PARTIAL EXAMINATION
OCTOBER 27, 2023 – TWO HOURS
HINTS

Exercise # 1. Let:

(i) $N \geq 3$.

(ii) $\Omega \subset \mathbb{R}^N$ a bounded Lipschitz open set.

(iii) $a \in L^{(2N)/(N+2)}(\Omega)$.

(iv) $f \in C^1(\mathbb{R}, \mathbb{R})$ a Lipschitz function.

Set

$$F(u) := \int_\Omega a(x) f(u(x))\, dx, \ \forall\, u \in H^1(\Omega).$$

Prove that $F \in C^1(H^1(\Omega), \mathbb{R})$, and that

$$F'(u)\varphi = \int_\Omega a(x) f'(u(x))\varphi(x)\, dx, \ \forall\, u \in H^1(\Omega), \ \forall\, \varphi \in H^1(\Omega).$$

Hints. Let $p := (2N)/(N+2)$ and $q := (2N)/(N-2)$ and $X := L^q(\Omega)$. Let G be given by the same formula as F, but for $u \in X$. By the Sobolev embedding $i : H^1(\Omega) \to X$, we have $F = G \circ i$, and is suffices to prove that G is C^1 and its differential is given by the same formula as the one of F.

Fix some $u \in X$ and a sequence $(\varphi_j) \subset X \setminus \{0\}$ such that $\|\varphi_j\|_q \to 0$. Up to a subsequence, we have $\varphi_j \to 0$ a.e. Write $\varphi_j = t_j \psi_j$, with $\|\psi_j\|_q = 1$ and $t_j > 0$, $t_j \to 0$. Let η_j be such that

$$af(u + \varphi_j) - af(u) = af'(u + \eta_j)\varphi_j, \ |\eta_j| \leq |\varphi_j|.$$

By dominated convergence and the fact that p and q are conjugated, we have successively (up to a subsequence)

$$af'(u + \eta_j) - af'(u) \to 0 \text{ in } L^q(\Omega),$$
$$\frac{af(u + \varphi_j) - af(u) - af'(u)\varphi_j}{t_j} = [af'(u + \eta_j) - af'(u)]\psi_j \to 0 \text{ in } L^1(\Omega),$$

whence the differentiability of G at u.

In order to prove the continuity of G', it suffices to prove that $L^q(\Omega)u \mapsto af'(u) \in L^p(\Omega)$ is continuous. This follows by dominated convergence. □

Exercise # 2. Recall the Sobolev embedding $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$.

a) Preliminary question. Prove that, for each $\varepsilon > 0$, there exists some $C_\varepsilon < \infty$ such that

$$t \leq \varepsilon t^3 + C_\varepsilon, \ \forall\, t \geq 0.$$

b) Let $\Omega \subset \mathbb{R}^3$ be a bounded open set. Let $f \in L^{6/5}(\Omega)$ and $\lambda \in \mathbb{R}$. Prove that the equation

$$-\Delta u + u^5 = \lambda u + f \text{ in } \mathscr{D}'(\Omega)$$

has a distributional solution $u \in H^1_0(\Omega)$.

Hints. b) Set

$$J(u) := J_1(u) + J_2(u),\ \forall\, u \in H_0^1(\Omega),$$
$$J_1(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{6}\int_\Omega u^6 - \int_\Omega fu,\ J_2(u) := -\frac{\lambda}{2}\int_\Omega u^2.$$

The name of the game is to prove that J has a minimum point. We know that J_1 is convex and continuous, while J_2 is weakly continuous (by the compactness of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$). We find that J is weakly l.s.c. In order to conclude, it suffices to prove that J is coercive. With $\varepsilon|\lambda| = 1/3$, we find that

$$\begin{aligned} J(u) \geq & \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{|\lambda| C_\varepsilon}{2} - \|f\|_{6/5}\|u\|_6 \\ \geq & \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \frac{|\lambda| C_\varepsilon}{2} - K_1\|f\|_{6/5}\|u\|_{H_0^1(\Omega)}. \quad \square \end{aligned}$$

