Inequalities and other stories

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A muggle's approach to inequalities

To the memory of Mihai Onucu Drimbe (1952–2014)
A wonderful teacher, a passionated researcher, and a dear friend

A few years ago, my colleague Bodo Lass proposed me to give lectures at the Club de Mathématiques Discrètes, a twenty years old Lyon-based initiative gathering passionated high school students for week-end or week-long sessions of intensive mathematical training. (Bodo is Club's linchpin.) Though the Club sessions tend to look like training for Olympiads, Bodo gave me carte blanche, so that I chose a path that can be loosely defined as "a problem-based muggle's introduction to analysis through inequalities". This came somehow as a reaction to the excess of magic that tends to encompass such training (the Holy Hand Grenade of Antioch syndrome), conjugated with an excessive attention paid to problems that can be solved in little time.

Along the years, I have accumulated too much material to be discussed in one session, so I decided to gather it in a written text. Its purpose is twofold: to provide an essentially self-contained source (including the solutions to all the problems) to a junior reading group or to colleagues who want to teach classical analysis to advanced high school students, and to complement my lectures at the Club, allowing along the way motivated students to consolidate their knowledge and work independently on the problems presented.

The August 2025 version of this text is only about inequalities. I plan to add later extra chapters, dealing with topics as polynomials, systems, and some of the problems that struck me when I was a high school student myself.

A few legal disclaimers. This is by no means a monograph on inequalities, and I do not mention the most useful inequalities in advanced analysis, which often involve integrals of functions. Nevertheless, in selecting the questions proposed, their relevance and (necessarily subjective) aesthetic criteria played an important role. Speaking of aesthetics, I tried to avoid rhinestones and glitter. The text has almost no claim of originality, at least concerning most of the statements and proofs. Occasionally, the proofs use standard undergraduate level analysis, which is part of every bachelor level curriculum in mathematics, but I have tried to limit the use of such tools. I provided the references whenever I was aware of the original source, but I have certainly missed many of them.

An important source of inspiration for the text was the book (in Romanian) "Inequalities. Ideas and methods" of my former teacher Mihai Onucu Drimbe, a charismatic figure who passed on his passion for mathematics to me and to whom I dedicate this text.

Finally, a warm recommandation to the young readers: try to solve the problems first, not to read the solutions right away.

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Chapter 1

Basic methods and inequalities

Overview. In this chapter, we present some must-know inequalities and a few of the most significant basic methods used in proving inequalities.

1.1 Must-know inequalities

Definition 1.1. Let $x_1, x_2, \ldots, x_n > 0$. The numbers

$$A = A(x_1, x_2, \ldots, x_n) \coloneqq \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{j=1}^{n} x_j}{n}, \tag{1.1}$$

$$G = G(x_1, x_2, \ldots, x_n) \coloneqq \sqrt[n]{x_1 x_2 \cdots x_n} = x_1^{1/n} x_2^{1/n} \cdots x_n^{1/n} = \sqrt[n]{\prod_{j=1}^{n} x_j}, \tag{1.2}$$

$$H = H(x_1, \ldots, x_n) \coloneqq \frac{n}{\dfrac{1}{x_1} + \dfrac{1}{x_2} + \cdots + \dfrac{1}{x_n}}, \tag{1.3}$$

are respectively the *arithmetic mean* (AM), *geometric mean* (GM), and *harmonic mean* (HM) of $x_1, x_2, \ldots, x_n$.

We have the (AM-GM) and (GM-HM) inequalities

$$A \geq G, \tag{AM-GM}$$
$$G \geq H, \tag{GM-HM}$$

with equality iff all the x_j's are equal.

The following is obvious.

Problem 1.1. (GM-HM) for $x_1, x_2, \ldots, x_n$ is equivalent to (AM-GM) applied to $\dfrac{1}{x_1}, \dfrac{1}{x_2}, \ldots, \dfrac{1}{x_n}$.

Definition 1.2. Two numbers $1 < p, q < \infty$ are *conjugate* (or *conjugate exponents*) if they satisfy one of the following equivalent conditions

$$\frac{1}{p} + \frac{1}{q} = 1, \text{ or } q = \frac{p}{p-1}, \text{ or } p = \frac{q}{q-1}, \text{ or } pq = p + q. \tag{1.4}$$

By extension, 1 and ∞ are also conjugate exponents.

When $1 < p, q < \infty$, we have the Hölder inequality (H)

$$\left|\sum_{j=1}^{n} a_j b_j\right| \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} \left(\sum_{j=1}^{n} |b_j|^q\right)^{1/q}, \ \forall\, a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}. \tag{H}$$

When $p = 2$ (and thus $q = 2$), the Hölder inequality takes the form of the Cauchy-Schwarz inequality (CS)

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \leq \sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2, \ \forall\, a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}. \tag{CS}$$

When $1 < p < \infty$, we have the Minkowski inequality (M)

$$\left(\sum_{j=1}^{n} |a_j + b_j|^p\right)^{1/p} \leq \left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |b_j|^p\right)^{1/p}, \quad \forall\, a_1, \dots, a_n, b_1, \dots, b_n \in \mathbb{R}. \tag{M}$$

The following is easily obtained by taking square on both sides of (M) with $p = 2$.

Problem 1.2. The Minkowki inequality (M) with $p = 2$ is equivalent to the (CS) inequality.

The next two must-know inequalities are *rearrangement inequalities*, involving *ordered lists* of numbers, $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$, and *permutations* $\sigma \in S_n$. We have the rearrangement inequality (R)

$$\sum_{j=1}^{n} a_j b_{n-j+1} \leq \sum_{j=1}^{n} a_j b_{\sigma(j)} \leq \sum_{j=1}^{n} a_j b_j, \ \forall\, \sigma \in S_n, \tag{R}$$

and the Chebyshev inequality (C)

$$\sum_{j=1}^{n} a_j b_j \geq \frac{1}{n} \sum_{j=1}^{n} a_j \sum_{j=1}^{n} b_j. \tag{C}$$

The above inequalities are reversed if the lists are ordered $a_1 \le a_2 \le \cdots \le a_n$, $b_1 \ge b_2 \ge \cdots \ge b_n$.

A common feature of these inequalities is that they are *homogenous*: for example, proving (AM-GM) for $x_1, x_2, \ldots, x_n$ is equivalent to proving the same inequality for $tx_1, tx_2, \ldots, tx_n$ for *some* $t > 0$.

Problem 1.3. Use the homogeneity of the inequalities to reduce them to the following special cases:

(1) (AM-GM) when $x_1 x_2 \cdots x_n = 1$ *or* when $x_1 + x_2 + \cdots + x_n = n$.

(2) (H) when $\sum_{j=1}^n |a_j|^p = 1$ *and* $\sum_{j=1}^n |b_j|^q = 1$.

(3) (CS) when $\sum_{j=1}^n b_j^2 = 1$.

1.2 Induction

Problem 1.4. Prove (AM-GM) by *Cauchy induction*, that is, using the following scheme:

(1) Prove (AM-GM) for $n = 2, 4, \ldots, 2^k, \ldots$

(2) Given n and some integer k such that $n < 2^k$, prove (AM-GM) for n starting from (AM-GM) for 2^k.

(3) On the way, prove that equality holds in (AM-GM) iff $x_1 = \cdots = x_n$.

Problem 1.5. Prove (AM-GM) by induction.

Problem 1.6. Prove (CS) by induction.

1.3 Convexity

Convexity provides a very fruitful insight to inequalities. Many of the must-know inequalities, if interpreted correctly, are occurrences of the Jensen inequality that we recall below.

Definition 1.3. If $I \subset \mathbb{R}$ is an interval, a function $f : I \to \mathbb{R}$ is *convex* iff it satisfies the Jensen inequality (J)

$$f((1-t)x + ty) \le (1-t)f(x) + tf(y),\ \forall\, x, y \in I,\ \forall\, 0 \le t \le 1. \tag{J}$$

A convex function is *strictly convex* if equality in (J) for some $0 < t < 1$ implies that $x = y$.

A function is *concave* if it satisfies the reverse inequality. Similarly for strictly concave. In general, a function is neither convex, nor concave.

A straightforward induction leads to the following generalization of (J).

Problem 1.7. (1) If $f : I \to \mathbb{R}$ is convex, then it satisfies the general(ized) Jensen inequality (GJ)

$$f\left(\sum_{j=1}^{n} \lambda_j x_j\right) \le \sum_{j=1}^{n} \lambda_j f(x_j), \ \forall\, n \ge 2, \ \forall\, x_1, \ldots, x_n \in I, \\ \forall\, \lambda_1, \ldots, \lambda_n \in [0,1] \text{ such that } \sum_{j=1}^{n} \lambda_j = 1. \tag{GJ}$$

(2) If f is strictly convex and equality occurs in (GJ) for some $0 < \lambda_1, \ldots, \lambda_n < 1$, then $x_1 = x_2 = \cdots = x_n$.

The inequality is reversed if f is concave.

Definition 1.4. A *convex combination* of $x_1, \ldots, x_n$ is a sum of the form $\sum_{j=1}^{n} \lambda_j x_j$, with $\lambda_1, \ldots, \lambda_n \in [0,1]$ such that $\sum_{j=1}^{n} \lambda_j = 1$.

In order to effectively use (GJ), one needs examples of convex and concave functions. They can be found in most of calculus books. A good executive summary can be found on Wikipedia [12]. For us to start, the following examples are sufficient.

a) $\mathbb{R} \ni x \mapsto e^x$ is strictly convex.

b) If $1 < p < \infty$, $\mathbb{R} \ni x \mapsto |x|^p$ is strictly convex. In particular, if $1 < p < \infty$ $[0,\infty) \ni x \mapsto x^p$ is strictly convex.

c) $(0,\infty) \ni x \mapsto \ln x$ is strictly concave. So is $(-1,\infty) \ni x \mapsto \ln(1+x)$.

d) If n is an integer, $(-\infty, 0] \ni x \mapsto x^{2n+1}$ is strictly concave.

e) Affine functions $\mathbb{R} \ni x \mapsto ax + b$ are both convex and concave. In particular, linear functions $\mathbb{R} \ni x \mapsto ax$ are both convex and concave.

f) The functions $\mathbb{R} \ni x \mapsto (x-a)^+$ and $\mathbb{R} \ni x \mapsto (x-a)^-$ are convex. Recall that $x^+ := \begin{cases} x, & \text{if } x \ge 0 \\ 0, & \text{if } x \le 0 \end{cases}$ and $x^- := \begin{cases} 0, & \text{if } x \ge 0 \\ -x, & \text{if } x \le 0 \end{cases}$.

g) If $f : I \to \mathbb{R}$ satisfies $f'' \ge 0$, respectively $f'' > 0$ possibly except at the endpoints, then f is convex, respectively strictly convex.

Problem 1.8. Prove (AM-GM), with the equality case, using the strict convexity of $\mathbb{R} \ni x \mapsto e^x$.

Problem 1.9. (1) Prove (H) using the strict convexity of $\mathbb{R} \ni x \mapsto |x|^p$ (with $1 < p < \infty$). Hint: take, in (GJ), $\lambda_j \coloneqq C|b_j|^q$, for some appropriate constant C.

(2) Prove that equality occurs in (H) iff

$$\text{the vectors } (a_j)_{1\le j\le n} \text{ and } (\operatorname{sgn} b_j |b_j|^{q-1})_{1\le j\le n} \text{ are proportional.} \tag{1.5}$$

A few comments about (1.5).

a) By symmetry of (H) in p and q, equality occurs in (H) iff

$$\text{the vectors } (b_j)_{1\le j\le n} \text{ and } (\operatorname{sgn} a_j |a_j|^{p-1})_{1\le j\le n} \text{ are proportional.} \tag{1.6}$$

b) In the special case where $p = q = 2$, (1.5) reads as follows: equality occurs in (CS) iff the vectors $(a_j)_{1\le j\le n}$ and $(b_j)_{1\le j\le n}$ are proportional.

The perspective of convexity allows to see the (AM-GM) and (GM-HM) inequalities as special cases of an infinite family of inequalities. First, a definition.

Definition 1.5. Let $x_1, x_2 \dots, x_n > 0$. For $r \in \mathbb{R}\setminus\{0\}$, we define the *generalized mean*

$$M_r = M_r(x_1, x_2, \dots, x_n) \coloneqq \left(\frac{\sum_{j=1}^n x_j^r}{n} \right)^{1/r}. \tag{1.7}$$

Set also

$$M_0 = M_0(x_1, x_2, \dots, a_n) \coloneqq x_1^{1/n} \dots x_n^{1/n}. \tag{1.8}$$

This scale of means includes AM, GM, and HM:

$$A = M_1,\ G = M_0,\ H = M_{-1}.$$

Another important mean is the *quadratic mean* $M_2 = \sqrt{\dfrac{\sum_{j=1}^n x_j^2}{n}}$.

These inequalities are related by the following noticeable means inequality:

$$\text{if } r_1 < r_2, \text{ then } M_{r_1} \le M_{r_2}, \tag{MI}$$

with equality iff $x_1 = x_2 = \dots = x_n$.

Problem 1.10. Prove (MI) via the following strategy.

(1) Reduce the study of (MI) to the case where $0 \le r_1 < r_2$.

(2) When $r_1 = 0$ and $r_2 > 0$, reduce (MI) to the (AM-GM) inequality.

(3) If $0 < r_1 < r_2$, reduce (MI) to (GJ) applied to the function $(0, \infty) \ni x \mapsto x^r$, where $r \coloneqq r_2/r_1$.

Remark 1.1. One may consider, more generally, numbers $0 \le \lambda_1, \ldots, \lambda_n \le 1$ such that $\sum_{j=1}^n \lambda_j = 1$ and the more general means

$$M_r \coloneqq \left(\sum_{j=1}^n \lambda_j x_j^r \right)^{1/r}, \text{ if } r \neq 0,$$
$$M_0 \coloneqq \prod_{j=1}^n x_j^{\lambda_j}.$$

((1.7) and (1.8) correspond to the choice $\lambda_j = 1/n$, $\forall\, j$.) Then (MI) still holds. This is proved by repeating the argument in Problem 1.10.

The following general means inequality (corresponding to $r_2 = 1$ and $r_1 = 0$) is frequently used.

$$\sum_{j=1}^n \lambda_j x_j \ge \prod_{j=1}^n x_j^{\lambda_j},\ \forall\, x_j > 0,\ \lambda_j \ge 0 \text{ s.t. } \sum_{j=1}^n \lambda_j = 1. \tag{GMI}$$

It is less obvious to recognize (M) as an occurrence of (J). The starting point is that, when $1 < p < \infty$, the function

$$[0, \infty) \ni x \mapsto f(x) \coloneqq (1 + x^p)^{1/p} \tag{1.9}$$

is strictly convex. This fact can be proved in calculus, and we take it for granted.

Problem 1.11. (1) Prove by induction that, if (M) holds for $n = 2$, then it holds for every n.

(2) When $n = 2$, use homogeneity to reduce (M) to the special case where $|a_1| + |b_1| = 1$.

(3) In the above special case, use (J) for the function f defined in (1.9) to derive (M).

1.4 Majorization

The Jensen inequalities (J) and (GJ) lead to more general inequalities, involving more points. To give a flavor of this, let us prove that, for $f : [0, 3] \to \mathbb{R}$, f convex, we have

$$f(1) + f(2) \leq f(0) + f(3). \tag{1.10}$$

Indeed, we have

$$1 = \frac{2}{3} \times 0 + \frac{1}{3} \times 3 \text{ and } 2 = \frac{1}{3} \times 0 + \frac{2}{3} \times 3. \tag{1.11}$$

Applying twice (J), we find that

$$\begin{aligned} f(1) + f(2) =& f\left(\frac{2}{3} \times 0 + \frac{1}{3} \times 3\right) + f\left(\frac{1}{3} \times 0 + \frac{2}{3} \times 3\right) \\ \leq& \frac{2}{3}f(0) + \frac{1}{3}f(3) + \frac{1}{3}f(0) + \frac{2}{3}f(3) = f(0) + f(3), \end{aligned}$$

whence (1.10).

Here is an easy generalization of the above. (Deeper results, soon after.)

Problem 1.12. Let $f : I = [\alpha, \beta] \to \mathbb{R}$ be convex.

(1) Assume that $a, b, c, d \in I$, $a \leq c \leq b$, and $a + b = c + d$. Prove that $f(c) + f(d) \leq f(a) + f(b)$. If in addition, f is strictly convex, characterize the equality cases.

(2) Assume that f is strictly convex. Given $S \in \mathbb{R}$:

  (a) Prove that the couple (x, y) achieves

$$\max\{f(x) + f(y);\ x, y \in I,\ x + y = S\}$$

  iff either x or y is an endpoint of I.

  (b) Prove that the couple (x, y) achieves the min in the above iff $x = y = S/2$.

In this section, we characterize the inequalities of the type (1.10) that are valid *for all convex functions*. Some intuition is provided by (1.11), which can be rewritten, using vectors and a matrix, as

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{pmatrix} \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

The properties of the above matrix are captured in the following definition.

Definition 1.6 (Doubly stochastic matrix)**.** A matrix $A = (a_{jk})_{1\leq j,k\leq n}$ is *doubly stochastic* (*DS*) iff

$$a_{jk} \geq 0,\ \forall\, j,k, \tag{1.12}$$

$$\sum_{k=1}^{n} a_{jk} = 1,\ \forall\, j, \tag{1.13}$$

$$\sum_{j=1}^{n} a_{jk} = 1,\ \forall\, k. \tag{1.14}$$

The next definition, less intuitive, will be crucial in the first main result of this section, Theorem 1.1 below.

Definition 1.7 (Majorization)**.** Let $x_1,\ldots,x_n,y_1,\ldots,y_n \in \mathbb{R}$. By definition, $(x_1,\ldots,x_n)$ *majorizes* $(y_1,\ldots,y_n)$ iff

$$x_1 \leq x_2 \leq \cdots \leq x_n,\ y_1 \leq y_2 \leq \cdots \leq y_n, \tag{1.15}$$

$$x_1 \leq y_1, \tag{1.16}$$

$$x_1 + x_2 \leq y_1 + y_2, \tag{1.17}$$

$$\vdots$$

$$x_1 + x_2 + \cdots + x_{n-1} \leq y_1 + y_2 + \cdots + y_{n-1}, \tag{1.18}$$

$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n. \tag{1.19}$$

Theorem 1.1. *Let I be an interval and let $x_1,\ldots,x_n,y_1,\ldots,y_n$ in I satisfy (1.15). Then the following are equivalent:*

(1) For every convex function $f : I \to \mathbb{R}$, we have

$$\sum_{j=1}^{n} f(y_j) \leq \sum_{j=1}^{n} f(x_j). \tag{1.20}$$

(2) $(x_1,\ldots,x_n)$ majorizes $(y_1,\ldots,y_n)$.

(3) There exists a doubly stochastic matrix $A = (a_{jk})_{1\leq j,k\leq n}$ such that

$$y_j = \sum_{k=1}^{n} a_{jk}x_k,\ \forall\, j. \tag{1.21}$$

Moreover, the condition (1.15) can be relaxed to

$$y_1 \leq y_2 \leq \cdots \leq y_n. \tag{1.22}$$

Remark 1.2. In the implication "(3) $\Rightarrow$ (1)", the assumptions $x_1 \leq x_2 \leq \cdots \leq x_n$ and $y_1 \leq y_2 \leq \cdots \leq y_n$ are not used. Therefore, the valid inequalities of the form (1.20) are characterized by doubly stochastic matrices, which are magic squares with sum 1. It is thus interesting to know how to build magic squares. There actually exists a "recipe" to cook *all* magic squares. See Theorem 3.1.

Remark 1.3. The implication "(2) $\Rightarrow$ (1)" is known as the Karamata theorem [9].

In the proof of Theorem 1.1, two of the implications are easy.

Problem 1.13. Prove the implications "(1) $\Rightarrow$ (2)" and "(3) $\Rightarrow$ (1)". Hint for the first implication: use linear functions and functions of the form $x \mapsto (x-a)^-$.

Problem 1.14. Prove that, indeed, in Theorem 1.1, the assumption (1.15) can be replaced with the weaker assumption (1.22).

The remaining implication required to complete the proof of the theorem, "(2) $\Rightarrow$ (3)", is more difficult to obtain. We present below a rather natural proof.

