, $a \ge 0$, $a \not\equiv 0$. Prove that the problem

$$\begin{cases} -\Delta u = a\dfrac{u}{\sqrt{1+u^2}} & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \end{cases}$$

has a weak solution $u \in H^1(\Omega)$.

E **Rabinowitz linking theorem** Let X, X^-, X^+, and J be as above.

For fixed $R > \rho > 0$ and $z \in X^+ \setminus \{0\}$, let

$$K := \{u = x + tz;\ x \in X^-,\ t \geq 0,\ \|u\| \leq R\},$$
$$K_0 := \partial K \text{ (where } K \text{ is considered as a subset of } Y \oplus \mathbb{R}z),$$
$$L := \{x \in X^+;\ \|x\| = \rho\}.$$

Assume that:

(i) $\max_{K_0} J < \min_{L} J$.

(ii) J satisfies the $(\mathrm{PS})_c$ condition, where

$$c := \inf\{\max J \circ g;\ g \in C(K, X), g(x) = x \text{ if } x \in K_0\}.$$

Then J has a critical point x such that $J(x) = c$.

Useful reference: [7, Theorem 2.12]

Theorem. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $\lambda \in \mathbb{R}$, and $1 < p < \dfrac{N+2}{N-2}$. The equation

$$-\Delta u = \lambda u + |u|^{p-1} u \text{ in } \Omega$$

has a non trivial solution $u \in H_0^1(\Omega)$.

Useful reference: [7, Theorem 2.18, Corollary 2.19]

F For the record:

Theorem (**Lusternik/Ljusternik**, **Schnirelman**, Rabinowitz) Let X be a Banach space and G a discrete subgroup of X spanning an N-dimensional subspace of X.

Let $J \in C^1(X, \mathbb{R})$ be such that:

(i) $J(x + g) = J(x), \forall\, x \in X, \forall\, g \in G$.

(ii) $J : X/G \to \mathbb{R}$ satisfies the $(\mathrm{PS})_c$ condition at any level $c \in \mathbb{R}$.

(iii) J is bounded from below.

Then J has at least $N + 1$ critical orbits, i.e., there exist $x_1, \ldots, x_{N+1} \in X$ such that:

(a) $J'(x_j) = 0, \forall\, j$.

(b) $x_j - x_k \notin G$ if $j \neq k$.

Useful reference: [2, Section 4.6].

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