Exercise # 3. Let $\Omega \subset \mathbb{R}^N$ be a bounded *connected* open set, and let $\lambda_1(\Omega)$ be the best constant in Poincaré's inequality, i.e., the largest constant C such that

$$C\int_\Omega u^2 \leq \int_\Omega |\nabla u|^2,\ \forall\, u \in H_0^1(\Omega).$$

We take for granted the following result: if $u \in H_0^1(\Omega)$ is such that

$$\lambda_1(\Omega)\int_\Omega u^2 = \int_\Omega |\nabla u|^2,$$

then u is smooth and either $u \equiv 0$, or $u(x) \neq 0, \forall\, x \in \Omega$.

Let $a \in C(\overline{\Omega})$ be such that $a(x) \leq 1, \forall\, x \in \Omega$, and $a \not\equiv 1$. Prove that there exists some $\varepsilon > 0$ such that

$$(\lambda_1(\Omega) + \varepsilon)\int_\Omega au^2 \leq \int_\Omega |\nabla u|^2,\ \forall\, u \in H_0^1(\Omega).$$

Hints. Argue by contradiction and consider a sequence $(u_j) \subset H_0^1(\Omega)$ such that

$$1 = \int_\Omega |\nabla u_j|^2 \geq \lambda_1(\Omega)\int_\Omega (u_j)^2 \geq \lambda_1(\Omega)\int_\Omega a(u_j)^2 > (1 - 1/j)\int_\Omega |\nabla u_j|^2 = 1 - 1/j.$$

Up to a subsequence $u_j \rightharpoonup u$ in $H_0^1(\Omega)$ and $u_j \to u$ in $L^2(\Omega)$. Since $\lambda_1(\Omega)\int_\Omega u^2 = 1$, we find that $\int_\Omega |\nabla u|^2 \geq 1 = \lim_j \int_\Omega |\nabla u_j|^2$, and thus $u_j \to u$ in $H_0^1(\Omega)$. Finally,

$$1 = \int_\Omega |\nabla u|^2 = \lambda_1(\Omega)\int_\Omega u^2 = \int_\Omega au^2.$$

The second equality implies that $u(x) \neq 0, \forall\, x \in \Omega$. The last one yields the contradiction $a \equiv 1$. $\square$

Exercise # 4. Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let p be such that $\dfrac{2N}{N+1} < p < N$. Set $r := \dfrac{Np}{2N - p}$. Sketch a proof of the following results.

a) If $u, v \in W_0^{1,p}(\Omega)$, then $uv \in W_0^{1,r}(\Omega)$.

b) If $u_j \rightharpoonup u$ and $v_j \rightharpoonup v$ in $W_0^{1,p}(\Omega)$, then $u_j v_j \rightharpoonup uv$ in $W_0^{1,r}(\Omega)$.

Proof. a) Let $q := (Np)/(N-p)$, so that

$$1/r = 1/p + 1/q \text{ and } W_0^{1,p}(\Omega) \hookrightarrow L^q(\Omega). \tag{1}$$

Let $u, v \in W^{1,p}(\Omega)$. Consider $(u_j), (v_j) \subset C_c^\infty(\Omega)$ such that $u_j \to u$, $v_j \to v$ in $W_0^{1,p}(\Omega)$. Then

$$\nabla(u_j v_j) = u_j \nabla v_j + v_j \nabla u_j \to u\nabla v + v\nabla u \ \text{ in } L^r(\Omega) \text{ (using (1))}.$$

By the Sobolev embedding, we also have $u_j v_j \to uv \in L^{q/2}(\Omega)$, and thus (since $q/2 > r$), $u_j v_j \to uv \in L^r(\Omega)$. This yields $uv \in W_0^{1,r}(\Omega)$ and the Leibniz rule $\nabla(uv) = u\nabla v + v\nabla u$.

b) By the chain rule and (1), the sequence $(u_j v_j)$ is bounded in $W_0^{1,r}(\Omega)$, so it has, up to a subsequence, a weak limit which is also an a.e. limit. On the other hand, up to a subsequence $u_j v_j \to uv$ a.e. By uniqueness, $u_j v_j \rightharpoonup uv$. $\square$