Proof of "(2) $\Rightarrow$ (3)". The proof is by induction on $n \geq 2$, the case $n = 2$ being essentially already settled in Problem 1.12.

Step 0. Proof when $n = 2$. Since

$$2y_1 \leq y_1 + y_2 = x_1 + x_2 \leq 2x_2,$$

we have $y_1 \leq x_2$. By assumption, we also have $x_1 \leq y_1$, and thus $x_1 \leq y_1 \leq x_2$. Therefore, there exists some $\lambda \in [0, 1]$ such that $y_1 = (1-\lambda)x_1 + \lambda x_2$. Since

$$y_2 = x_1 + x_2 - y_1 = x_1 + x_2 - [(1-\lambda)x_1 + \lambda x_2] = \lambda x_1 + (1-\lambda)x_2,$$

we find that (1.21) holds for the DS matrix $A := \begin{pmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{pmatrix}$.

Next, assume that $n \geq 3$ and that "(2) $\Rightarrow$ (3)" holds for $n-1$.

We prove in four steps that "(2) $\Rightarrow$ (3)" holds for n. Step 1 is a preliminary reduction to the special case $y_n = 0$. We appeal to the induction assumption in Step 2; to be able to do so, we rely on the assumption $y_n = 0$. The key step is Step 3: it consists of constructing the first $(n-1)$ lines of the DS matrix that we want to find; this is the heart of the proof, and the assumption $y_n = 0$ will again be used. In Step 4, we complete the DS matrix.

Step 1. We may assume that $y_n = 0$. Indeed, if $x_1, \ldots, x_n, y_1, \ldots, y_n$ satisfy (1.15)–(1.19), then so do $x_1 - c, \ldots, x_n - c, y_1 - c, \ldots, y_n - c$, $\forall\, c \in \mathbb{R}$, and it clearly suffices to prove item (3) for these new numbers. Choosing $c := y_n$, we may thus assume that $y_n = 0$. Subtracting (1.18) from (1.19), we find that $x_n \geq y_n = 0$.

Step 2. Use of the induction assumption. The systems of numbers $x_1, \ldots, x_{n-2}, x_{n-1} + x_n$, respectively $y_1, \ldots, y_{n-2}, y_{n-1}$, satisfy the assumptions (1.15)–(1.19) (with n replaced with

$n-1$). Indeed, the condition $x_{n-2} \le x_{n-1} + x_n$ holds since $x_{n-2} \le x_{n-1}$ and $x_n \ge 0$, and all the other conditions are clearly satisfied. By the induction assumption, there exists a DS matrix $B = (b_{jk})_{1\le j,k\le n-1}$ such that

$$y_j = \sum_{k=1}^{n-1} b_{jk}x_k + b_{j(n-1)}x_n, \; \forall\, 1 \le j \le n-1. \tag{1.23}$$

In a matrix form, (1.23) reads

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1(n-1)} & b_{1(n-1)} \\ b_{21} & b_{22} & \dots & b_{2(n-1)} & b_{2(n-1)} \\ \vdots & \vdots & & \vdots & \vdots \\ b_{(n-1)1} & b_{(n-1)2} & \dots & b_{(n-1)(n-1)} & b_{(n-1)(n-1)} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}. \tag{1.24}$$

Let us denote by $C = (c_{jk})_{1\le j\le n-1, 1\le k\le n}$ the matrix in (1.24). Unfortunately, C does not look like the first part of a DS matrix, since the sum of the entries on the line j is $1 + b_{j(n-1)}$, and in general this sum is > 1. Now comes the key argument: it is possible to replace, in (1.24), C with a matrix with *smaller* coefficients, such that the sum of the entries on each line is exactly 1. This is the done in the next step.

Step 3. There exist numbers a_{jk}, $1 \le j \le n-1$, $1 \le k \le n$, such that

$$0 \le a_{jk} \le c_{jk}, \; \forall\, j, \; \forall\, k, , \tag{1.25}$$

$$\sum_{k=1}^{n} a_{jk} = 1, \; \forall\, j, \tag{1.26}$$

$$y_j = \sum_{k=1}^{n} a_{jk}x_k, \; \forall\, j. \tag{1.27}$$

Here is the heuristics. There are two obvious ways to replace the entries c_{jk} with *smaller* entries with the sum on each line equal to 1. A first one consists of replacing the last entry on each line, $b_{j(n-1)}$, with 0. The second one consists of dividing the entries of the line j with $1 + b_{j(n-1)}$, which is the sum of the entries of that line. While both procedures lead to entries satisfying (1.25) and (1.26), there is no reason that any of them will lead to entries satisfying (1.27). However, *an appropriate combination of them will do it*.

We now proceed to the rigorous argument. Set, for $1 \le j \le n-1$ and $1 \le k \le n$,

$$d_{jk} := \begin{cases} b_{jk}, & \text{if } 1 \le k \le n-1 \\ 0, & \text{if } k = n \end{cases}, \tag{1.28}$$

$$e_{jk} := \frac{c_{jk}}{1 + b_{j(n-1)}}, \tag{1.29}$$

$$s_j := \sum_{k=1}^{n} d_{jk}x_k = y_j - b_{j(n-1)}x_n, \tag{1.30}$$

$$t_j := \sum_{k=1}^{n} e_{jk}x_k = \frac{1}{1 + b_{j(n-1)}} y_j. \tag{1.31}$$

Using (1.28) and (1.29), we see that the following hold for $1 \le j \le n-1$ and $1 \le k \le n$:

$$0 \le d_{jk}, e_{jk} \le c_{jk}, \ \forall\, j, \ \forall\, k, \tag{1.32}$$

$$\sum_{k=1}^{n} c_{jk} = \sum_{k=1}^{n} d_{jk} = 1, \ \forall\, j. \tag{1.33}$$

Now comes the key argument, valid thanks to our special choice $y_n = 0$. Since $y_j \le 0$ (recall that $y_j \le y_n = 0$) and $x_n \ge 0$, (1.30) and (1.31) imply that

$$s_j \le y_j \le t_j, \ \forall\, j. \tag{1.34}$$

Therefore, there exists some $\lambda_j \in [0, 1]$ satisfying $y_j = (1 - \lambda_j)s_j + \lambda_j t_j$. We then set

$$a_{jk} \coloneqq (1 - \lambda_j)d_{jk} + \lambda_j e_{jk}, \ \forall\, j, k. \tag{1.35}$$

Using (1.32)–(1.35), it is straightforward to see that (1.25)–(1.27) hold.

Step 4. Completion of the matrix $(a_{jk})_{1\le j\le n-1, 1\le k\le n}$ to a DS matrix. Set $a_{nk} \coloneqq 1 - \sum_{j=1}^{n-1} a_{jk}, \forall\, k$. In view of (1.25), of the definition of a_{nk}, and of the fact that $\sum_{j=1}^{n-1} b_{jk} = 1$ (since B is doubly stochastic), we have $a_{nk} \ge 0, \ \forall\, k$. Using (1.25)–(1.27) and (1.19), we find that $A \coloneqq (a_{jk})_{1\le j,k\le n}$ is doubly stochastic and that (1.21) holds. QED

Now comes the final bouquet of this section, and one of the most useful results in this chapter. We will find necessary and sufficient conditions for the validity of the following natural generalization of (1.20): given an interval I, points $x_1, \ldots, x_n, y_1, \ldots, y_m \in I$ and constants $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m > 0$, we want the inequality

$$\sum_{j=1}^{m} \beta_j f(y_j) \le \sum_{k=1}^{n} \alpha_k f(x_k) \tag{1.36}$$

to hold for each convex function $f : I \to \mathbb{R}$.

Without loss of generality, we may assume that

$$x_1 \le x_2 \le \cdots \le x_n, y_1 \le y_2 \le \cdots \le y_m. \tag{1.37}$$

Les us first note that, by taking $f \equiv 1$ and $f \equiv -1$, we find the necessary condition for the validity of (1.36):

$$S \coloneqq \sum_{k=1}^{n} \alpha_k = \sum_{j=1}^{m} \beta_j. \tag{1.38}$$

We consider, on $[0, S]$, two auxiliary functions that, as we will see below, "encode" the analogues of (1.16)–(1.19).

We associate, with $x_1, \dots, x_n, \alpha_1, \dots, \alpha_n$, the (only) function $g : [0, S] \to \mathbb{R}_+$ with the following properties: (i) $g(0) = 0$; (ii) on $[0, \alpha_1]$, g is affine, with slope x_1; (iii) on $[\alpha_1, \alpha_1 + \alpha_2]$, g is affine with slope x_2; etc. (Thus, on the last interval, $[\alpha_1 + \cdots + \alpha_{n-1}, S]$, g is affine with slope x_n.) Equivalently: (j) on $[0, \alpha_1]$, g varies with constant slope from 0 to $\alpha_1 x_1$; (jj) on $[\alpha_1, \alpha_2]$, g varies with constant slope from $\alpha_1 x_1$ to $\alpha_1 x_1 + \alpha_2 x_2$; etc.

We associate, using the same recipe, with $y_1, \dots, y_m, \beta_1, \dots, \beta_m$, a function $h : [0, S] \to \mathbb{R}_+$.

The next problem will provide us some insight about the role of g and h.

Problem 1.15. Note that, in the setting of (1.20), we have $m = n$, $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = 1$, $S = n$.

Prove that

$$[(1.16) - (1.19)] \Leftrightarrow [g(t) \le h(t),\ \forall\, t \in [0, S],\ \text{and } g(S) = h(S)].$$

We have the following far-reaching generalization of Theorem 1.1.

Theorem 1.2. Let I be an interval and let $x_1, \dots, x_n, y_1, \dots, y_m \in I$ satisfy (1.37). Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m > 0$ satisfy (1.38). Then the following are equivalent:

(1) For every convex function $f : I \to \mathbb{R}$, (1.36) holds.

(2) We have

$$g(t) \le h(t),\ \forall\, t \in [0, S],\ \text{and } g(S) = h(S). \tag{1.39}$$

(3) There exists a matrix $A = (a_{jk})_{1 \le j \le m, 1 \le k \le n}$ such that:

$$a_{jk} \ge 0,\ \forall\, j,\ \forall\, k, \tag{1.40}$$

$$\sum_{k=1}^{n} a_{jk} = 1,\ \forall\, j, \tag{1.41}$$

$$\sum_{j=1}^{m} \beta_j a_{jk} = \alpha_k,\ \forall\, k, \tag{1.42}$$

$$y_j = \sum_{k=1}^{n} a_{jk} x_k,\ \forall\, j. \tag{1.43}$$

Remark 1.4. The equivalence "(1) $\Leftrightarrow$ (2)" in Theorem 1.2 appears implicitly in Karamata [9].

The following is obvious.

Problem 1.16. In the setting of Theorem 1.2, we have "$(3) \Rightarrow (1)$", even without assuming (1.37).

Remark 1.5. In practice, the implication "$(3) \Rightarrow (1)$" is the most useful one. In order to prove (1.36), it suffices to "cook" the matrix A.

In both Theorems 1.1 and 1.2, the most difficult part is "$(2) \Rightarrow (3)$". Though conceptually important, this implication has little applications in basic problems.

In a special case (already very useful!), we are now in position to give a full proof of Theorem 1.2.

Problem 1.17. Prove Theorem 1.2 when the scalars $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ are *rational*. Hint: reduce first the problem to the case of *integer* scalars. Next prove that this can be seen as a special case of Theorem 1.1.

We now turn to the general case. The proof of Theorem 1.2 is based on the following simple scheme:

Theorem 1.1⇒(Problem 1.17) Theorem 1.2 for rational scalars $\alpha_1, \ldots, \beta_m$
⇒(by approximation) Theorem 1.2 for real scalars $\alpha_1, \ldots, \beta_m$.

The second part of the proof is slightly long, and we do not give all details. However, an advanced reader should be able to fill in the blanks.

Sketch of proof of Theorem 1.2 using Problems 1.16 and 1.17. *Step 1.* Testing (1.36) on particular functions. First, by letting, in (1.36), $f(x) \equiv x$ and $f(x) \equiv -x$, we find that

$$T \coloneqq \sum_{k=1}^{m} \alpha_k x_k = \sum_{j=1}^{n} \beta_j y_j. \tag{1.44}$$

Note that this is the same as

$$g(S) = h(S). \tag{1.45}$$

Next, let, for $z \in \mathbb{R}$, $f(x) \equiv (x-z)^-$. Testing (1.36) for this f, we find that

$$u(z) \coloneqq \sum_{k=1}^{n} \alpha_k (x_k - z)^- \geq v(z) \coloneqq \sum_{j=1}^{m} \beta_j (y_j - z)^-, \ \forall\, z \in \mathbb{R}. \tag{1.46}$$

It turns out that u can be explicitly calculated:

$$u(z) = \begin{cases} 0, & \text{if } z \leq x_1 \\ \alpha_1 z - \alpha_1 x_1, & \text{if } x_1 \leq z \leq x_2 \\ (\alpha_1 + \alpha_2) z - (\alpha_1 x_1 + \alpha_2 x_2), & \text{if } x_2 \leq z \leq x_3 \\ \vdots & \\ Sz - T, & \text{if } z \geq x_n \end{cases}, \tag{1.47}$$

where the last line relies on (1.38) and (1.44). A similar formula holds for v.

Step 2. "(1) $\Rightarrow$ (2)" We prove here that (1.46), (1.47), and the analogue of (1.47) for v, imply item (2). Here it will seem that we use some magic, but the idea comes from more advanced mathematics (the use of the Legendre transform). Let $t \in [0, S]$. Since $u(z) \geq v(z)$, $\forall\, z \in \mathbb{R}$, we have

$$tz - u(z) \leq tz - v(z),\ \forall\, z \in \mathbb{R},$$

and thus

$$\max\{tz - u(z);\ z \in \mathbb{R}\} \leq \max\{tz - v(z);\ z \in \mathbb{R}\},\ \forall\, 0 \leq t \leq S. \tag{1.48}$$

We will see that the left-hand side, respectively right-hand side, of (1.48) equals $g(t)$, respectively $h(t)$. (This will complete Step 2.) Indeed, using (1.47) we find that

$$tz - u(z) = \begin{cases} tz, & \text{if } z \leq x_1 \\ (t - \alpha_1)z + \alpha_1 x_1, & \text{if } x_1 \leq z \leq x_2 \\ (t - (\alpha_1 + \alpha_2))z + (\alpha_1 x_1 + \alpha_2 x_2), & \text{if } x_2 \leq z \leq x_3 \\ \vdots & \\ (t - S)z + T, & \text{if } z \geq x_n \end{cases}, \tag{1.49}$$

and thus $\max(tz - u(z))$ is achieved: (i) when $z = x_1$, if $0 \leq t \leq \alpha_1$; (ii) when $z = x_2$, if $\alpha_1 \leq t \leq \alpha_1 + \alpha_2$, etc. Calculating the value of this maximum, one sees that, indeed, (1.48) is the same as $g(t) \leq h(t)$, $\forall\, 0 \leq t \leq S$.

Step 3. From real scalars $\alpha_1, \ldots, \beta_m$ to rational scalars. This is the part of the proof that we do not rigorously prove or state; however, the argument we present may be easily made rigorous. First, by multiplying all the scalars with the same positive constant, we may assume that S is rational. We now note that (1.39) asserts that the graphs of g and h, which are piecewise affine functions, touch at the end points, and are one below the other. By "slightly" decreasing x_1 and "slightly" increasing x_n (make a picture!), we may assume that the two graphs do not to touch in $(0, S)$. We may now "slightly" change the scalars such that the graphs still do not touch except at the endpoints, and the new scalars are *rational*.

To summarize, given $\varepsilon > 0$, there are new points $\overline{x}_k$, $\overline{y}_j$, and new rational scalars $\overline{\alpha}_k$, $\overline{\beta}_j$, all depending on ε, such that $|\overline{x}_k - x_k| < \varepsilon$, $\forall\, k$, etc., and the corresponding functions $\overline{g}$ and $\overline{h}$ satisfy (1.39).

Step 4. Use of Problem 1.17 to prove "(2) $\Rightarrow$ (3)". Consider a matrix (depending on ε) $\overline{A} = (\overline{a}_{jk})_{1 \leq j \leq m,\, 1 \leq k \leq n}$ satisfying the analogues of (1.40)–(1.43) for $\overline{x}_1, \ldots, \overline{y}_m, \overline{\alpha}_1, \ldots, \overline{\beta}_m$. By (1.40) and (1.41), all the coefficients of the matrix are between 0 and 1. Recalling that a bounded sequence of real numbers contains a convergent subsequence, we obtain that, when $\varepsilon \to 0$ and possibly up to passing to a subsequence, we have $\overline{a}_{jk} \to a_{jk}$, and the matrix $A := (a_{jk})_{1 \leq j \leq m,\, 1 \leq k \leq n}$ has all the required properties.

Finally, "(3) $\Rightarrow$ (1)" is the content of Problem 1.16. QED

Problem 1.18. Prove, without using Theorem 1.2, that

$$(1.36) \Rightarrow x_1 \le y_1 \text{ and } x_n \ge y_m.$$

1.5 Rearrangement

In this section, whose presentation is inspired by the expository text (on statistics!) of Block *et al.* [2], we prove the rearrangement inequalities (R) and (C). By broadening the perspective, we will see them as special cases of more general inequalities. The main result here is the following.

Theorem 1.3. Let $\sigma, \tau \in S_n$. Then the following are equivalent:

(1) For each ordered lists $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$, we have

$$S_\sigma := \sum_{j=1}^{n} a_j b_{\sigma(j)} \le S_\tau := \sum_{j=1}^{n} a_j b_{\tau(j)}. \tag{1.50}$$

(2) For each $2 \le k, \ell \le n$, we have

$$\text{Card}\,\{j \ge k;\, \sigma(j) \ge \ell\} \le \text{Card}\,\{j \ge k;\, \tau(j) \ge \ell\}. \tag{1.51}$$

When the lists are ordered $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \ge b_2 \ge \cdots \ge b_n$, the validity of (1.50) is equivalent to

$$\text{Card}\,\{j \ge k;\, \sigma(j) \ge \ell\} \ge \text{Card}\,\{j \ge k;\, \tau(j) \ge \ell\},\ \forall\, 2 \le k, \ell \le n. \tag{1.52}$$

Problem 1.19. Prove Theorem 1.3 by expressing (1.50) in terms of the quantities $x_j := a_j - a_{j-1}$ and $y_j := b_j - b_{j-1}$, $2 \le j \le n$.

Problem 1.20. Prove that (R) is a special case of Theorem 1.3.

Next, a definition.

Definition 1.8. The lists $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are *identically ordered* if

$$(a_j - a_k)(b_j - b_k) \ge 0,\ \forall\, j, k, \tag{1.53}$$

respectively *oppositely ordered* if

$$(a_j - a_k)(b_j - b_k) \le 0,\ \forall\, j, k.$$

It turns out that, up to a permutation, identically ordered lists are ordered lists.

Problem 1.21. Assume that $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are identically ordered. Prove that there exists a permutation $\sigma \in S_n$ such that $a_{\sigma(1)} \leq a_{\sigma(2)} \leq \cdots \leq a_{\sigma(n)}$ and $b_{\sigma(1)} \leq b_{\sigma(2)} \leq \cdots \leq b_{\sigma(n)}$.

Similar result if the lists are oppositely ordered.

Problem 1.22. Prove the following generalization of (C): if the lists $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are identically ordered, then we have the general(ized) Chebyshev inequality (GC)

$$\sum_{j=1}^{n} a_j b_j \geq \frac{1}{n} \sum_{j=1}^{n} a_j \sum_{j=1}^{n} b_j. \tag{GC}$$

If the lists are oppositely ordered, then the inequality is reversed.

Hint: find n permutations $\sigma_1, \ldots, \sigma_n \in S_n$ such that

$$\sum_{j=1}^{n} a_j \sum_{k=1}^{n} b_k = \sum_{j=1}^{n} \sum_{k=1}^{n} a_j b_{\sigma_k(j)}.$$

Finally, a worked beautiful inequality, whose solution combines rearrangement and convexity. Its starting point is the following natural would-be inequality: given $n \geq 3$ and $x_1, \ldots, x_n > 0$, is it true that

$$\sum_{j=1}^{n} \frac{x_j}{x_{j+1} + x_{j+2}} \geq \frac{n}{2} \tag{1.54}$$

(with the convention $x_{n+1} = x_1$, $x_{n+2} = x_2$)?

