Lecture # 5
A GLIMPSE OF THE VARIATIONAL CONVERGENCE

A If X is a metric space and $f_j, f_0 : X \to \overline{\mathbb{R}}$, then $f_j \xrightarrow{\Gamma} f_0$ ((f_j) *Γ-converges to* f_0) if and only if:

$$[(x_j) \subset X, x \in X, x_j \to x] \implies f(x) \le \liminf f_j(x_j),$$
$$\forall\, x \in X, \exists\, (x_j) \subset X \text{ such that } x_j \to x \text{ and } f_j(x_j) \to f(x).$$

B **Lemma.** Let X be a Banach space. Let $f_j : X \to (-\infty, \infty]$, $j \ge 0$, be convex, l. s. c. and such that

$$f_j(0) = 0,\ \forall\, j,$$
$$\lim_{\|x\|\to\infty} \inf_j \frac{f_j(x)}{\|x\|} = \infty,$$
$$[f_j(x_j) \le C < \infty,\ \forall\, j \ge 1] \implies [(x_j) \text{ is relatively compact}].$$

Then

$$f_j \xrightarrow{\Gamma} f \iff f_j^*(x^*) \to f^*(x^*),\ \forall\, x^* \in X^*.$$

C (Example of relaxation of convex functionals) Let $a : \mathbb{R} \to (0, \infty)$ be continuous and 1-periodic. Set

$$F_\varepsilon(u) := \frac{1}{2}\int_0^1 a(t/\varepsilon)(u'(t))^2\, dt,\ \forall\, u \in H_0^1((0,1)).$$

Then, in $L^2((0,1))$, $F_\varepsilon \xrightarrow{\Gamma} F_0$, where

$$F_0(u) := \frac{b}{2}\int_0^1 (u'(t))^2\, dt,\ \forall\, u \in H_0^1((0,1)),\ \text{and } b := \int_0^1 1/a(t)\, dt.$$

D (Example of relaxation of the van der Waals functional) Let $W \in C^2(\mathbb{R}; [0,\infty))$ be such that: (i) $W^{-1}(0) = \{0,1\}$; (ii) $W''(0) > 0$, $W''(1) > 0$; (iii) $\lim_{|s|\to\infty} W(s) = \infty$. Set

$$F_\varepsilon(u) := \varepsilon \int_0^1 (u'(t))^2\, dt + \frac{1}{\varepsilon}\int_0^1 W(u(t))\, dt,\ \forall\, u \in H^1((0,1)).$$

Then $F_\varepsilon \xrightarrow{\Gamma} F_0$ in $L^1((0,1))$, where

$$F_0(u) = \begin{cases} \beta \# J, & \text{if } u : (0,1) \to \{0,1\} \text{ and } u' = \sum_{a\in J} \pm\delta_a \\ \infty, & \text{otherwise} \end{cases},$$

and $\beta := 2\displaystyle\int_0^1 \sqrt{W(s)}\, ds$.