We will discuss special cases of this inequality in Problem 2.19. This inequality holds true for $n = 3, \ldots, 13$, but is wrong when $n \geq 24$. (For the full picture concerning its validity and the methods used in proving or disproving it, see the expository text of Clausing [4].) The next result, due to Drinfeld [6], shows that (1.54) is "not far from being true" for every n.

Problem 1.23. Let $f : \mathbb{R} \to \mathbb{R}$ be a convex function such that

$$f(x) \leq \min\left(\frac{2}{e^x + e^{x/2}}, \frac{1}{e^x}\right), \ \forall\, x \in \mathbb{R}. \tag{1.55}$$

Then

$$\sum_{j=1}^{n} \frac{x_j}{x_{j+1} + x_{j+2}} \geq f(0)\frac{n}{2}, \forall\, n \geq 3, \ \forall\, x_1, \ldots, x_n > 0. \tag{1.56}$$

Proof. In all the proof, we use the convetions $x_{n+1} = x_1$, $y_{n+2} = y_2$, etc.

Step 1. Change of unknowns. This step ressembles arguing by homogeneity. Set $y_j := \dfrac{x_{j+1}}{x_j}$, so that

$$\sum_{j=1}^{n} \frac{x_j}{x_{j+1} + x_{j+2}} = \sum_{j=1}^{n} \frac{1}{x_{j+1}/x_j + x_{j+2}/x_j} = \sum_{j=1}^{n} \frac{1}{y_j(1 + y_{j+1})},$$
$$\prod_{j=1}^{n} y_j = 1.$$

Step 2. Use of (R). The numbers $y_1, \ldots, y_n$ need not be ordered. Consider a permutation $\sigma \in S_n$ such that $y_{\sigma(1)} \geq \cdots \geq y_{\sigma(n)}$. Then

$$\frac{1}{y_{\sigma(1)}} \leq \cdots \leq \frac{1}{y_{\sigma(n)}} \text{ and } \frac{1}{1 + y_{\sigma(1)}} \leq \cdots \leq \frac{1}{1 + y_{\sigma(n)}}. \tag{1.57}$$

Set $z_j := y_{\sigma(j)}$, $\forall\, j$. From (1.57) and the first inequality in (R), we find that

$$\sum_{j=1}^{n} \frac{x_j}{x_{j+1} + x_{j+2}} \geq \sum_{j=1}^{n} \frac{1}{y_{\sigma(j)}\big(1 + y_{\sigma(n+1-j)}\big)} = \sum_{j=1}^{n} \frac{1}{z_j(1 + z_{n+1-j})}.$$

Therefore, it suffices to prove the following

$$\left[z_1, \ldots, z_n > 0, \prod_{j=1}^{n} z_j = 1\right] \Rightarrow \sum_{j=1}^{n} \frac{1}{z_j(1 + z_{n+1-j})} \geq f(0)\frac{n}{2}. \tag{1.58}$$

(The fact that $z_1 \leq \cdots \leq z_n$ will not be used in the proof of (1.58).)

Step 3. Use of calculus. Consider the sum

$$S_j := \frac{1}{z_j(1 + z_{n+1-j})} + \frac{1}{z_{n+1-j}(1 + z_j)}.$$

Set $t_j := z_j z_{n+1-j}$, so that $\prod_{j=1}^{n} t_j = 1$. When t_j is fixed, the sum S_j, considered as a function of z_j: (i) is constant 1 when $t_j = 1$; (ii) has a maximum equal to $\dfrac{2}{t_j + \sqrt{t_j}}$ when $t_j < 1$; (iii) has an infimum equal to $\dfrac{1}{t_j}$ when $t_j > 1$. We find that, in any case, we have

$$\sum_{j=1}^{n} \frac{1}{z_j(1 + z_{n+1-j})} = \frac{1}{2}\sum_{j=1}^{n} S_j \geq \frac{1}{2}\sum_{j=1}^{n} \min\left(\frac{2}{t_j + \sqrt{t_j}}, \frac{1}{t_j}\right). \tag{1.59}$$

Step 4. Use of (GJ). Write $t_j = e^{u_j}$, so that $\sum_{j=1}^{n} u_j = 0$. Then (1.59) reads

$$\sum_{j=1}^{n} \frac{1}{z_j(1 + z_{n+1-j})} \geq \frac{1}{2}\sum_{j=1}^{n} \min\left(\frac{2}{e^{u_j} + e^{u_j/2}}, \frac{1}{e^{u_j}}\right) \geq \frac{1}{2}\sum_{j=1}^{n} f(u_j), \tag{1.60}$$

where the last inequality uses (1.55). Combining (1.60) with (GJ) (here, we need the convexity of f), we find that

$$\sum_{j=1}^{n} \frac{x_j}{x_{j+1}+x_{j+2}} \geq \sum_{j=1}^{n} \frac{1}{z_j(1+z_{n+1-j})} \geq \frac{1}{2}\sum_{j=1}^{n} f(u_j) \geq \frac{n}{2} f\left(\sum_{j=1}^{n} \frac{u_j}{n}\right) = f(0)\frac{n}{2}. \quad \text{QED}$$

Remark 1.6. It is not clear why (1.56) implies that (1.54) is "not far from being true". It turns out that one may construct f satisfying (1.55) such that $f(0) = 0.989133\ldots$ Thus, although (1.54) is wrong, we have

$$\sum_{j=1}^{n} \frac{x_j}{x_{j+1}+x_{j+2}} \geq 0.989133\frac{n}{2}, \forall\, n \geq 2,\ \forall\, x_1, \ldots, x_n > 0.$$

1.6 Quadratic trinomials and forms

This section is more about tricks than methods, in the sense that the applications, though spectacular, are limited. The common theme is that we consider quadratic trinomials or *quadratic forms*, i.e., expressions of the form

$$Q(x_1, \ldots, x_n) = \sum_{j,k=1}^{n} a_{j,k} x_j x_k.$$

A first direction consists of considering the discriminant of a quadratic trinomial. The flagship of this approach is the following.

Problem 1.24. Prove the (CS) formula, including the equality case, by considering the sign of the quadratic trinomial

$$T(x) := \sum_{j=1}^{n} (a_j x - b_j)^2,\ \forall\, x \in \mathbb{R}. \tag{1.61}$$

Here is an even more spectacular result. To motivate it, we start from (CS), in which we have equality iff the vectors $(a_j)_{1\leq j\leq n}$ and $(b_j)_{1\leq j\leq n}$ are proportional. Intuitively, one expects that, if these vectors are "almost proportional", and in particular "almost equal", then we have "almost equality" in (CS). This is quantified by the following inequality, due to Pólya and Szegö. (This is *not* the famous Pólya-Szegö inequality.) If $\alpha > 1$, then

$$\begin{aligned} &\left[a_1, \ldots, a_n, b_1, \ldots, b_n > 0, \frac{1}{\alpha} \leq \frac{a_j}{b_j} \leq \alpha, \forall\, j\right] \\ &\quad \Rightarrow \left(\sum_{j=1}^{n} a_j b_j\right)^2 \geq \frac{4\alpha^2}{(\alpha^2+1)^2} \sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2. \end{aligned} \tag{1.62}$$

Note that, when $\alpha = 1$, we have $\frac{4\alpha^2}{(\alpha^2+1)^2} = 1$, and we recover the equality in (CS).

Problem 1.25. Let $a_1, \ldots, a_n, b_1, \ldots b_n$ and α be as in (1.62). Consider

$$f(t) := \sum_{j=1}^{n} \left(a_j^2 t^2 - 2a_j b_j t + \frac{4\alpha^2}{(\alpha^2+1)^2} b_j^2 \right), \ \forall\, t \in \mathbb{R}.$$

Derive (1.62) by determining the sign of $f\left(\frac{2\alpha}{\alpha^2+1}\right)$.

We next informally discuss the Gauss method from the limited perspective of deciding whether a quadratic form is non-negative. Here is a worked example. Consider the inequality

$$ab + bc + ca \le a^2 + b^2 + c^2, \ \forall\, a, b, c \in \mathbb{R}. \tag{1.63}$$

Proof of (1.63) *via the Gauss method.* We will form squares of linear forms as follows

$$\begin{aligned}(a^2+b^2+c^2) - (ab+bc+ca) &= (a^2 - ab - ac) + (b^2 + c^2 - bc) \\ &= \left(a - \frac{1}{2}b - \frac{1}{2}c\right)^2 + \left(\frac{3}{4}b^2 + \frac{3}{4}c^2 - \frac{3}{2}bc\right) \\ &= \left(a - \frac{1}{2}b - \frac{1}{2}c\right)^2 + \frac{3}{2}(b-c)^2 \ge 0.\end{aligned}$$

QED

As a side remark, (1.63) can be obtained by other means.

Problem 1.26. Prove (1.63) using (CS) or (R).

The interest of the Gauss method lies mainly in its generality, and in the fact that it can also decide whether a quadratic form is *not* always ≥ 0. Here is an example.

Problem 1.27. Consider, for $n \ge 3$, the inequality

$$n \sum_{j=1}^{n} x_j (x_{j+1} + x_{j+2}) \le 2 \left(\sum_{j=1}^{n} x_j \right)^2, \ \forall\, x_1, \ldots, x_n \ge 0 \tag{1.64}$$

(with the convention $x_{n+1} = x_1$, $x_{n+2} = x_2$). Prove that (1.64) holds iff $n \le 6$.

1.7 Calculus

One can get useful information by studying the variations of a function. Two basic examples: (i) if f is non-decreasing on $[a, b]$, then $f(a) \leq f(x) \leq f(b)$, $\forall\, x \in [a, b]$; (ii) if f is convex on $[a, b]$, then $f(x) \leq \max(f(a), f(b))$, $\forall\, x \in [a, b]$; (iii) similarly, if f is strictly convex on $[a, b]$ and $f(a) = f(b)$, then $f(x) < f(a)$, $\forall\, x \in (a, b)$.

A textbook example is the following.

Problem 1.28. Let $1 < p, q < \infty$ be conjugated exponents. Prove the Young inequality (Y)

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},\ \forall\, a, b \geq 0, \tag{Y}$$

by studying the function

$$[0, \infty) \ni a \mapsto \frac{a^p}{p} + \frac{b^q}{q} - ab.$$

(Another textbook proof of (Y) relies on (J).)

We now present a more involved worked example, generalizing the Pólya-Szegö inequality (1.62) and quantifying the "almost equality" case in the Hölder inequality (H).

Problem 1.29. Let $1 < p, q < \infty$ be conjugate exponents. Let $\alpha > 1$. Define

$$C = C_{\alpha,p} := \frac{p^p \alpha^{2p-2}(\alpha^{2p-2} - 1)^{p-1}(\alpha^2 - 1)}{(p-1)^{p-1}(\alpha^{2p} - 1)^p}. \tag{1.65}$$

Then

$$\begin{aligned} &\left[a_1, \ldots, a_n, b_1, \ldots, b_n > 0, \frac{1}{\alpha} \leq \frac{a_j}{b_j^{q-1}} \leq \alpha, \forall\, j\right] \\ &\qquad \Rightarrow \left(\sum_{j=1}^n a_j b_j\right)^p \geq C \sum_{j=1}^n a_j^p \left(\sum_{j=1}^n b_j^q\right)^{p-1}. \end{aligned} \tag{1.66}$$

Noting that, when $p = 2$, we have $q = 2$ and $C_{\alpha,2} = \dfrac{4\alpha^2}{(\alpha^2+1)^2}$, we see that, indeed, (1.66) generalizes (1.62).

Proof. *Step 1.* Reduction to a minimization problem. By homogeneity, we may assume that $\sum_{j=1}^n b_j^q = 1$. Let $\lambda_j := b_j^q$ and $x_j := \dfrac{a_j}{b_j^{q-1}}, \forall\, j$. Since $a_j b_j = \lambda_j x_j$ and $a_j^p = \lambda_j x_j^p$, (1.66) amounts to

$$\left[\lambda_1, \ldots, \lambda_n > 0, \sum_{j=1}^n \lambda_j = 1, \frac{1}{\alpha} \le x_j \le \alpha, \forall\, j\right] \Rightarrow \frac{\left(\sum_{j=1}^n \lambda_j x_j\right)^p}{\sum_{j=1}^n \lambda_j x_j^p} \ge C. \tag{1.67}$$

Step 2. A calculus part. Set

$$f(x_1, \ldots, x_n) := \frac{\left(\sum_{j=1}^n \lambda_j x_j\right)^p}{\sum_{j=1}^n \lambda_j x_j^p}. \tag{1.68}$$

We study the variations of f in the variable x_j, the other ones being fixed. With no loss of generality, we take $j = 1$, and let $S := \sum_{j=2}^n \lambda_j x_j$, $T := \sum_{j=2}^2 \lambda_j x_j^p$, so that

$$f(x_1, \ldots, x_n) = \frac{(\lambda_1 x_1 + S)^p}{\lambda_1 x_1^p + T}.$$

The derivative of f with respect to x_1 is

$$g(x_1, \ldots, x_n) := \frac{p\lambda_1 x_1^{p-1}}{(\lambda_1 x_1^p + T)^2}\left[\left(\lambda_1 + \frac{S}{x_1}\right)^{p-1} - 1\right], \ \frac{1}{\alpha} \le x_1 \le \alpha.$$

We deduce that the following can happen: (i) if $\lambda_1 + \dfrac{S}{\alpha} \ge 1$, then $g \ge 0$, f increases with x_1, and its minimum is achieved when $x_1 = \dfrac{1}{\alpha}$; (ii) if $\lambda_1 + \alpha S \le 1$, then $g \le 0$, f decreases with x_1, and its minimum is achieved when $x_1 = \alpha$; (iii) in the remaining case, g has exactly one zero $x_0 \in (1/\alpha, \alpha)$, f increases on $[1/\alpha, x_0]$, decreases on $[x_0, \alpha]$, and achieves its minimum when either $x_1 = 1/\alpha$, or $x_1 = \alpha$. In all cases, f is minimal when either $x_1 = 1/\alpha$, or $x_1 = \alpha$.

Iterating the above argument, we see that, for every list $(x_1 \ldots, x_n)$, there exists a list $(y_1, \ldots, y_n) \in \{1/\alpha, \alpha\}^n$ such that $f(x_1, \ldots, x_n) \ge f(y_1, \ldots, y_n)$. We have thus reduced the initial problem to the following:

$$f(y_1, \ldots, y_n) \ge C, \ \forall\, y_1, \ldots, y_n \in \{1/\alpha, \alpha\}. \tag{1.69}$$

Step 3. A second calculus part. Let, for $(y_1, \ldots, y_n) \in \{1/\alpha, \alpha\}^n$, $a := \sum_{y_j = \alpha} \lambda_j$, so that

$$f(y_1, \ldots, y_n) = \frac{(a\alpha + (1-a)/\alpha)^p}{a\alpha^p + (1-a)/\alpha^p} = \frac{\left[(\alpha^2 - 1)a + 1\right]^p}{(\alpha^{2p} - 1)a + 1}.$$

In view of (1.68), (1.69) amounts to

$$h(a) := \frac{\left[(\alpha^2 - 1)a + 1\right]^p}{(\alpha^{2p} - 1)a + 1} \ge C, \ \forall\, a \in [0, 1]. \tag{1.70}$$

By studying the variation of h, one finds that h has a minimum at

$$a_0 := \frac{(\alpha^{2p}-1)-p(\alpha^2-1)}{(p-1)(\alpha^{2p}-1)(\alpha^2-1)},$$

and we have $h(a_0) = C$. QED

We have just met, in Step 3, a “proof by intimidation” or “tedious proof”, relying on a bunch of long calculations that, most likely, you will not check. To inspire you second thoughts: complex calculations may lead to new ideas and, more importantly, the mastery of analysis requires the one of calculations.

Another worked example. The idea of this inequality is due to Panaitopol. Its proof relies on a classical textbook inequality, the Bernoulli inequality

$$(1+x)^\alpha \geq 1+\alpha x,\ \forall\, \alpha \geq 1,\ \forall\, x \geq -1. \tag{B}$$

Problem 1.30. Let $\alpha > 1$ and $n \geq 1$. Prove that

$$[1 \leq x_j \leq \alpha,\ \forall\, 1 \leq j \leq n] \Rightarrow \left(\sum_{j=1}^n x_j\right)\left(\sum_{j=1}^n \frac{1}{x_j}\right)^\alpha \leq n^{\alpha+1}. \tag{1.71}$$

Proof. The case $n = 1$ is clear, so let $n \geq 2$.

Step 1. Reduction to the comparison of two values. We study the variation of the left-hand side of (1.71) as a function of, say, x_n, with $x_1, \dots, x_{n-1}$ fixed. In order to simplify the calculations, we write x instead of x_n, and denote

$$A := \sum_{j=1}^{n-1} x_j,\ B := \sum_{j=1}^{n-1} \frac{1}{x_j},\ f(x) := (A+x)\left(B+\frac{1}{x}\right)^\alpha,$$

so that our inequality becomes $f(x) \leq n^{\alpha+1}$, $\forall\, 1 \leq x \leq \alpha$. Now

$$\begin{aligned} f'(x) =& \left(B+\frac{1}{x}\right)^\alpha - \frac{\alpha}{x^2}(A+x)\left(B+\frac{1}{x}\right)^{\alpha-1} \\ =& \frac{1}{x^2}\left(B+\frac{1}{x}\right)^{\alpha-1} \underbrace{(Bx^2-(\alpha-1)x-\alpha A)}_{Q(x)}. \end{aligned} \tag{1.72}$$

Let us note that $Q(1) < 0$ (since $B \leq n$, while $A \geq n$). Since Q is a quadratic trinomial, we find that one the two happens: (i) f decreases on $[1,\alpha]$; (ii) there exists some $1 < x_0 < \alpha$ such that f decreases on $[1, x_0]$ and increases on $[x_0, \alpha]$. In both cases, f achieves its maximum either at $x = 1$, or at $x = \alpha$.

Step 2. We have $f(1) > f(\alpha)$. This amounts to

$$\left(\frac{B+1}{B+1/\alpha}\right)^\alpha > \frac{A+\alpha}{A+1}. \tag{1.73}$$

To prove (1.73), we rely on (B), which implies

$$\left(\frac{B+1}{B+1/\alpha}\right)^{\alpha} = \left(1+\frac{1-1/\alpha}{B+1/\alpha}\right)^{\alpha} \geq 1+\frac{\alpha-1}{B+1/\alpha} > 1+\frac{\alpha-1}{A+1} = \frac{A+\alpha}{A+1},$$

where the last inequality uses the fact that $B \leq n \leq A$.

Step 3. Conclusion. The above analysis shows that $f(x) < f(1), \forall\, x > 1$. By symmetry of the expression we investigate, we find that

$$\left(\sum_{j=1}^{n} x_j\right)\left(\sum_{j=1}^{n}\frac{1}{x_j}\right)^{\alpha} \leq \left(\sum_{j=1}^{n} 1\right)\left(\sum_{j=1}^{n}\frac{1}{1}\right)^{\alpha} = n^{\alpha+1},$$

with equality iff $x_j = 1, \forall\, j$. QED

Finally, a more involved worked example. Its starting point is the following standard calculus fact. If $\alpha \in \mathbb{R}$ and $|x| < 1$, then we have the power series expansion

$$(1+x)^{\alpha} = 1+\frac{\alpha}{1!}x+\frac{\alpha(\alpha-1)}{2!}x^2+\frac{\alpha(\alpha-1)\alpha-2}{3!}x^3+\cdots. \tag{1.74}$$

Problem 1.31. Let $1 \leq p \leq 2$. Prove that

$$(1+x)^p+(1-x)^p \leq 2+(2^p-2)x^2,\ \forall\, -1 \leq x \leq 1. \tag{1.75}$$

Proof. (1.75) holds with equality when $x = \pm 1$ or when $p \in \{1, 2\}$. We may therefore assume that $|x| < 1$ and $1 < p < 2$. Using (1.74), we find that

$$\begin{aligned}&(1+x)^p+(1-x)^p\\ &=2\left[1+\frac{p(p-1)}{2!}x^2+\sum_{k\geq 2}\frac{p(p-1)(2-p)\cdots(2k-1-p)}{(2k)!}x^{2k}\right].\end{aligned} \tag{1.76}$$

Now comes the key observation. Under the assumption $1 < p < 2$, the coefficient in front of x^{2k} is positive, $\forall\, k$. Fix some integer $K \geq 2$. Then (by positivity of the coefficients when $k > K$)

$$\begin{aligned}&\frac{p(p-1)}{2!}x^2+\sum_{k=2}^{K}\frac{p(p-1)(2-p)\cdots(2k-1-p)}{(2k)!}x^{2k}\\ &\qquad\leq \frac{(1+x)^p+(1-x)^p}{2}-1.\end{aligned} \tag{1.77}$$

Letting $x \nearrow 1$ in (1.77), we find that

$$\frac{p(p-1)}{2!}+\sum_{k=2}^{K}\frac{p(p-1)(2-p)\cdots(2k-1-p)}{(2k)!} \leq 2^{p-1}-1,\ \forall\, K \geq 2. \tag{1.78}$$

From (1.76) and (1.78), we find that

$$\begin{aligned}
&(1+x)^p + (1-x)^p \\
&= \lim_{K\to\infty} 2\left[1 + \frac{p(p-1)}{2!}x^2 + \sum_{k=2}^{K} \frac{p(p-1)(2-p)\cdots(2k-1-p)}{(2k)!}x^{2k}\right] \\
&\le \lim_{K\to\infty} 2\left[1 + \frac{p(p-1)}{2!}x^2 + \sum_{k=2}^{K} \frac{p(p-1)(2-p)\cdots(2k-1-p)}{(2k)!}x^{2}\right] \\
&\le 2 + (2^p - 2)x^2.
\end{aligned}$$

QED

We will see in more depth, in Section 3.3, how textbook calculus methods can be successfully used to prove rather difficult inequalities.

1.8 Optimization

A more adapted title of this section would be "constrained optimization". I will be very informal on the calculus part, which can be found in elementary calculus courses, and will not precisely define important notions as compact sets or continuous or differentiable functions.

To start with, we explain this approach on the familiar (AM-GM) inequality

$$\sqrt[n]{x_1 \cdots x_n} \le \frac{x_1 + \cdots + x_n}{n},\ \forall\, x_1, \ldots, x_n > 0,$$

which, in view of its homogeneity, can be restated as

$$f(x_1, \ldots, x_n) \coloneqq \sqrt[n]{x_1 \cdots x_n} \le 1,\ \forall\, x_1, \ldots, x_n > 0 \text{ s.t. } x_1 + \cdots + x_n = n. \quad (1.79)$$

In turn, proving (1.79) amounts to finding $\max f$ under the *constraints*

$$x_1, \ldots, x_n > 0,\ x_1 + \cdots + x_n = n.$$

Treating such problems relies on two calculus results: (i) existence of a point of constrained maximum/minimum; (ii) an equation ("Fritz John conditions") satisfied at such points, for which the Wikipedia executive summary [14] is sufficient as a first reading.

To start with, let us consider the existence of a constrained maximum of f. For this purpose, it is more convenient to replace (1.79) with the seemingly stronger result

$$f(x_1, \ldots, x_n) \le 1,\ \forall\, x_1, \ldots, x_n \ge 0 \text{ s.t. } x_1 + \cdots + x_n = n. \quad (1.80)$$

The advantage of this formulation is that $\max f$ under the constraints in (1.80) is attained. This comes from the calculus fact that "a continuous function on a compact set has a maximum point and a minimum point". As a user's guide, a set of the form

$$K := \{x \in \mathbb{R}^n;\ f_\ell(x) \geq 0,\ \ell = 1, \ldots, m\}$$

is compact provided: (i) each f_ℓ is continuous; (ii) there exists some constant M such that

$$x = (x_1, \ldots, x_n) \in K \Rightarrow [|x_j| \leq M, j = 1, \ldots, n]. \tag{1.81}$$

(In our case, (1.81) holds with $M = n$.)

For further use, let us note that, for our specific f, a maximum point $x = (x_1, \ldots, x_n)$ necessarily satisfies $x_j > 0$, $\forall\, j$.

Let us now broaden the perspective and consider a constrained optimization problem in its general form

$$\begin{aligned} &\text{maximize} f(x),\ \text{with } x = (x_1, \ldots, x_n),\ \text{under the constraints} \\ &\qquad g_1(x) = 0, \ldots, g_\ell(x) = 0,\ h_1(x) \geq 0, \ldots, h_m(x) \geq 0. \end{aligned} \tag{1.82}$$

Assume that x solves (1.82) and that, at x, we have

$$h_1(x) = 0, \ldots, h_p(x) = 0,\ \text{while } h_{p+1}(x) > 0, \ldots, h_m(x) > 0. \tag{1.83}$$

(In the optimization jargon, the constraints $h_1, \ldots h_p$ are *active*, while the constraints $h_{p+1}, \ldots h_n$ are *inactive*.)

Let us define the *gradient* of a function f of $x = (x_1, \ldots, x_n)$ as

$$\nabla f(x) := \begin{pmatrix} \dfrac{\partial f}{\partial x_1}(x) \\ \dfrac{\partial f}{\partial x_2}(x) \\ \vdots \\ \dfrac{\partial f}{\partial x_n}(x) \end{pmatrix}$$

(aka "nabla f"), where $\dfrac{\partial f}{\partial x_j}(x)$ stands for the derivative of f with respect to x_j, the other variables being fixed. For example, if $f(x, y) := xy^2$, then $\nabla f(x, y) = \begin{pmatrix} y^2 \\ 2xy \end{pmatrix}$.

We have the following important result.

Theorem 1.4 (Fritz John conditions)**.** Assume that x solves (1.82) and that (1.83) holds. Then there exist scalars $\lambda_j, j = 0, \ldots, \ell$, and $\mu_k, k = 1, \ldots, p$, not all zero, such that

$$\lambda_0 \nabla f(x) = \sum_{j=1}^{\ell} \lambda_j \nabla g_j(x) + \sum_{k=1}^{p} \mu_k \nabla h_k(x). \tag{1.84}$$

In many practical cases, one may prove that $\lambda_0 \neq 0$ and then, by homogeneity, we may assume that $\lambda_0 = 1$.

With this important tool in our pocket, let us come back to the proof of (1.80). Since at a maximum point x we have $x_j > 0, \forall\, j$, (1.84) reads that there exist λ_0, λ_1, not both zero, such that

$$\frac{1}{n}\lambda_0 \frac{f(x)}{x_j} = \lambda_1,\ \forall\, j.$$

We find first that $\lambda_0 \neq 0$ (since otherwise $\lambda_1 = 0$), and next that all the x_j's are equal. Finally, the constraint $x_1 + \cdots + x_n = n$ implies that $x = (1, \ldots, 1)$, and we find that the maximum is indeed 1.

Let us now work a more involved example, taken, as many problems in this text, from the Kvant magazine [16].

Problem 1.32. We have

$$(x_1 + \cdots + x_n)^2 \leq (n-1)\big(x_1^2 + \cdots + x_n^2\big) + n\sqrt[n]{x_1^2 \cdots x_n^2}, \quad \forall\, x_1, \ldots, x_n > 0. \tag{1.85}$$

Proof. By homogeneity, (1.85) amounts to proving that $m \geq 1$, where

$$m := \min f(x),\ f(x) := (n-1)\big(x_1^2 + \cdots + x_n^2\big) + n\sqrt[n]{x_1^2 \cdots x_n^2},$$
$$\text{under the constraints } x_1, \ldots, x_n \geq 0,\ x_1 + \cdots + x_n = 1.$$

If $x = (x_1, \ldots, x_n)$ achieves m, then: either (i) there exists some j such that $x_j = 0$; or $x_j > 0, \forall\, j$. If (i) occurs, we may assume that $x_n = 0$, and then (1.85) amounts to

$$(x_1 + \cdots + x_{n-1})^2 \leq (n-1)\big(x_1^2 + \cdots + x_{n-1}^2\big),$$

which is a special case of (CS).

More interesting is (ii). In this case, Theorem 1.4 asserts that there exist λ_0, λ_1, not both zero, such that, with $P := \sqrt[n]{x_1 \cdots x_n}$, we have

$$\lambda_0\left(2(n-1)x_j + 2\frac{P^2}{x_j}\right) = 2\lambda_1,\ \forall\, j. \tag{1.86}$$

We cannot have $\lambda_0 = 0$, and thus we may assume $\lambda_0 = 1$. Then (1.86) implies that

$$(n-1)x_j^2 - \lambda_1 x_j + P^2 = 0, \forall\, j.$$

Since the equation $(n-1)X^2 - \lambda_1 X + P^2 = 0$ has at most two distinct roots, we find that the coordinates of x can take at most two distinct values: say ℓ of them equal a, and $n-\ell$ equal b. We are thus led to the following *simpler* problem, involving *two* coordinates instead of n: prove that, for each integer $0 \le \ell \le n$, we have

$$(\ell a + (n-\ell)b)^2 \le (n-1)\big(\ell a^2 + (n-\ell)b^2\big) + n a^{2\ell/n} b^{2(n-\ell)/n}, \ \forall\, a, b > 0. \tag{1.87}$$

This inequality is clear when $\ell = 0$ or $\ell = n$. Assuming that $1 \le \ell \le n-1$, (1.87) reads

$$ab \le \frac{\ell(n-\ell-1)}{2\ell(n-\ell)} a^2 + \frac{(n-\ell)(\ell-1)}{2\ell(n-\ell)} b^2 + \frac{n}{2\ell(n-\ell)} a^{2\ell/n} b^{2(n-\ell)/n}, \ \forall\, a, b > 0. \tag{1.88}$$

In turn, (1.88) amounts to a convexity inequality. Indeed, by homogeneity we may assume that $a = 1$ and then, with $k(x) := b^x$, (1.88) reads

$$k(1) \le \frac{\ell(n-\ell-1)}{2\ell(n-\ell)} k(0) + \frac{(n-\ell)(\ell-1)}{2\ell(n-\ell)} k(2) + \frac{n}{2\ell(n-\ell)} a^{2\ell/n} k\left(\frac{2(n-\ell)}{n}\right),$$

which is a special case of (GJ) for the convex function k. QED

Chapter 2

Worked inequalities

Overview. Unlike in real life, the problems in this chapter come with a hint. A first hint is provided by the name of the section.

2.1 Induction

Problem 2.1. Prove by induction the general Chebyshev inequality (GC).

2.2 Convexity. Majorization

Problem 2.2. Let $p > 1$ and $a \geq b > 0$. Prove that

$$(a^p - (a^p - b^p)t^p)^{1/p} \geq a - (a-b)t, \ \forall\, 0 \leq t \leq 1. \tag{2.1}$$

Hint: if $f : [\alpha, \beta] \to \mathbb{R}$ is convex and $f(\alpha) = f(\beta)$, then $f(t) \leq f(\alpha)$, $\forall\, \alpha \leq t \leq b$.

Definition 2.1. A *norm* on $\mathbb{R}^n$ is a function $N : \mathbb{R}^n \to [0, \infty)$ such that

$$N(tx) = |t|N(x), \ \forall\, x \in \mathbb{R}^n, \ \forall\, t \in \mathbb{R}, \tag{2.2}$$
$$N(x) = 0 \Rightarrow x = 0, \tag{2.3}$$
$$N(x+y) \leq N(x) + N(y), \ \forall\, x, y \in \mathbb{R}^n. \tag{2.4}$$

Similar definition when $\mathbb{R}^n$ is replaced by a real or complex vector space.

Problem 2.3. Let $\varphi : \mathbb{R} \to (0, \infty)$ be a convex even function such that

$$(0, \infty) \ni t \mapsto \frac{\varphi(t)}{t} \text{ is non-increasing and } \lim_{t\to\infty} \frac{\varphi(t)}{t} := \ell > 0. \tag{2.5}$$

(1) Prove that φ is non-decreasing on $[0, \infty)$.

(2) Set, for every $(u, v) \in \mathbb{R}^2$,

$$\Phi_2(u, v) := \begin{cases} |v|\,\varphi\left(\frac{|u|}{|v|}\right), & \text{if } v \neq 0 \\ \ell|u|, & \text{if } v = 0 \end{cases}.$$

Prove that Φ_2 is a norm on $\mathbb{R}^2$.

(3) Let $n \geq 3$. Let $N : \mathbb{R}^{n-1} \to [0, \infty)$ be a norm. Hint: a convex function on $\mathbb{R}$ is continuous. Set

$$\Phi_n(x_1, \ldots, x_n) := \Phi_2(N(x_1, \ldots, x_{n-1}), x_n), \ \forall\, x_1, \ldots, x_n \in \mathbb{R}.$$

Prove that Φ_n is a norm on $\mathbb{R}^n$.

(4) Derive (again) the Minkowski inequality (M).

Problem 2.4.

(1) Let $a, a_1, \ldots, a_k \in \mathbb{R}$. Prove the equivalence between:

(a) There exist

$$\lambda_1, \ldots, \lambda_k \text{ such that } \lambda_j \in [0, 1],\ \forall\, j,\ \sum_{j=1}^{k} \lambda_j = 1, \tag{2.6}$$

$$x^a \leq \sum_{j=1}^{k} \lambda_j x^{a_j},\ \forall\, x > 0. \tag{2.7}$$

(b) We have, for the above λ_j's,

$$a = \sum_{j=1}^{k} \lambda_j a_j. \tag{2.8}$$

(2) Given $x = (x_1, \ldots, x_n) \in (0, \infty)^n$ and $\alpha = (\alpha^1, \ldots, \alpha^n) \in \mathbb{R}^n$, set

$$x^\alpha := x_1^{\alpha^1} \cdots x_n^{\alpha^n}.$$

Let $\alpha, \alpha_1, \ldots, \alpha_k \in \mathbb{R}^n$. Prove the equivalence between:

(a) There exist

$$\lambda_1, \ldots, \lambda_k \text{ such that } \lambda_j \in [0, 1],\ \forall\, j,\ \sum_{j=1}^{k} \lambda_j = 1, \tag{2.9}$$

$$x^\alpha \leq \sum_{j=1}^{k} \lambda_j x^{\alpha_j},\ \forall\, x \in (0, \infty)^n. \tag{2.10}$$

(b) We have, for the above λ_j's,

$$\alpha = \sum_{j=1}^{k} \lambda_j \alpha_j. \tag{2.11}$$

Hints: (i) the function $\alpha \mapsto x^\alpha$ is convex (what this could mean)?; (ii) find a minimum point for $x \mapsto \sum_{j=1}^{k} \lambda_j x^{\alpha_j} - x^\alpha$.

Problem 2.5. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a convex function, and $a, b, c \in I$ be such that $a \le b \le c$. Prove that

$$(c-b)f(a) + (a-c)f(b) + (b-a)f(c) \ge 0. \tag{2.12}$$

Problem 2.6. Let $I \subset \mathbb{R}$ be an interval, $f : I \to \mathbb{R}$ be a convex function, and $x_1, \ldots, x_n \in I$ be such that $x_1 \le x_2 \le \cdots \le x_n$. Set $x_{n+1} := x_1$. Prove that

$$\sum_{j=1}^{n} x_j f(x_{j+1}) \ge \sum_{j=1}^{n} x_{j+1} f(x_j). \tag{2.13}$$

Hint: induction.

Problem 2.7. Let $f : [0, \infty) \to (0, \infty)$ be a convex, continuous, non-decreasing function such that $f(x) > 0$, $\forall\, x > 0$. Find the best lower bound of the form

$$\frac{1}{\displaystyle\sum_{j=1}^{n} \frac{1}{f(x_j+1)}} - \frac{1}{\displaystyle\sum_{j=1}^{n} \frac{1}{f(x_j)}} \ge C_n,\ \forall\, x_1, \ldots, x_n > 0. \tag{2.14}$$

Same question if we only assume f non-decreasing and such that $f(x) > 0$, $\forall\, x > 0$.

Hints: (i) the Karamata theorem; (ii) the Chebyshev inequality.

Next, a very popular inequality.

Problem 2.8. Let $\lambda_1, \ldots, \lambda_n \in [0,1]$, $\mu_1, \ldots, \mu_n \in [0,1]$ be such that $\sum_{j=1}^{n} \lambda_j = \sum_{j=1}^{n} \mu_j = 1$ and $(\lambda_1, \ldots, \lambda_n)$ majorizes $(\mu_1, \ldots, \mu_n)$.

Given n numbers $x_1, \ldots, x_n$ and $\sigma \in S_n$, set

$$y_\sigma := \sum_{j=1}^{n} \lambda_j x_{\sigma(j)},\ z_\sigma := \sum_{j=1}^{n} \mu_j x_{\sigma(j)}.$$

If $f : I \to \mathbb{R}$ is convex and $x_1, \ldots, x_n \in I$, prove the Muirhead inequality

$$\sum_{\sigma \in S_n} f(z_\sigma) \le \sum_{\sigma \in S_n} f(y_\sigma). \tag{2.15}$$

Hints: start with the case where the μ_k's are permutations of the λ_j's. Then use the Birkhoff-von Neumann Theorem 3.1.

Problem 2.9. Let $n \geq 3$. Find

$$\max\left\{\sum_{j=1}^{n} x_j^3; x_j \in [0,1], \sum_{j=1}^{n} x_j = 0\right\}. \tag{2.16}$$

(Take for granted the fact that max is achieved in (2.16).) Hints: (i) the function $x \mapsto x^3$ is strictly convex on $[0, \infty)$, strictly concave on $(-\infty, 0]$; (ii) Problem 1.12.

Problem 2.10. If $0 < a_1 \leq \cdots \leq a_n$ and $\sigma \in S_n$, prove that

$$\begin{aligned}&(a_1 + a_1)(a_2 + a_2)\cdots(a_n + a_n)\\ &\leq \big(a_1 + a_{\sigma(1)}\big)\big(a_2 + a_{\sigma(2)}\big)\cdots\big(a_n + a_{\sigma(n)}\big)\\ &\leq (a_1 + a_n)(a_2 + a_{n-1})\cdots(a_n + a_1).\end{aligned} \tag{2.17}$$

Hint: broaden the perspective and try to prove a (much more) general result. Life will be easier!

Problem 2.11. Broaden the perspective in order to prove the inequality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{9}{a+b+c} \geq \frac{4}{a+b} + \frac{4}{b+c} + \frac{4}{c+a}, \ \forall\, a, b, c > 0. \tag{2.18}$$

Problem 2.12. Broaden the perspective in order to prove and improve the following inequality due to Drimbe.

$$\sum_{j=1}^{n} \sqrt{a_j c_j - b_j^2} \leq \sqrt{\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} c_j\right) - \left(\sum_{j=1}^{n} b_j\right)^2} \tag{2.19}$$

if $a_j > 0,\ c_j > 0,\ a_j c_j > b_j^2,\ \forall\, j$.

The following inequality is due to Drimbe. The idea of the proof is from [5].

Problem 2.13. If $f : I \to \mathbb{R}$ is such that f and f' are convex, then

$$\begin{aligned}&f(a) + f(b) + f(c) + 3f\left(\frac{a+b+c}{3}\right)\\ &\geq f\left(\frac{2a+b}{3}\right) + f\left(\frac{2b+a}{3}\right) + f\left(\frac{2a+c}{3}\right) + f\left(\frac{2c+a}{3}\right)\\ &\quad + f\left(\frac{2b+c}{3}\right) + f\left(\frac{2c+b}{3}\right), \ \forall\, a, b, c \in I.\end{aligned} \tag{2.20}$$

Hints. Assume that $a \leq b \leq c$. Use the convexity of f to prove that

$$f(a) + f\left(\frac{a+b+c}{3}\right) \geq f\left(\frac{2a+b}{3}\right) + f\left(\frac{2a+c}{3}\right). \tag{2.21}$$

Then use the convexity of f' to complete (2.21) to (2.20).

Problem 2.14. Broaden the perspective in order to prove the following result. If $b_1 \geq b_2 \geq \cdots \geq b_n > 0, a_1 \geq b_1, a_1 a_2 \geq b_1 b_2, \ldots, a_1 \cdots a_n \geq b_1 \cdots b_n$, then $\sum_{j=1}^{n} a_j \geq \sum_{j=1}^{n} b_j$.

2.3 Rearrangement

Problem 2.15. Formulate and prove the analogues of (R) for identically ordered, respectively oppositely ordered lists.

The following beautiful problem is taken from Drimbe [5].

Problem 2.16. Prove the (AM-GM) inequality using rearrangement inequalities. Hint: consider appropriate oppositely ordered lists $a_1, \ldots, a_n$ and $b_1 \ldots, b_n$ such that $a_j b_j = G$, $\forall\, j$.

2.4 Quadratic trinomials and forms

Problem 2.17. Assume that $a_1^2 > a_2^2 + \cdots a_n^2$. Prove the Aczél inequality

$$(a_1 b_1 - a_2 b_2 - \cdots - a_n b_n)^2 \geq \left(a_1^2 - a_2^2 - \cdots - a_n^2\right)\left(b_1^2 - b_2^2 - \cdots - b_n^2\right), \quad \forall\, b_1, \ldots, b_n \in \mathbb{R}, \tag{2.22}$$

and find the equality case.

Problem 2.18. Let $0 < a < A$ and $0 < b < B$. Assume that

$$a \leq a_j \leq A,\ b \leq b_j \leq B,\ \forall\, 1 \leq j \leq n. \tag{2.23}$$

(1) Prove the Pólya-Szegö inequality

$$\left(\sum_{j=1}^{n} a_j b_j\right)^2 \geq \frac{4abAB}{(ab + AB)^2} \sum_{j=1}^{n} a_j^2 \sum_{j=1}^{n} b_j^2. \tag{2.24}$$

(2) Prove that the constant $\dfrac{4abAB}{(ab + AB)^2}$ is optimal.

Problem 2.19. Let $3 \leq n \leq 6$. Prove the inequality

$$\sum_{j=1}^{n} \frac{x_j}{x_{j+1} + x_{j+2}} \geq \frac{n}{2},\ \forall\, x_1, \ldots, x_n > 0, \tag{2.25}$$

with the convention $x_{n+1} = x_1$ and $x_{n+2} = x_2$. Hints: (i) Problem 1.27; (ii) (CS).

(The case $n = 3$ is known as the Nesbitt inequality. The case $n = 6$ and the method of proof suggested above are due to Mordell.)

Problem 2.20. Prove that

$$a^2 + b^2 + c^2 + ab + bc + ca + a + b + c \geq -\frac{3}{8},\ \forall\, a, b, c \in \mathbb{R}, \tag{2.26}$$

and find the equality case.

2.5 Calculus

Problem 2.21. Prove that

$$\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\le\frac{1}{abc}, \forall\, a,b,c>0, \tag{2.27}$$

and characterize the equality case. Hints: (i) reduce, by homogeneity, (2.27) to the special case where $ab=1$ and $c\ge 1$; (ii) study the sum of the last two-terms on the left-hand side under the constraint $ab=1$; (iii) using exponentials may help.

Bonus: a tricky solution.

Problem 2.22. Prove that

$$(a+b+c+d)^2>8(ac+bd),\ \forall\, a<b<c<d. \tag{2.28}$$

Problem 2.23. For $n\ge 2$, find

$$m\coloneqq\min\left\{\sum_{j=1}^{n}\frac{x_j^{n-2}}{1-x_j^{n-1}};\ 0<x_j<1,\ \sum_{j=1}^{n}x_j^{n-1}=1\right\}.$$

Problem 2.24.

(1) Let $0<x<1$. Prove that

$$(0,\infty)\ni t\mapsto f(t)\coloneqq\frac{1-x^t}{t}$$

is decreasing. Hint: take for granted the inequality $e^y>1+y,\ \forall\, y\ne 0$.

(2) Let $1<p<2$ and let q be the conjugate of p. Use the previous item to prove that

$$\frac{p(p-1)}{2k(2k-p)}-\frac{p-1}{2k-p}x^{(2k-1)q-2k}+\frac{p-1}{2k}x^{2kq-2k}>0,\ \forall\, k\ge 1,\ \forall\, 0<x<1. \tag{2.29}$$

(3) Let $1<p<2$ and let q be the conjugate of p. Prove the Clarkson inequality

$$(1+x)^p+(1-x)^p\ge 2(1+x^q)^{p-1},\ \forall\, 0\le x\le 1. \tag{2.30}$$

Hints: (i) formula (1.74); (ii) the previous item.

Problem 2.25. This problem echoes Problem 1.29, dealing with the "almost equality" case in (H). Let $1 < p, q < \infty$ be conjugate exponents. By symmetry, we assume that $p \geq 2$. Let $a_1, \ldots, a_n, b_1 \ldots, b_n > 0$, $\alpha \geq 1$, and assume that

$$\frac{1}{\alpha} \leq \frac{a_j}{b_j^{q-1}} \leq \alpha, \ \forall\, j. \tag{2.31}$$

If $\alpha = 1$, then (1.4) and (2.31) yield

$$\frac{1}{p}\sum_{j=1}^{n} a_j^p + \frac{1}{q}\sum_{j=1}^{n} b_j^q = \sum_{j=1}^{n} a_j b_j. \tag{2.32}$$

On the other hand, (Y) shows that we always have "$\geq$" in (2.32). We quantify below how much we deviate from (2.32) under the assumption (2.31). Let

$$f(a) \coloneqq \frac{1}{a}\left(\frac{1}{p}a^p + \frac{1}{q}\right), \ \forall\, a > 0.$$

(1) When $p = 2$, prove that

$$\max_{1/\alpha \leq a \leq \alpha} f(a) = \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right). \tag{2.33}$$

(2) Derive the Diaz-Metcalf inequality

$$\sum_{j=1}^{n} a_j^2 + \sum_{j=1}^{n} b_j^2 \leq \left(\alpha + \frac{1}{\alpha}\right)\sum_{j=1}^{n} a_j b_j \text{ if } a_1, \ldots, b_n > 0 \text{ and } \frac{1}{\alpha} \leq \frac{a_j}{b_j} \leq \alpha, \ \forall\, j.$$

(3) When $p > 2$, write $a = e^x$ and set $g(x) \coloneqq f(a) = f(e^x)$, $\forall\, x \in \mathbb{R}$. Prove that $g(x) > g(-x)$, $\forall\, x > 0$. Hint:

$$e^y = 1 + \frac{y}{1!} + \frac{y^2}{2!} + \cdots, \ \forall\, y \in \mathbb{R}.$$

(4) Derive the inequality

$$\begin{gathered}\frac{1}{p}\sum_{j=1}^{n} a_j^p + \frac{1}{q}\sum_{j=1}^{n} b_j^q \leq \frac{1}{\alpha}\left(\frac{1}{p}\alpha^p + \frac{1}{q}\right)\sum_{j=1}^{n} a_j b_j \\ \text{if } p \geq 2,\ a_1, \ldots, b_n > 0, \text{ and } \frac{1}{\alpha} \leq \frac{a_j}{b_j^{q-1}} \leq \alpha, \ \forall\, j.\end{gathered} \tag{2.34}$$

(5) Equality cases in (2.34)?

2.6 Optimisation

Problem 2.26.

(1) Let $0 < x < 1$ and $1/2 < \alpha < 1$. Prove that

$$(1+x)^{\alpha} + (1-x)^{\alpha} > (1+x)^{1-\alpha} + (1-x)^{1-\alpha}. \tag{2.35}$$

Hint: formula (1.74).

(2) Let $1 < p < 2$. Prove that

$$(a+b)^p \leq a^p + b^p + (2^p - 2)(ab)^{p/2}, \forall\, a, b > 0. \tag{2.36}$$

Hint: consider a constrained minimization problem.

(3) Prove that (2.36) improves both the following special case of (J)

$$\left(\frac{a+b}{2}\right)^p \leq \frac{1}{2}a^p + \frac{1}{2}b^p,\ \forall\, a, b > 0, \tag{2.37}$$

and, for x sufficiently close to 0, the Clarkson inequality (2.30).
Hint: use the calculus inequality

$$(1+x)^{\alpha} \leq 1 + \alpha x,\ \forall\, 0 < \alpha < 1,\ \forall\, x \geq -1. \tag{2.38}$$

Problem 2.27. Solve the system

$$\begin{cases} 2^x + 2^y = 3 \\ x^2 + y^2 = 1 \end{cases}. \tag{2.39}$$

Problem 2.28. Let $n \geq 3$. Prove that

$$\prod_{j=1}^{n} a_j \sum_{j=1}^{n} \frac{1}{a_j} \leq \frac{1}{n^{(n-3)/2}} \left(\sum_{j=1}^{n} a_j^2\right)^{(n-1)/2},\ \forall\, a_1, \ldots, a_n > 0. \tag{2.40}$$

Hint: we have $\left(1 + \dfrac{1}{k}\right)^k < e = 2.71828...,\forall\, k \geq 1.$

Problem 2.29. Broaden the perspective of (2.40). Rewrite (2.40) as

$$M_0^n \leq M_{-1} M_2^{n-1},\ \forall\, a_1, \ldots, a_n > 0 \tag{2.41}$$

(with notation as in (1.7)–(1.8)). "Embed" (2.41) into a family of inequalities involving M_0, M_{-r}, and M_s, with $r, s > 0$.

Problem 2.30. To motivate this problem, we start from the inequality

$$\sqrt{a_1 a_2} + \sqrt{b_1 b_2} \le \sqrt{(a_1 + a_2)(b_1 + b_2)},\ \forall\, a_1, a_2, b_1, b_2 > 0, \tag{2.42}$$

which can be easily proved by taking squares and using (AM-GM).

Given $0 < p < \infty$ and an integer $n \ge 2$, consider the following potential generalization of (2.42).

$$\left(\prod_{j=1}^{n} a_j\right)^{1/p} + \left(\prod_{j=1}^{n} b_j\right)^{1/p} \le \left(\prod_{j=1}^{n} (a_j + b_j)\right)^{1/p}, \quad \forall\, a_1, \dots, a_n, b_1 \dots, b_n > 0, \tag{2.43}$$

which, after setting $a_j = x_j^p$, $b_j = y_j^p$, is equivalent to

$$\left(\prod_{j=1}^{n} x_j + \prod_{j=1}^{n} y_j\right)^{p} \le \prod_{j=1}^{n} (x_j^p + y_j^p),\ \forall\, x_1, \dots, x_n, y_1 \dots, y_n > 0. \tag{2.44}$$

Prove that (2.44) (and thus (2.43)) holds iff $p \le n$. Hint: take for granted the existence of a minimizer of a constrained minimization problem naturally associated with (2.44).

When $p = n$, (2.43) is known as the Huygens inequality.

2.7 Conditional inequalities

The common theme here is the study of inequalities valid for numbers satisfying typical conditions such as being ordered, being side lengths of a triangle, etc. A basic question will be: how to translate such assumptions into more tractable ones?

Problem 2.31. Let $I \subset \mathbb{R}$ be an interval, $f : I \to [0, \infty)$ be a monotonic function, and $a, b, c \in I$. Prove the Schur inequality

$$(a - b)(a - c)f(a) + (b - c)(b - a)f(b) + (c - a)(c - b)f(c) \ge 0. \tag{2.45}$$

The following is a must-know property.

Problem 2.32. Let $a, b, c > 0$. Prove that a, b, c are the side lengths of a triangle iff there exist numbers $u, v, w > 0$ such that

$$a = u + v, b = v + w, c = w + u. \tag{2.46}$$

Problem 2.33. If a, b, c are the side lengths of a triangle, then

$$a^3 + b^3 + c^3 \geq a(b-c)^2 + b(c-a)^2 + c(a-b)^2 + 3abc. \tag{2.47}$$

Hint: Problems 2.32 and 2.4.

Though inequality (2.49) below looks quite simple, I do not know any "obvious" proof of it. The proof below is from Drimbe [5].

Problem 2.34. (1) If a, b, c are the side lengths of an acute triangle, prove that

$$\left(a^2 + b^2 - c^2\right)\left(b^2 + c^2 - a^2\right) \leq (a+b-c)^2(b+c-a)^2. \tag{2.48}$$

(2) If $a, b, c > 0$, prove that

$$\begin{aligned} &\left(a^2 + b^2 - c^2\right)\left(b^2 + c^2 - a^2\right)\left(c^2 + a^2 - b^2\right) \\ &\quad \leq (a+b-c)^2(b+c-a)^2(c+a-b)^2. \end{aligned} \tag{2.49}$$

The following was already noticed in the proof of Theorem 1.3; see the solution of Problem 1.19.

Problem 2.35. A (finite or infinite) sequence $a_1, a_2, a_3, \ldots,$ is non-decreasing iff there exist non-negative numbers $x_2, \ldots,$ such that $a_2 = a_1 + x_2, a_3 = a_1 + x_2 + x_3, \ldots$

Problem 2.36. By definition, a (finite or infinite) sequence $a_1, a_2, a_3, \ldots,$ is convex iff $a_j \leq \dfrac{a_{j-1} + a_{j+1}}{2}$ for each $j \geq 2$ for which this inequality makes sense.

Reformulate the convexity property in terms of non-negative numbers.

Problem 2.37. Prove the following analogue of Theorem 1.3.

Let $n \geq 3$ and $\sigma, \tau \in S_n$. Then the following are equivalent:

(1) For each convex lists $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, we have

$$S_\sigma := \sum_{j=1}^{n} a_j b_{\sigma(j)} \leq S_\tau := \sum_{j=1}^{n} a_j b_{\tau(j)}. \tag{2.50}$$

(2) We have $\sigma(1) = \tau(1)$, $\sigma^{-1}(1) = \tau^{-1}(1)$, and, for each $3 \leq k, \ell \leq n$,

$$\operatorname{Card}\{j \geq k;\ \sigma(j) \geq \ell\} \leq \operatorname{Card}\{j \geq k;\ \tau(j) \geq \ell\}. \tag{2.51}$$

Problem 2.38. (1) Prove that

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{1}{1+ab}, \ \forall\, a, b > 0. \tag{2.52}$$

Case of equality?

(2) If $ab \geq \frac{1}{4}$, improve the above inequality to

$$\frac{1}{(1+a)^2} + \frac{1}{(1+b)^2} \geq \frac{2}{(1+\sqrt{ab})^2}. \tag{2.53}$$

Case of equality?

Hint: work with $S \coloneqq a + b$ and $P \coloneqq ab$, and use the fact that these quantities are not independent.

Problem 2.39. Prove that

$$a^2 + b^2 + c^2 - ab - bc - ca \leq a^2b + b^2c + c^2a - 3abc, \ \forall\, a, b, c \geq 1. \tag{2.54}$$

Problem 2.40. If a, b, c, d are the side lengths of a quadrilateral, prove that

$$\begin{aligned}&(a+b+c-d)(-a+b+c+d)(a-b+c+d)(a+b-c+d)\\ &\leq (a+b)(b+c)(c+d)(d+a).\end{aligned} \tag{2.55}$$

Problem 2.41. If $a, b, c, d > 0$ are such that $2a > b$, $2b > c$, $2c > d$, and $2d > a$, prove that

$$(2a-b)(2b-c)(2c-d)(2d-a) \leq abcd. \tag{2.56}$$

Chapter 3

More on inequalities

3.1 Magic squares

Definition 3.1. A square matrix $P = (p_{jk})_{1 \le j,k \le n}$ is a *permutation matrix* iff on each line and column, exactly one entry is 1, the other ones being 0.

Equivalently, there exists a permutation $\sigma \in S_n$ such that $p_{jk} = \begin{cases} 1, & \text{if } k = \sigma(j) \\ 0, & \text{if } k \neq \sigma(j) \end{cases}$. The matrix is then denoted P_σ.

Theorem 3.1 (Birkhoff-von Neumann theorem)**.** A matrix A is doubly stochastic iff: there exist $\lambda_\sigma \in [0, 1]$, $\forall\, \sigma \in S_n$, such that $\sum_{\sigma \in S_n} \lambda_\sigma = 1$ and $A = \sum_{\sigma \in S_n} \lambda_\sigma P_\sigma$.

Moreover, when $n \ge 4$ we may choose the coefficients λ_σ such that at most $n^2 + 1$ of them are non-zero.

In other words, when $n \ge 4$, the magic squares of sum 1 are precisely the (convex) combinations of at most $n^2 + 1$ permutation matrices.

Remark 3.1. Despite what its name suggests, this result was first obtained by König [10].

Problems 3.1–3.4, leading to the constructive proof of the first part of Theorem 3.1 we propose below, are inspired by Hurlbert [8].

Problem 3.1. Prove that a matrix as in Theorem 3.1 is doubly stochastic.

Problem 3.2. (1) A DS matrix has at least n non-zero entries.

(2) A DS matrix with exactly n non-zero entries is a permutation matrix.

Problem 3.3. Let A be a DS matrix. A *cycle* (of length 2ℓ) in A is a sequence of mutually distinct entries $a_{j_1k_1}, a_{j_1k_2}, a_{j_2k_2}, a_{j_2k_3}, \ldots, a_{j_\ell k_1}$, all different of 0 and 1. (Thus: the first two entries are on the same line, the second and the third on the same column, the third and the fourth on the same line,..., the last one and the first one on the same column.)

Prove that, if A is a DS matrix which is not a permutation matrix, then it contains a cycle.

Problem 3.4. (1) Let A be a DS matrix which is not a permutation matrix, and consider a cycle $a_{j_1k_1}, a_{j_1k_2}, a_{j_2k_2}, a_{j_2k_3}, \ldots, a_{j_\ell k_1}$ as above. By replacing the cycle with $a_{j_1k_1} - \alpha, a_{j_1k_2} + \alpha, a_{j_2k_2} - \alpha, a_{j_2k_3} + \alpha, \ldots, a_{j_\ell k_1} + \alpha$, respectively $a_{j_1k_1} + \beta, a_{j_1k_2} - \beta, a_{j_2k_2} + \beta, a_{j_2k_3} - \beta, \ldots, a_{j_\ell k_1} - \beta$, for appropriate $\alpha, \beta > 0$, write $A = (1-t)B + tC$, where $0 < t < 1$, where B, C are DS, and have more zero entries than A.

(2) Prove Theorem 3.1 by backward induction on the number of zero entries of A.

At this stage, we know that Theorem 3.1 holds, but in principle we need as many scalars λ_σ as permutation matrices, i.e., $n!$. To reduce this number, we use the following result.

Theorem 3.2 (Steinitz lemma)**.** Let V be a vector space of dimension k. Let $m \geq k+1$ and $x_1, \ldots, x_m \in V$. If x is a convex combination of $x_1, \ldots, x_m$, then there exist $x_{j_1}, \ldots, x_{j_{k+1}}$ such that x is a convex combination of $x_{j_1}, \ldots, x_{j_{k+1}}$.

For example, a convex combination of ten points in the plane is a convex combination of three of them.

Problem 3.5. Prove that the Steinitz lemma implies the second part of Theorem 3.1.

We next present a proof of the Steinitz lemma, relying only on the following textbook fact. Given $m \geq k+1$ vectors in a vector space of dimension k, one of them is a linear combination of the others.

Proof of Theorem 3.2. We prove that, if $m > k+1$, one may select at most $(m-1)$ points among $x_1, \ldots, x_m$ such that x still is a convex combination of the remaining points.

By subtracting x_1 from all the x_j's and from x, we may assume that $x_1 = 0$. By the fact recalled above, one of the vectors $x_2, \ldots, x_m$ is a linear combination of the remaining ones. With no loss of generality, we may assume that there exist scalars μ_j such that

$$x_m = \sum_{j=2}^{m-1} \mu_j x_j. \tag{3.1}$$

On the other hand, by assumption we may write

$$x = \sum_{j=1}^{m} \lambda_j x_j, \text{ where } \lambda_j \geq 0, \forall j, \text{ and } \sum_{j=1}^{m} \lambda_j = 1. \tag{3.2}$$

We may assume that $\lambda_j > 0$, $\forall j$, for otherwise we may remove x_j from the convex combination.

For every $t \in \mathbb{R}$, we have (using (3.1), (3.2), and the assumption $x_1 = 0$)

$$x = \left(\lambda_1 - \sum_{j=2}^{m-1} t\mu_j + t\right) x_1 + \sum_{j=2}^{m-1} (\lambda_j + t\mu_j) x_j + (\lambda_m - t) x_m. \tag{3.3}$$

We note that the coefficients were chosen such that their sum is 1.

Now comes the key argument. For small $t > 0$, all the coefficients in (3.3) are positive (since each λ_j is > 0). On the other hand, for $t > \lambda_m$ the last coefficient is negative. Therefore, the set

$$M := \{t > 0; \text{ all the coefficients in (3.3) are positive}\}$$

is non-empty and bounded. If we take $t := \sup A$, it is routine that all the coefficients in (3.3) are ≥ 0, and for at least one j the coefficient in front of x_j vanishes. Thus, for this t and j, (3.3) represents x as a convex combination of $x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_m$. QED

Finally, we propose an improvement of the second part of Theorem 3.1.

Problem 3.6. Prove that a DS matrix A can be written as $A = \sum_{\sigma \in S_n} \lambda_\sigma P_\sigma$, where $\lambda_\sigma \in [0, 1]$, $\forall \sigma \in S_n$, $\sum_{\sigma \in S_n} \lambda_\sigma = 1$, and at most $(n-1)^2 + 1$ of the coefficients λ_σ are non-zero. Hint: identify the set of DS matrices with a part of a vector space of dimension $(n-1)^2$.

3.2 Equal variables method

The method we explain in this section was devised by Cîrtoaje [3]. Since it leads to a very large number of possible statements, we rather focus on the method and the heuristics, and present only one statement and one application. Many others can be found in [3] and the references therein.

A typical problem to which the method applies is

$$\begin{aligned}&\text{maximize} \sum_{j=1}^{n} f(x_j), \text{ under the constraints}\\&\qquad a < x_j < b,\ \forall\, j,\ \sum_{j=1}^{n} g(x_j) = S,\ \sum_{j=1}^{n} h(x_j) = T.\end{aligned} \tag{3.4}$$

Here, $n \geq 3$, and $f, g, h : (a, b) \to \mathbb{R}$ are smooth functions. In order to simplify the presentation, we consider the special case where $h = \mathrm{id}$, but the method can be adapted to the case of a general h. From now, we thus consider the following special case of (3.4)

$$\begin{aligned}&\text{maximize} \sum_{j=1}^{n} f(x_j), \text{ under the constraints}\\&\qquad a < x_j < b,\ \forall\, j,\ \sum_{j=1}^{n} g(x_j) = S,\ \sum_{j=1}^{n} x_j = T.\end{aligned} \tag{3.5}$$

Assume that $X = (x_1, \ldots, x_n)$ solves (3.5), and thus in particular that the set of competitors in (3.5) is non-empty. With no loss of generality, we may assume that $x_1 \leq x_2 \leq \cdots \leq x_n$. The equal variables method consists of finding conditions on f and g such that, at a maximum point in (3.5), we must have

$$x_1 \leq x_2 = \cdots = x_n. \tag{3.6}$$

(As a variant, we may want to have $x_1 = \cdots = x_{n-1} \leq x_n$.) When the method can be successfully implemented, it thus reduces the study of an inequality for n variables, $x_1, \ldots, x_n$, to the study of one for two variables, x_1 and x_2.

The approach is by contradiction: we assume that (3.6) does not hold, and obtain that X does not solve (3.5). Since (3.6) does not hold, there exists some j such that $x_j \leq x_{j+1} < x_{j+2}$. Set

$$Y \coloneqq (x_j, x_{j+1}, x_{j+2}),\ \overline{S} \coloneqq g(x_j) + g(x_{j+1}) + g(x_{j+2}),\ \overline{T} \coloneqq x_j + x_{j+1} + x_{j+2}.$$

Since X solves (3.5), we find that Y solves

$$\begin{aligned}&\text{maximize}\,(f(x) + f(y) + f(z)),\ \text{under the constraints}\\&\qquad a < x, y, z < b,\ g(x) + g(y) + g(z) = \overline{S},\ x + y + z = \overline{T}.\end{aligned} \tag{3.7}$$

Consider now the system

$$\begin{cases} g(x) + g(z) = \overline{S} - g(y) \\ x + z = \overline{T} - y \end{cases}. \tag{3.8}$$

When $y = x_{j+1}$, the system has the solution $(x, z) = (x_j, x_{j+2})$, with $x \le y < z$. We will now impose a condition ensuring that, for y "close" to x_{j+1}, the system has a solution (x, z) "close" to (x_j, x_{j+2}). This relies on an advanced calculus theorem (the inverse function theorem), about which the Wikipedia executive summary [15] provides some insight. More specifically, we have the following consequence of the inverse function theorem. Consider the system

$$\begin{cases} G(x,z) = k(y) \\ H(x,z) = \ell(y) \end{cases},$$

where all the functions are smooth. If: (i) for $y = \overline{y}$, the system has a solution $(\overline{x}, \overline{z})$; (ii) the determinant

$$\begin{vmatrix} \dfrac{\partial G}{\partial x}(\overline{x},\overline{z}) & \dfrac{\partial G}{\partial z}(\overline{x},\overline{z}) \\ \dfrac{\partial H}{\partial x}(\overline{x},\overline{z}) & \dfrac{\partial H}{\partial z}(\overline{x},\overline{z}) \end{vmatrix} = \frac{\partial G}{\partial x}(\overline{x},\overline{z}) \times \frac{\partial H}{\partial z}(\overline{x},\overline{z}) - \frac{\partial G}{\partial z}(\overline{x},\overline{z}) \times \frac{\partial H}{\partial x}(\overline{x},\overline{z})$$

(known as "Jacobian determinant") is different from 0, then: for y close to $\overline{y}$, the system has smooth solution $(x(y), z(y))$ such that $x(\overline{y}) = \overline{x}$ and $z(\overline{y}) = \overline{z}$.

In the case of (3.8), the Jacobian determinant at is

$$\begin{vmatrix} g'(\overline{x}) & g'(\overline{z}) \\ 1 & 1 \end{vmatrix} = g'(\overline{x}) - g'(\overline{z}),$$

and thus a natural condition on g in order to apply the above is to assume that g' is strictly monotonic. Since the discussion below depends on the monotonicity of g' and we want just to consider a single illustration of the method, we assume, e.g., that

$$g' \text{ is (strictly) decreasing} \tag{3.9}$$

(which, in particular, implies that g is strictly concave). Under this assumption, the inverse function theorem applies (with $\overline{y} = x_{j+1}$, $k(x) = \overline{S} - g(x)$, and $\ell(x) = \overline{T} - x$), and thus $x(y)$ and $z(y)$ as above exist.

Consider, for y close to x_{j+1}, the quantity

$$F(y) := f(x(y)) + f(y) + f(z(y)),$$

whose maximum is achieved when $y = x_{j+1}$ (since Y solves (3.7) and $(x(y), y, z(y))$ is a competitor in (3.7)).

In order to contradict this fact (and thus achieve the implementation of the method), we will calculate $F'(y)$ and impose conditions ensuring that F does not have a maximum at $y = x_{j+1}$. For this purpose, we start from the system (3.8), which yields

$$\begin{cases} g(x(y)) + g(z(y)) = \overline{S} - g(y) \\ x(y) + z(y) = \overline{T} - x \end{cases},$$

and therefore we have

$$\begin{cases} g'(x(y))x'(y) + g'(z(y))z'(y) = -g'(y) \\ x'(y) + z'(y) = -1 \end{cases}. \tag{3.10}$$

When y is sufficiently close to x_{j+1}, we have $y < z(y)$ and $x(y) < z(y)$ (since $x_{j+1} < z(x_{j+1}) = x_{j+2}$ and $x(x_{j+1}) = x_j < z(x_{j+1}) = x_{j+2}$). Thanks to (3.9), for such y we thus have $g'(x(y)) > g'(z(y))$. Using this information, we may solve (3.10) and find that

$$\begin{cases} x'(y) = -\dfrac{g'(y) - g'(z(y))}{g'(x(y)) - g'(z(y))} < 0 \\ z'(y) = -\dfrac{g'(x(y)) - g'(y)}{g'(x(y)) - g'(z(y))} \end{cases}. \tag{3.11}$$

Therefore, we have (writing, for simplicity, $x = x(y)$, $z = z(y)$)

$$\begin{aligned} F'(y) =& f'(x)x'(y) + f'(y) + f'(z)z'(y) \\ =& f'(y) - \frac{g'(y) - g'(z)}{g'(x) - g'(z)} f'(x) - \frac{g'(x) - g'(y)}{g'(x) - g'(z)} f'(z). \end{aligned} \tag{3.12}$$

Using the above and (3.9), we find that $F'(y)$ and $K(x(y), y, z(y))$ have the same sign, where

$$K(x, y, z) \coloneqq f'(y) - \frac{g'(y) - g'(z)}{g'(x) - g'(z)} f'(x) - \frac{g'(x) - g'(y)}{g'(x) - g'(z)} f'(z). \tag{3.13}$$

In turn, the expression of K has the flavor of the Jensen inequality. In order to make this more obvious, we express K differently. By assumption (3.9), $g' : (a, b) \to g'((a, b))$ is invertible. Set $k \coloneqq (g')^{-1}$ and $r \coloneqq f' \circ k$. Let $a < x, y < z < b$. Then, with $\alpha \coloneqq g'(x), \beta \coloneqq g'(y), \gamma \coloneqq g'(z)$, we have $\alpha, \beta > \gamma$ and

$$\begin{aligned} K(x, y, z) =& r(\beta) - \frac{\beta - \gamma}{\alpha - \gamma} r(\alpha) - \frac{\alpha - \beta}{\alpha - \gamma} r(\gamma) \\ =& r\left(\frac{\beta - \gamma}{\alpha - \gamma}\alpha + \frac{\alpha - \beta}{\alpha - \gamma}\gamma\right) - \frac{\beta - \gamma}{\alpha - \gamma} r(\alpha) - \frac{\alpha - \beta}{\alpha - \gamma} r(\gamma), \end{aligned}$$

so that

$$F'(y) = r\left(\frac{\beta - \gamma}{\alpha - \gamma}\alpha + \frac{\alpha - \beta}{\alpha - \gamma}\gamma\right) - \frac{\beta - \gamma}{\alpha - \gamma} r(\alpha) - \frac{\alpha - \beta}{\alpha - \gamma} r(\gamma). \tag{3.14}$$

(In (3.14), we have $\alpha = \alpha(y) = g'(x(y))$, $\beta = \beta(y) = g'(y)$, $\gamma = \gamma(y) = g'(z(y))$, and $\alpha, \beta > \gamma$.)

This suggests the following statement [3].

Theorem 3.3 (Equal variables theorem)**.** Consider the maximization problem (3.5). Assume that: (i) g' is strictly decreasing; (ii) $f' \circ ((g')^{-1})$ is strictly concave.

Let $n \geq 3$. If $X = (x_1, \ldots, x_n)$, with $x_1 \leq x_2 \leq \cdots \leq x_n$, solves (3.5), then $x_1 \leq x_2 = \cdots = x_n$.

Proof. We use the same notation as above. If $x_j < x_{j+1}$, then $\alpha(x_{j+1}) > \beta(x_{j+1}) > \gamma(x_{j+1})$. Therefore, (3.14), the assumption (ii), and (J) imply that $F'(x_{j+1}) > 0$, and this contradicts the fact that F achieves its maximum at $y = x_{j+1}$.

Consider now the case where $x_j = x_{j+1}$, and thus $x(x_{j+1}) = x_{j+1}$ and $\alpha(x_{j+1}) = \beta(x_{j+1})$. By the first equation in (3.11), we have $x'(x_{j+1}) < 0$, and thus, for $y > x_{j+1}$ close to x_j, we have $x(y) < x(y_{j+1}) = x_{j+1} < y$ and therefore $\alpha(y) > \beta(y)$. For such y, we have, arguing as above, $F'(y) > 0$. This is again a contradiction. QED

Problem 3.7. Prove the following version of the equal variable theorem. Assume that: (i′) g' is strictly monotonic; (ii) $f' \circ ((g')^{-1})$ is strictly concave.

Let $n \geq 3$. If $X = (x_1, \ldots, x_n)$, with $x_1 \leq x_2 \leq \cdots \leq x_n$, solves (3.5), then $x_1 \leq x_2 = \cdots = x_n$.

A very quick worked application.

Problem 3.8. Let $n = 3$ or 4. Prove that

$$\left(\sum_{j=1}^{n} x_j\right)^n \leq (n-1)^{n-1} \sum_{j=1}^{n} x_j^n + n\big(n^{n-1} - (n-1)^{n-1}\big) \prod_{j=1}^{n} x_j, \quad \forall\, x_1, \ldots, x_n > 0. \tag{3.15}$$

Bonus. Prove that (3.15) still holds for $n = 5, 6, \ldots$

Proof. Step 1. Use of the equal variables method. Set $y_j := x_j^n$, $g(x) := \ln x$, and $f(x) := x^{1/n}$. Then (3.15) amounts to

$$\left[y_j > 0,\ \forall\, j,\ \sum_{j=1}^{n} g(y_j) = S,\ \sum_{j=1}^{n} y_j = T\right] \Rightarrow \sum_{j=1}^{n} f(y_j) \leq \Big((n-1)^{n-1} T + n\Big(n^{n-1} - (n-1)^{n-1}\Big) e^{S/n}\Big)^{1/n}. \tag{3.16}$$

Clearly, the assumptions of Theorem 3.3 are satisfied. Moreover, using the "a continuous function on a compact set has a maximum point" argument, we see that, if the set of competitors in (3.16) is non-empty, then the maximum of $\sum_{j=1}^{n} f(y_j)$ is achieved. We find that, if $Y =$

$(y_1, \ldots, y_n)$, with $y_1 \le y_2 \le \cdots \le y_n$, maximizes $\sum_{j=1}^n f(y_j)$ in (3.16), then we have $y_1 \le y_2 = \cdots = y_n$. Therefore, it suffices to check the validity of (3.16) when $y_1 \le y_2 = \cdots = y_n$. Going back to (3.15), we find that it suffices to check its validity when $x_1 \le x_2 = \cdots = x_n$. Moreover, with no loss of generality, we may assume, by a homogeneity argument, that $x_n = 1$.

Step 2. Proof of (3.15). We have reduced (3.15) to proving that

$$((n-1)+x)^n \le (n-1)^n + (n-1)^{n-1}x^n + n\Big(n^{n-1} - (n-1)^{n-1}\Big)x, \quad \forall\, 0 < x \le 1. \tag{3.17}$$

When $n = 3$, (3.17) amounts to $x(x-1)^2 \ge 0$. On the other hand, when $n = 4$, (3.17) amounts to $x(13x+20)(x-1)^2 \ge 0$.

Step 3. Hint for the bonus. By (J), we have

$$x^j \le \frac{j-1}{n-1}x^n + \frac{n-j}{n-1}x,\ \forall\, 1 \le j \le n.$$

Therefore, we have

$$((n-1)+x)^n \le (n-1)^n + \sum_{j=1}^n \binom{n}{j}(n-1)^{n-j}\left[\frac{j-1}{n-1}x^n + \frac{n-j}{n-1}x\right]. \tag{3.18}$$

To conclude, it suffices to check (using basing combinatorics) that the right-hand sides of (3.17) and (3.18) coincide. QED

3.3 Refined means inequalities

The general theme of this section is the following: given $r_{-1} < r_0 < r_1$, "refine" the (MI) inequality $M_{r_0} \le M_{r_1}$ to an "optimal" inequality involving also $M_{r_{-1}}$. To make this more concrete, we start with a very simple example.

Problem 3.9. (1) If $x, y > 0$, prove that their means satisfy

$$G \le \frac{1}{2}A + \frac{1}{2}H. \tag{3.19}$$

(2) Prove that (3.19) is "optimal", in the sense that, if

$$G \le \theta A + (1-\theta)H, \forall\, x, y > 0, \tag{3.20}$$

then $\theta \ge \dfrac{1}{2}$ and (3.20) is a consequence of (3.19).

Before proceeding further, let us note that, since

$$\frac{1}{2}A + \frac{1}{2}H \le \frac{1}{2}A + \frac{1}{2}G,$$

(3.19) is indeed stronger than $G \le A$.

Compared to the general theme stated at the beginning of the section, in Problem 3.9 we have $r_{-1} = -1$, $r_0 = 0$, $r_1 = 1$, and the refinement of $M_0 \le M_1$ is (3.19), which reads $M_0 \le \frac{1}{2}M_{-1} + \frac{1}{2}M_1$.

We already have at our disposal another result in the same vein, Problem 3.8, since (3.15) can be rewritten as

$$M_1^n \le \theta M_n^n + (1-\theta)M_0^n, \forall\, x_1, \ldots, x_n > 0, \text{ with } \theta = \left(\frac{n-1}{n}\right)^{n-1}. \tag{3.21}$$

Yet another example is provided by Problem 1.32: (1.85) can be rewritten as

$$M_1^2 \le \theta M_2^2 + (1-\theta)M_0^2,\ \forall\, x_1, \ldots, x_n > 0, \text{ with } \theta = \frac{n-1}{n}. \tag{3.22}$$

The next problem addresses the matter of the optimality of the two above estimates.

Problem 3.10. Prove that (3.21) is optimal, in the sense that, if (3.21) holds for some θ, then $\theta \ge \left(\frac{n-1}{n}\right)^{n-1}$.

Same question for (3.22).

We next consider similar inequalities, and use calculus, as well as the equal variables method from the previous section, to tackle them. As we will see, the answers depend not only on the form of the inequality we are looking for, but also on the relation between the r_j', n, and possibly another parameter, q. As a warning: these are some of the very few instances where the optimal answer is known. (Some others can be found in [11].) However, in general, the answer is widely open. While one can usually prove the existence of an optimal θ, and even to characterize the optimal θ as the maximal value of a numeric function (see the solution to Problem 3.18), its precise value can be difficult to find.

In Problems 3.11–3.17, we let $n = 2$. In this case, we denote $x = x_1$, $y = x_2$. The case $n \ge 3$, which is more difficult, is more superficially tackled in Problems 3.18–3.22. To start with, a simple inequality.

Problem 3.11. If $0 < r, q < \infty$ and $x, y > 0$, prove the inequality

$$M_0^q \le \frac{1}{2}M_r^q + \frac{1}{2}M_{-r}^q, \tag{3.23}$$

and establish its optimality.

Next, a more involved example.

Problem 3.12. If $r > 0$ and $x, y > 0$, prove the inequality

$$M_0^r \leq \frac{1}{r+1} M_r^r + \frac{r}{r+1} M_{-1}^r = \frac{1}{r+1} M_r^r + \left(1 - \frac{1}{r+1}\right) M_{-1}^r, \tag{3.24}$$

and establish its optimality. Hints: (i) take $x = e^t$, with $t > 0$, and $y = 1/x$; (ii) for this choice, prove that the right-hand side of (3.24) increases with t; (iii) it may be useful to use the textbook power series expansions

$$\sinh s = \frac{s^1}{1!} + \frac{s^3}{3!} + \frac{s^5}{5!} + \cdots, \ \forall s \in \mathbb{R}, \tag{3.25}$$

$$\cosh s = \frac{s^0}{0!} + \frac{s^2}{2!} + \frac{s^4}{4!} + \cdots, \ \forall s \in \mathbb{R}. \tag{3.26}$$

Problem 3.13. Let $0 < r \leq 1$, $q > 0$, and $x, y > 0$. Prove the inequality

$$M_0^q \leq \frac{1}{r+1} M_r^q + \frac{r}{r+1} M_{-1}^q, \tag{3.27}$$

and establish its optimality. A possible strategy is the following. (1) Reduce the problem to the case where $x > 1$ and $y = 1/x$. (2) Reduce the problem to a case where $q = 0$, and use calculus.

Bonus. Prove that the above result does not hold when $r > 1$.

Here is a similar result.

Problem 3.14. Let $1 < r \leq 2$, $q \geq 1$, and $x, y > 0$. Prove the inequality

$$M_0^q \leq \frac{1}{r+1} M_r^q + \frac{r}{r+1} M_{-1}^q, \tag{3.28}$$

and establish its optimality. Hints: (i) reduce the problem to the case where $x = e^t > 1$, $y = e^{-t}$, and $q = 1$; (ii) prove that $\sinh(rt) > r \sinh(t)$ when $r > 1$ and $t > 0$; (iii) use the identity $\cosh(2s) = 2\cosh^2 s - 1$, combined with the fact that $r \leq 2$.

Bonus. Prove that (3.28) does not hold for $p = 3$ and $q = 1$. Hint: let $y = 1/x$ and express (3.28) in terms of $z := x + \dfrac{1}{x}$.

Next, a more tricky inequality.

Problem 3.15. Let $1 \leq r \leq 2$ and $x, y > 0$. Prove the inequality

$$M_1^r \leq \frac{1}{2^{r-1}} M_r^r + \frac{2^{r-1} - 1}{2^{r-1}} M_0^r = \frac{1}{2^{r-1}} M_r^r + \left(1 - \frac{1}{2^{r-1}}\right) M_0^r, \tag{3.29}$$

and establish its optimality.

Proof. Step 1. Initial reductions. When $r = 1$ or $r = 2$, (3.29) becomes an identity. We may therefore assume that $1 < r < 2$. By homogeneity and symmetry, we may assume that $x \geq y$ and $x + y = 2$. Writing $x = 1 + t$, $1 = 1 - t$, we have to prove that $f(t) \geq 2^r$, $\forall\, 0 \leq t \leq 1$, where

$$f(t) \coloneqq (1+t)^r + (1-t)^r + \alpha P(t),\ \alpha \coloneqq 2^r - 2,\ P(t) \coloneqq \left(1 - t^2\right)^{r/2}.$$

Step 2. Heuristics. Since $f(0) = f(1) = 2^r$, f cannot be monotonic. In order to conclude, one option is to prove that f is strictly concave. (This, combined with the fact that $f(0) = f(1)$, would imply that $f(t) > f(0)$, $\forall\, 0 < t < 1$, and allow us to conclude.) We will take a different route, that we now explain. Since "a continuous function on a compact set has a maximum point and a minimum point", f achieves its minimum at some point t in $[0, 1]$. If $t = 0$ or $t = 1$, then we are done. We will prove by contradiction that t cannot be in $(0, 1)$. For, otherwise, we have $f'(t) = 0$ and $m \coloneqq f(t) \leq f(0) = 2^r$, i.e.,

$$\begin{cases} r(1+t)^{r-1} - r(1-t)^{r-1} & = \alpha r t \dfrac{P(t)}{1-t^2} \\ (1+t)^r + (1-t)^r & = m - \alpha P(t) \end{cases}. \tag{3.30}$$

Multiplying by $\dfrac{1-t^2}{r}$ the first equality in (3.30) and treating the two equalities as a system with the unknowns $(1+t)^r$ and $(1-t)^r$, we successively find

$$\begin{cases} (1-t)(1+t)^r - (1+t)(1-t)^r & = \alpha t P(t) \\ (1+t)^r + (1-t)^r & = m - \alpha P(t) \end{cases},$$

$$\begin{cases} (1+t)^r = \dfrac{m(1+t) - \alpha P(t)}{2} \\ (1-t)^r = \dfrac{m(1-t) - \alpha P(t)}{2} \end{cases},$$

$$(1+t)^r - (1-t)^r = mt \leq 2^r t.$$

In order to obtain a contradiction, it suffices to prove that $g(t) > 0$, $\forall\, 0 < t < 1$, where

$$g(t) \coloneqq (1+t)^r - (1-t)^r - 2^r t.$$

Step 3. Conclusion. Since $1 < r < 2$, we have

$$g''(t) = r(r-1)\Big[(1+t)^{r-2} - (1-t)^{r-2}\Big] < 0,\ \forall\, 0 < t < 1,$$

and thus g is strictly concave. Since $g(0) = g(1) = 0$, we find that $g(t) > 0$, $\forall\, 0 < t < 1$.

Step 4. Optimality. Let θ be such that

$$M_1^r \leq \theta M_r^r + (1-\theta) M_0^r,\ \forall\, x, y > 0. \tag{3.31}$$

Testing (3.31) with $x = 2$, $y = 0$, we find that $\theta \geq 1/2^{r-1}$. QED

The above approach can be also successfully used when $r > 2$.

Problem 3.16. Let $r > 2$ and $x, y > 0$. Prove the inequality

$$M_1^r \le \frac{1}{r} M_r^r + \frac{r-1}{r} M_0^r = \frac{1}{r} M_r^r + \left(1 - \frac{1}{r}\right) M_0^r, \tag{3.32}$$

and establish its optimality.

An easy consequence of (3.29).

Problem 3.17. Let $1 \le r \le 2$, $0 < q \le r$, and $x, y > 0$. Prove the inequality

$$M_1^q \le \frac{1}{2^{q-q/r}} M_r^q + \left(1 - \frac{1}{2^{q-q/r}}\right) M_0^q, \tag{3.33}$$

and establish its optimality. Hint: Problem 2.2.

Remark 3.2. One may apply the method in Problem 3.17 to the case where $r > 2$ (relying on Problem 3.16 instead of Problem 3.15), and obtain the following counterpart of (3.33) with $r > 2$ and $0 < q \le r$:

$$M_1^q \le \frac{1}{r^{q/r}} M_r^q + \left(1 - \frac{1}{r^{q/r}}\right) M_0^q. \tag{3.34}$$

However, in general, (3.34), though true, is not optimal. For example, when $q = 1$, one has the (optimal) inequality [11]

$$M_1^q \le \frac{1}{2^{1-1/r}} M_r^q + \left(1 - \frac{1}{2^{1-1/r}}\right) M_0^q. \tag{3.35}$$

Since $2^{1-1/r} > r^{1/r}$ when $r > 2$ (this amounts to $2^r - 2r > 0$ for $r > 2$, which can be easily proved by calculus), (3.35) is an improvement of (3.34). The optimality of (3.35) can be proved as in Problem 3.17.

We next briefly illustrate the case where $n \ge 3$. Here, the equal variables method explained in the previous section will play an important role.

Problem 3.18. Let $n = 3$. Prove that the optimal constant θ in the inequality

$$M_0 \le \theta M_1 + (1 - \theta) M_{-1} \tag{3.36}$$

is given by

$$\theta := \frac{3}{2} \max_{x>0} \frac{x^2(x+2)}{(x^2+x+1)^2}. \tag{3.37}$$

(Numerically, we have $\theta = 0.5260499\ldots$)

Proof. Step 1. Use of the equal variables method. Let $g(x) \coloneqq 1/x$, $f(x) \coloneqq \ln x$. Then (3.36) reads

$$\left[x_1, x_2, x_3 > 0,\ \sum_{j=1}^{3} g(x_j) = S,\ \sum_{j=1}^{3} x_j = T\right] \Rightarrow \sum_{j=1}^{3} f(x_j) \leq 3 \ln \left[\theta T + (1-\theta)\frac{3}{S}\right]. \tag{3.38}$$

By Problem 3.7, (3.38) holds provided it holds when $x_1 \leq x_2 = x_3$. Turning back to (3.36), we find that it holds iff it holds when $x_1 \leq x_2 = x_3$.

Step 2. Reduction to the study of a real function. Since, by homogeneity, we may further assume that $M_0 = 1$ (and then, with $x \coloneqq x_2$, we have $x_1 = 1/x^2$ and $x_2 = x_3 = x$), we find that (3.36) holds iff

$$1 \leq \theta M_1 + (1-\theta)M_{-1} = \theta \frac{1/x^2 + 2x}{3} + (1-\theta)\frac{3}{x^2 + 2/x},\ \forall\, x > 0. \tag{3.39}$$

Since (3.39) clearly holds when $x = 1$, we may assume that $x \neq 1$, and then (3.39) is equivalent to

$$\theta \geq \frac{1 - M_{-1}}{M_1 - M_{-1}} = \frac{3x^2\left(x^3 - 3x + 2\right)}{(2x^3+1)(2+x^3) - 9x^3},\ \forall\, x > 0,\ x \neq 1.$$

Using the fact that, for $x > 0$, $x \neq 1$, we have (by intimidation and after simplification with $(x-1)^2$)

$$\frac{3x^2\left(x^3 - 3x + 2\right)}{(2x^3+1)(2+x^3) - 9x^3} = \frac{3}{2}\frac{x^2(x+2)}{\left(x^2+x+1\right)^2},$$

we find that the smallest θ satisfying (3.36) is indeed given by (3.37). Numerically, one finds that equality in (3.36) is obtained for $x_0 = 1.3614687\ldots$, with $\theta = 0.5260499\ldots$

Note that there exists an explicit formula for θ. Indeed, setting

$$f(x) \coloneqq \frac{x^2(x+2)}{\left(x^2+x+1\right)^2},$$

the equation $f'(x) = 0$ amounts to a fourth order equation that it it possible to solve explicitly. This gives first the value of x_0, then the optimal θ. QED

Now comes the final bouquet of this section: a result that encompasses Problem 3.8 and Problem 1.32. We consider here only the case where $n \geq 3$. However, since the case where $n = 2$ was treated in Problem 3.15, the result below still holds when $n = 2$.

Problem 3.19. Let $n \geq 3$ and $1 < r \leq n$. Prove the inequality

$$M_1^r \leq \theta M_r^r + (1-\theta)M_0^r, \text{ with } \theta = \left(\frac{n-1}{n}\right)^{r-1}, \tag{3.40}$$

and establish its optimality.

Proof. Step 1. Use of the equal variables method. Arguing as in Step 1 in Problem 3.8 (with, this time, $y_j := x_j^r$, $g(t) := \ln t$, and $f(x) := x^{1/r}$), we see that it suffices to prove (3.40) when $x_1 \leq x_2 = \cdots = x_n$.

Step 2. Further reductions. By homogeneity, we may assume that $\sum x_j = n$, and then, by Step 1, it suffices to prove that (3.40) holds when

$$x_1 = 1-(n-1)x,\ x_2 = \cdots = x_n = 1+x, \text{ with } 0 \leq x \leq \frac{1}{n-1}. \tag{3.41}$$

Step 3. Strategy of the proof. Let us denote $M_t(x)$ the value of M_t for x_j as in (3.41). We have to prove that

$$F(x) := \theta M_r^r(x) + (1-\theta)M_0^r(x) \geq 1,\ \forall\, 0 \leq x \leq \frac{1}{n-1}. \tag{3.42}$$

Since (3.42) holds when $x = 0$ (for any θ) and when $x = \dfrac{1}{n-1}$ (for this special θ), it suffices to prove that (3.42) holds when $0 < x < \dfrac{1}{n-1}$. Assume by contradiction that this is not the case. Since "a continuous function on a compact set has a maximum point and a minimum point", there exists thus some $0 < x < \dfrac{1}{n-1}$ such that $F(x) < 1$ and $F'(x) = 0$. We will obtain a contradiction by proving that

$$\left[0 < x < \frac{1}{n-1},\ F'(x) = 0\right] \implies [m := F(x) > 1] \tag{3.43}$$

(which is stronger than needed for a contradiction, since $m \geq 1$ would suffice for that purpose).

Step 4. A convenient formula for $F'(x)$. We have, slightly by intimidation,

$$\begin{aligned}
[M_r^r]'(x) =& r\frac{n-1}{n}\left[(1+x)^{r-1} - (1-(n-1)x)^{r-1}\right] \\
=& r\frac{n-1}{n}\frac{(1-(n-1)x)(1+x)^r - (1+x)(1-(n-1)x)^r}{(1+x)(1-(n-1)x)} \\
=& r\frac{n-1}{n}\left\{\frac{(1+x)^r - (1-(n-1)x)^r}{(1+x)(1-(n-1)x)} - \frac{nM_r^r(x)}{(1+x)(1-(n-1)x)}\right\} \\
=& r\frac{n-1}{n}\frac{(1+x)^r - (1-(n-1)x)^r}{(1+x)(1-(n-1)x)} - r(n-1)x\frac{M_r^r(x)}{(1+x)(1-(n-1)x)}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}[M_0^r]'(x) =& r\frac{n-1}{n}(1+x)^{r(n-1)/n-1}(1-(n-1)x)^{r/n}\\ &-r\frac{n-1}{n}(1+x)^{r(n-1)/n}(1-(n-1)x)^{r/n-1}\\ =& r\frac{n-1}{n}M_0^r(x)\left[\frac{1}{1+x}-\frac{1}{1-(n-1)x}\right]\\ =& -r(n-1)x\frac{M_0^r(x)}{(1+x)(1-(n-1)x)}.\end{aligned}$$

Substituting these formulas into the expression of $F'(x)$ we find that, at a point where $F'(x) = 0$, we have

$$\theta\frac{1}{n}[(1+x)^r-(1-(n-1)x)^r] = \theta x M_r^r(x) + (1-\theta)xM_0^r(x) = mx.$$

(with $m = F(x)$). This is equivalent to

$$\begin{aligned}(1+x)^r-(1-(n-1)x)^r =& n\theta x M_r^r(x) + n(1-\theta)xM_0^r(x)\\ =& \frac{nm}{\theta}x = \frac{n^r}{(n-1)^{r-1}}mx.\end{aligned}$$

Since we want to prove that $m > 1$, it suffices to prove that

$$G(x) \coloneqq (1+x)^r-(1-(n-1)x)^r - \frac{n^r}{(n-1)^{r-1}}x > 0,\ \forall\, 0 < x < \frac{1}{n-1}. \tag{3.44}$$

Step 5. Proof of (3.44) when $1 < r \le 2$. We have (using the facts that $n-1 > 1$ and $r-2 \le 0$)

$$G''(x) = r(r-1)\Big[(1+x)^{r-2}-(n-1)^2(1-(n-1)x)^{r-2}\Big] < 0,\ \forall\, 0 < x < \frac{1}{n-1},$$

and thus G is strictly concave. Since $G(0) = G(1/(n-1)) = 0$, we find that (3.44) holds.

Step 6. Proof of (3.44) when $2 < r \le n$. We have

$$\begin{aligned}G'''(x) = r(r-1)(r-2)\Big[(1+x)^{r-3}+(n-1)^3(1-(n-1)x)^{r-3}\Big] > 0,\\ \forall\, 0 < x < \frac{1}{n-1}.\end{aligned}$$

Since, on the other hand, $G''(0) < 0$ (here, we use $n \ge 3$) and $G''(1/(n-1)) > 0$, we find that there exists some $0 < a < \dfrac{1}{n-1}$ such that

$$G''(x) < 0 \text{ if } 0 \le x < a \text{ and } G''(x) > 0 \text{ if } a < x \le \frac{1}{n-1}. \tag{3.45}$$

Now comes the key point: since $r \le n$ (assumption that was not used up to now), we have

$$G'(1/(n-1)) = r\left(\frac{n}{n-1}\right)^{r-1} - \frac{n^r}{(n-1)^{r-1}} = (r-n)\left(\frac{n}{n-1}\right)^{r-1} \le 0. \tag{3.46}$$

From (3.45) and (3.46), we find that G is (strictly) decreasing on $[a, 1/(n-1)]$. Since (by our choice of θ) $G(1/(n-1)) = 0$, we find that $G(x) > 0$, $\forall\, a \le x < 1/(n-1)$. Finally, since: (i) G is strictly concave on $[0, a]$; (ii) $G(0) = 0$; (iii) $G(a) > 0$, we obtain, from (J), that $G(x) > 0$, $\forall\, 0 < x \le a$.

Step 7. The optimality of θ is obtained by taking $x_1 = \varepsilon > 0$, $x_2 = \cdots = x_n = 1$ and letting $\varepsilon \to 0$.
QED

Problem 3.20. Prove that (3.40) does not hold when $r > n$. Hint: test (3.40) with $x_1 \to 0$ and $x_2 = \cdots = x_n = 1$.

Problem 3.21. Let $n \ge 3$, $1 < r \le n$, and $0 < q \le r$. Prove the inequality

$$M_1^q \le \left(\frac{n-1}{n}\right)^{q-q/r} M_r^q + \left(1 - \left(\frac{n-1}{n}\right)^{q-q/r}\right) M_0^q, \tag{3.47}$$

and establish its optimality. Hint: Problem 3.17.

We end this section with an application of some of the above inequalities. The following function

$$N(x) := x \ln x,\ \forall\, x > 0,$$

is used in information theory and is known as the (negative) Shannon entropy. For an executive summary concerning the entropy $-N$, see Wikipedia [13]. It is easy to see that N is convex. Therefore, (GJ) with $\lambda_j = 1/n$, $\forall\, j$, yields

$$\sum_{j=1}^n x_j \ln x_j \ge \sum_{j=1}^n x_j \ln\left(\sum_{k=1}^n x_k/n\right) = \sum_{j=1}^n x_j \left(\ln\left(\sum_{k=1}^n x_k\right) - \ln n\right), \quad \forall\, x_1, \ldots, x_n > 0. \tag{3.48}$$

We have the following improvement of (3.48).

Problem 3.22. Prove that

$$\sum_{j=1}^n x_j \ln x_j \ge \sum_{j=1}^n x_j \left(\ln\left(\sum_{k=1}^n x_k\right) - \ln n\right) + n(M_1 - M_0) \ln\left(\frac{n}{n-1}\right),\ \forall\, x_1, \ldots, x_n > 0. \tag{3.49}$$

Hints: (i) start from (3.33) or (3.47) with $q = 1$; (ii) what happens when $r = 1$?

3.4 The Farkas lemma

Assume that $x, y \in \mathbb{R}$ satisfy

$$\begin{cases} x + y \leq 0 \\ 2x - 3y \leq 0 \end{cases}. \tag{3.50}$$

What inferences can one derive starting from (3.50)? For example, that

$$4x - y = 2(x + y) + (2x - 3y) \leq 0,$$

or any other inequality obtained as a linear combination with non-negative coefficients of the two lines in (3.50). Anything else? The answer is negative, for (3.50) and for any general system of the form (3.50). This is the content of the Farkas lemma [7].

Theorem 3.4 (Farkas lemma)**.** Let $n \geq 1$ and $m \geq 1$. Consider $m + 1$ linear inequalities, $\ell_j(x) \leq 0, j = 1, \ldots, m$, $\ell(x) \leq 0$, where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and, with $a^j = (a^j_1, \ldots, a^j_n), a = (a_1, \ldots, a_n) \in \mathbb{R}^n \backslash \{0\}$,

$$\ell_j(x) \coloneqq a^j \cdot x = \sum_{k=1}^n a^j_k x_k, \ \ell(x) \coloneqq a \cdot x = \sum_{k=1}^n a_k x_k, \ \forall\, x \in \mathbb{R}^n, \ \forall\, 1 \leq j \leq m.$$

If

$$\forall\, x \in \mathbb{R}^n, \ [\ell_j(x) \leq 0, \ \forall\, 1 \leq j \leq m] \Rightarrow \ell(x) \leq 0, \tag{3.51}$$

then there exist non-negative scalars $\lambda_1, \ldots, \lambda_m$ such that

$$\alpha = \sum_{j=1}^m \lambda_j \alpha^j. \tag{3.52}$$

(Equivalently, $\ell = \sum_{j=1}^m \lambda_j \ell_j$.)

The proof below is due to Bartl [1].

Proof. The proof is by induction on m.

Step 1. Proof when $m = 1$. Write $\alpha = \lambda^1 \alpha^1 + \xi$, with $\lambda^1 \in \mathbb{R}$ and $\xi \perp \alpha^1$. Testing (3.51) with $x \coloneqq \xi$, we find that $\xi \cdot \xi \leq 0$, and thus $\xi = 0$. Testing next (3.51) with $x \coloneqq -\xi$, we find that $\lambda^1 \geq 0$.

Step 2. The induction process. Assume that the result holds for $m - 1$. Then one of the two holds: (i) the implication

$$\forall\, x \in \mathbb{R}^n, \ [\ell_j(x) \leq 0, \ \forall\, 1 \leq j \leq m - 1] \Rightarrow \ell(x) \leq 0 \tag{3.53}$$

holds true. In this case, by the induction assumption, there exist $\lambda_j \geq 0$, $1 \leq j \leq m-1$, such that $\alpha = \sum_{j=1}^{m-1} \lambda_j \alpha^j = \sum_{j=1}^{m} \lambda_j \alpha^j$, where we have set $\lambda_m = 0$. In this case, we are done. (ii) The implication (3.53) does not hold, and thus

$$\exists\, z \in \mathbb{R}^n \text{ such that } \ell_j(z) \leq 0,\ \forall\, j = 1, \ldots, m-1,\ \ell(z) > 0. \tag{3.54}$$

By assumption (3.51), in this case we also have

$$\ell_m(z) > 0. \tag{3.55}$$

By replacing z with $\dfrac{1}{\ell_m(z)} z$, we may assume that $\ell_m(z) = 1$.

Step 3. The reduction step. Let

$$\varphi(y) := y - \ell_m(y)\, z,\ \forall\, y \in \mathbb{R}^n, \tag{3.56}$$

so that $\ell_m \circ \varphi = 0$. From this and (3.51) (with $x := \varphi(y)$), it follows that

$$\begin{aligned} &\forall\, y \in \mathbb{R}^n, [\ell_j \circ \varphi(y) \leq 0,\ j = 1, \ldots, m-1] \\ &\qquad \Rightarrow [\ell_j \circ \varphi(y) \leq 0,\ j = 1, \ldots, m] \Rightarrow \ell \circ \varphi(y) \leq 0. \end{aligned} \tag{3.57}$$

From (3.57) and the induction assumption, there exist $\lambda_1, \ldots, \lambda_{m-1} \geq 0$ such that

$$\ell \circ \varphi = \lambda_1\, \ell_1 \circ \varphi + \cdots + \lambda_{m-1}\, \ell_{m-1} \circ \varphi. \tag{3.58}$$

Inserting (3.56) in (3.58), and using (3.54), we find that $\lambda = \sum_{j=1}^{m} \lambda_j \ell_j$, where $\lambda_1, \ldots, \lambda_{m-1} \geq 0$ are as in (3.58) and

$$\lambda_m := \ell(z) - \lambda_1 \ell_1(z) - \cdots - \lambda_{m-1} \ell_{-1}(z) > 0.$$

QED

Chapter 4

Problems on inequalities

Overview. A selected list of inequalities that I find funny or relevant. Many of them are taken from the book of Drimbe [5], but the perspective is often different. Some are very easy. Like in real life, no hint this time.

Problem 4.1. Let $n \geq 3$. Then

$$\frac{x_1}{x_1 + x_2} + \frac{x_2}{x_2 + x_3} + \cdots + \frac{x_n}{x_n + x_1} < n - 1, \tag{4.1}$$

and the constant $n - 1$ is optimal.

Problem 4.2. Prove the (AM-GM) inequality, including the equality case, by establishing the inequality

$$\frac{a_1 + \cdots + a_n}{n} - \sqrt[n]{a_1 \cdots a_n} \geq \frac{2}{n^2(n-1)} \sum_{1 \leq j < k \leq n} \left(\sqrt{a_j} - \sqrt{a_k}\right)^2, \quad \forall\, n \geq 2,\ \forall\, a_1, \ldots, a_n > 0. \tag{4.2}$$

Problem 4.3. If $0 < a_1, \ldots, a_n < T$ and $a_1 + \cdots + a_n = S$, then

$$\sum_{j=1}^{n} \frac{a_j}{T - a_j} \geq \frac{nS}{nT - S}. \tag{4.3}$$

Problem 4.4. If $0 < c < a$ and $0 < c < b$, prove that

$$\sqrt{c(a-c)} + \sqrt{c(b-c)} \leq \sqrt{ab}. \tag{4.4}$$

Problem 4.5. We have

$$\frac{a^\alpha}{b^\alpha} + \frac{b^\alpha}{c^\alpha} + \frac{c^\alpha}{a^\alpha} \geq \frac{a}{c} + \frac{b}{a} + \frac{c}{b},\ \forall\, a, b, c > 0,\ \forall\, \alpha \geq 2. \tag{4.5}$$

Problem 4.6. Given $r, s > 0$ and $0 < \alpha \leq \beta$, find

$$\max\left\{a^{r+s} + b^{r+s} + c^{r+s} - a^r b^s - b^r c^s - c^r a^s;\ \alpha \leq a, b, c \leq \beta\right\}. \tag{4.6}$$

Problem 4.7. Prove that

$$\begin{aligned}&\frac{1}{(4x+y)^2}+\frac{1}{(4y+z)^2}+\frac{1}{(4z+x)^2}\\&\geq\frac{1}{(2x+2y+z)^2}+\frac{1}{(2y+2z+x)^2}+\frac{1}{(2z+2x+y)^2},\ \forall\, x,y,z>0.\end{aligned}\tag{4.7}$$

Problem 4.8. Prove that

$$\frac{a^3}{a^2+ab+b^2}+\frac{b^3}{b^2+bc+c^2}+\frac{c^3}{c^2+ca+a^2}\geq\frac{a+b+c}{3},\ \forall\, a,b,c>0.\tag{4.8}$$

Problem 4.9. If $r>0$, $a,b>0$, and $a\neq b$, prove that

$$\left(a^{2r+1}+b^{2r+1}\right)>a^rb^r(a+b).$$

Problem 4.10. Improve the conclusion of Problem 2.33 to

$$a^3+b^3+c^3\geq a(b-c)^2+b(c-a)^2+c(a-b)^2+3abc,\ \forall\, a,b,c>0.\tag{4.9}$$

The next problem is from Kvant.

Problem 4.11. Prove that

$$\begin{aligned}&a^4+b^4+c^4+d^4+2abcd\\&\geq a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2,\ \forall\, a,b,c,d\geq 0.\end{aligned}\tag{4.10}$$

Problem 4.12. Prove that

$$\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab}\leq 2,\ \forall\, 0\leq a,b,c\leq 1.\tag{4.11}$$

Problem 4.13. Prove that

$$\begin{aligned}&\left(a_1+\frac{1}{a_2}\right)\left(a_2+\frac{1}{a_3}\right)\cdots\left(a_n+\frac{1}{a_1}\right)\\&\leq\left(a_1+\frac{1}{a_1}\right)\left(a_2+\frac{1}{a_2}\right)\cdots\left(a_n+\frac{1}{a_n}\right),\ \forall\, a_1,\ldots,a_n>0.\end{aligned}\tag{4.12}$$

Problem 4.14. Prove that, for $n\geq 2$, and $0<a_1,a_2,\ldots,a_n<1$, we have

$$(1-a_1a_2\cdots a_n)^n\geq(1-a_1^n)(1-a_2^n)\cdots(1-a_n^n).\tag{4.13}$$

Problem 4.15. Let $\alpha\geq 1$. Find

$$\max\left\{\frac{(a+b)(b+c)(c+a)}{abc};\ \frac{1}{\alpha}\leq a,b,c\leq\alpha\right\}.\tag{4.14}$$

Problem 4.16. Let $k \geq 1, n \geq 3$ be integers. Prove that

$$\sum_{j=1}^{n-1} \frac{x_j - x_{j+1}}{x_j^k + x_j^{k-1}x_{j+1} + \cdots + x_j x_{j+1}^{k-1} + x_{j+1}^k} \geq \frac{x_1 - x_n}{x_1^k + x_1^{k-1}x_n + \cdots + x_1 x_n^{k-1} + x_n^k}, \forall\, x_1 \geq x_2 \geq \cdots \geq x_n > 0 \tag{4.15}$$

(with the convention $x_{n+1} = x_1$).

Problem 4.17. Let $k \geq 1, n \geq 3$ be integers. Prove that

$$\sum_{j=1}^{n-1} \frac{x_j^{k+1}}{x_j^k + x_j^{k-1}x_{j+1} + \cdots + x_j x_{j+1}^{k-1} + x_{j+1}^k} \geq \frac{1}{k+1}\sum_{j=1}^{n} x_j, \forall\, x_1, x_2, \ldots, x_n > 0 \tag{4.16}$$

(with the convention $x_{n+1} = x_1$).

Problem 4.18. If $\alpha > 1$, prove that

$$\frac{1}{a^\alpha + b^\alpha + 2\alpha - 2} + \frac{1}{b^\alpha + c^\alpha + 2\alpha - 2} + \frac{1}{c^\alpha + a^\alpha + 2\alpha - 2} \leq \frac{1}{2\alpha}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right), \ \forall\, a, b, c > 0. \tag{4.17}$$

Problem 4.19. If $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$, $b_1 \geq 0, b_1 + b_2 \geq 0, \ldots, b_1 + \cdots + b_n \geq 0$, prove that $a_1 b_1 + \cdots + a_n b_n \geq 0$.

Chapter 5

Solutions

5.1 Basic methods and inequalities

Problem 1.2. Taking squares of the both sides of (M) (with $p = 2$), we see that (M) is equivalent to

$$\sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}, \tag{5.1}$$

which follows from (CS). Conversely, if (5.1) holds, replacing, in (5.1), b_j with $-b_j$, $\forall\, j$, we obtain

$$-\sum_{j=1}^{n} a_j b_j \leq \left(\sum_{j=1}^{n} a_j^2\right)^{1/2} \left(\sum_{j=1}^{n} b_j^2\right)^{1/2}. \tag{5.2}$$

We obtain (CS) from (5.1) and (5.2). QED

Problem 1.3. We prove, e.g., item (2), the other ones being similar. Set $S \coloneqq \sum_{j=1}^{n} |a_j|^p$, respectively $T \coloneqq \sum_{j=1}^{n} |b_j|^q$. If $S = 0$, then $a_j = 0$, $\forall\, j$, and (H) is clear. Similarly if $T = 0$. We may therefore assume that $S > 0$ and $T > 0$. Set $a'_j \coloneqq \dfrac{a_j}{S^{1/p}}$ and $b'_j \coloneqq \dfrac{b_j}{T^{1/q}}$. Then $\sum_{j=1}^{n} \left|a'_j\right|^p = 1$ and $\sum_{j=1}^{n} \left|b'_j\right|^q = 1$. By assumption, we have

$$\left|\sum_{j=1}^{n} a'_j b'_j\right| \leq 1,$$

which implies (H). QED

Problem 1.4. When $n = 2$, we have to prove that $x_1 + x_2 \geq 2\sqrt{x_1 x_2}$, which amounts to $(\sqrt{x_1} - \sqrt{x_2})^2 \geq 0$. Thus (AM-GM) holds when $n = 2$, with equality iff $x_1 = x_2$.

Assuming (AM-GM) for $n = 2^k$, the case $n = 2^{k+1}$ (including the equality case) follows from

$$\sum_{j=1}^{2^{k+1}} x_j = \sum_{j=1}^{2^k} x_j + \sum_{j=2^k+1}^{2^{k+1}} x_j \geq 2^k \prod_{j=1}^{2^k} x_j^{1/2^k} + 2^k \prod_{j=2^k+1}^{2^{k+1}} x_j^{1/2^k} \geq 2^{k+1} \prod_{j=1}^{2^{k+1}} x_j^{1/2^{k+1}},$$

where the last inequality relies on the case $n = 2$.

Finally, let n and k be integers such that $2^k > n$. Applying (AM-GM) to the 2^k numbers $x_1, \dots, x_n, \underbrace{G, G, \dots, G}_{2^k-n}$ (with $G = G(x_1, \dots, x_n)$), we find that (with $A = A(x_1, \dots, x_n)$)

$$nA + (2^k - n)G \geq 2^k G^{n/2^k} G^{(2^k-n)/2^k} = 2^k G,$$

with equality iff $x_1 = \cdots = x_n = G$. This yields (AM-GM) for n, including the equality case. QED

Problem 1.5. The case $n = 2$ is treated as above. Assume that the inequality holds for n. By homogeneity, we may assume that $x_1 \cdots x_n x_{n+1} = 1$. By the induction assumption, we have

$$x_1 + x_2 + \cdots + x_{n-1} + x_n x_{n+1} \geq n,$$

and thus

$$x_1 + x_2 + \cdots + x_{n-1} + x_n + x_{n+1} \geq n - x_n x_{n+1} + x_n + x_{n+1}.$$

Therefore, it suffices to prove that

$$n - x_n x_{n+1} + x_n + x_{n+1} \geq n + 1,$$

which amounts to $(1-x_n)(1-x_{n+1}) \leq 0$. Now comes the key argument. Since $x_1 \cdots x_n x_{n+1} = 1$, there exist $i \neq j$ such that $x_i \leq 1$ and $x_j \geq 1$. By symmetry of (AM-GM), we may assume that $i = n$ and $j = n + 1$, and then we are done. QED

Problem 1.6. When $n = 2$, we have to prove that

$$(a_1 b_1 + a_2 b_2)^2 \leq (a_1^2 + a_2^2)(b_1^2 + b_2^2),$$

which amounts to $(a_1 b_2 - a_2 b_1)^2 \geq 0$.

Assume that (CS) holds for n, and write it in the condensed form $U^2 \leq ST$. We have to prove that

$$(U + a_{n+1} b_{n+1})^2 \leq (S + a_{n+1}^2)(T + b_{n+1}^2),$$

which is equivalent to

$$V \coloneqq ST + S a_{n+1}^2 + T b_{n+1}^2 - U^2 - 2U a_{n+1} b_{n+1} \geq 0. \tag{5.3}$$

By the induction assumption, we have

$$Ua_{n+1}b_{n+1} \le |U||a_{n+1}||b_{n+1}| \le \sqrt{ST}|a_{n+1}||b_{n+1}|,$$

and therefore the quantity V defined in (5.3) satisfies

$$\begin{aligned} V \ge& ST + Sa_{n+1}^2 + Tb_{n+1}^2 - U^2 - 2\sqrt{ST}|a_{n+1}||b_{n+1}| \\ =& ST - U^2 + \left(\sqrt{S}|a_{n+1}| - \sqrt{T}|b_{n+1}|\right)^2 \ge ST - U^2 \ge 0, \end{aligned}$$

where we have used again the induction assumption $U^2 \le ST$. QED

Problem 1.7. (1) The case $n = 2$ is (J) with $t := \lambda_1$. Assume that (GJ) holds for some $n \ge 2$ and consider $x_1, \ldots, x_{n+1}, \lambda_1, \ldots, \lambda_{n+1}$ that satisfy the conditions of the statement. If $\lambda_j = 0$ for some j, then we are back the case n and we are done. Otherwise, set $\mu := \lambda_n + \lambda_{n+1}$, and

$$x := \frac{\lambda_n}{\mu}x_n + \frac{\lambda_{n+1}}{\mu}x_{n+1} \in I.$$

By (J), we have

$$f(x) \le \frac{\lambda_n}{\mu}f(x_n) + \frac{\lambda_{n+1}}{\mu}f(x_{n+1}). \tag{5.4}$$

Using first (GJ) for $x_1, \ldots, x_{n-1}, x$ and $\lambda_1, \ldots, \lambda_{n-1}, \mu$, then (5.4), we find that

$$\begin{aligned} f\left(\sum_{j=1}^{n+1}\lambda_j x_j\right) =& f\left(\sum_{j=1}^{n-1}\lambda_j x_j + \mu x\right) \le \sum_{j=1}^{n-1}\lambda_j\, f(x_j) + \mu\, f(x) \\ &\le \sum_{j=1}^{n-1}\lambda_j\, f(x_j) + \mu\left(\frac{\lambda_n}{\mu}f(x_n) + \frac{\lambda_{n+1}}{\mu}f(x_{n+1})\right) = \sum_{j=1}^{n+1}\lambda_j\, f(x_j). \end{aligned} \tag{5.5}$$

(2) The proof is again by induction. The case $n = 2$ follows from the definition of the strict convexity. Assume that the conclusion holds for n. Equality in (GJ) amounts to equality between the first and the last term in (5.5), so that, in (5.5), the inequalities are equalities. This requires (by the induction assumption and the case $n = 2$) that $x_1 = \ldots = x_{n-1} = x$ and $x_n = x_{n+1} = x$, whence the conclusion. QED

Problem 1.8. Write $x_j = e^{y_j}$, with $y_j \in \mathbb{R}$. Let $f(x) := e^x$. Then (AM-GM) can be rewritten as

$$\sum_{j=1}^{n}\frac{1}{n}f(y_j) \le f\left(\sum_{j=1}^{n}\frac{1}{n}y_j\right), \tag{5.6}$$

which is a special case of (GJ) for the convex function f. In addition, f being strictly convex, equality in (5.6) (and thus in (AM-GM)) occurs iff $y_1 = \cdots = y_n$, and thus iff $x_1 = \cdots = x_n$. QED

Problem 1.9. (1) If $b_j = 0$, $\forall\, j$, then (H) holds, with equality. On the other hand, if only one b_j is non-zero, then (H) holds, with equality iff $a_i = 0$ if $i \neq j$.

We now investigate the remaining cases. We may assume, with no loss of generality, that there exists some $1 \le m \le n$ such that $b_j \neq 0$ if $j \le m$, while $b_j = 0$ if $m + 1 < j \le n$. Let, as in the solution of Problem 1.3,

$$T := \sum_{j=1}^{n} |b_j|^q = \sum_{j=1}^{m} |b_j|^q.$$

Set, for $1 \le j \le m$, $\lambda_j := \dfrac{|b_j|^q}{T}$, so that $0 < \lambda_j < 1$ and $\sum_{j=1}^{m} \lambda_j = 1$. Set $f(x) := |x|^p$. Raising (H) to the power p and using the identity $p/q = p - 1$, we find that (H) is equivalent to

$$f\left(\sum_{j=1}^{m} a_j b_j\right) \le T^{p-1} \sum_{j=1}^{n} f(a_j). \tag{5.7}$$

Given our choice of λ_j, and in order to write (5.7) as a special case of (GJ), we let

$$x_j := \frac{T a_j b_j}{|b_j|^q},\ \forall\, 1 \le j \le m, \tag{5.8}$$

so that $a_j b_j = \lambda_j x_j$ and (5.7) becomes

$$f\left(\sum_{j=1}^{m} \lambda_j x_j\right) \le T^{p-1} \sum_{j=1}^{m} f(a_j) + T^{p-1} \sum_{j=m+1}^{n} f(a_j). \tag{5.9}$$

Using the identity $pq = p + q$, we see that the specific f we consider satisfies, for $1 \le j \le m$,

$$T^{p-1} f(a_j) = \lambda_j f(x_j),$$

and thus (5.9) is equivalent to

$$f\left(\sum_{j=1}^{m} \lambda_j x_j\right) \le \sum_{j=1}^{m} \lambda_j f(x_j) + T^{p-1} \sum_{j=m+1}^{n} f(a_j). \tag{5.10}$$

By (GJ), the strict convexity of f, and the fact that $f(x) \ge 0$ with equality iff $x = 0$, we find that (5.10) (and thus (H)) holds. Moreover, in this case we have equality iff $x_1 = \cdots = x_m$ and $a_j = 0$ if $j > m$.

(2) Assume first that (1.5) holds. This is equivalent to one the two following: (i) $b_j = 0$, $\forall\, j$, or (ii) there exists some constant $t \in \mathbb{R}$ such that $a_j = t \operatorname{sgn} b_j |b_j|^{q-1}$, $\forall\, j$. Clearly, if (i) holds, then we have equality in (H). If (ii) holds, then (H) becomes

$$\left|\sum_{j=1}^{n} t|b_j|^q\right| \le \left(\sum_{j=1}^{n} |t|^p |b_j|^{p(q-1)}\right)^{1/p} \left(\sum_{j=1}^{n} |b_j|^q\right)^{1/q}. \tag{5.11}$$

Using the identities $p(q-1) = q$ and $1/p + 1/q = 1$, we find that (5.11) holds with equality. In conclusion, if (1.5) holds, then equality holds in (H).

Conversely, by the analysis of item (1), equality occurs iff: (i) $b_j = 0$, $\forall\, j$; or (ii) there exists one j such that $b_j \neq 0$ and, for $i \neq j$, $a_i = 0$ and $b_i = 0$; or (iii) up to a permutation of indices, there exists $1 \le m \le n$ such that $x_1 = \cdots = x_m$ (with x_j as in (5.8)), $a_j = 0$, and $b_j = 0$ if $j > m$. Clearly, (1.5) holds in cases (i) and (ii). Assume next that (iii) holds. Set $C \coloneqq x_1$. Then $x_j = C$, $\forall\, 1 \le j \le m$, which is equivalent to

$$a_j = \frac{C|b_j|^q}{Tb_j} = \frac{C}{T}\operatorname{sgn} b_j |b_j|^{q-1},\ \forall\, 1 \le j \le m. \tag{5.12}$$

Since, clearly, (5.12) still holds when $j > m$, we find that (1.5) is valid. QED

Problem 1.10. (1) Since

$$M_{-r}(x_1, x_2, \ldots, x_n) = \frac{1}{M_r(1/x_1, 1/x_2, \ldots, 1/x_n)}, \tag{5.13}$$

if (MI) holds for every non-negative r_1 and r_2, then it holds for any r_1 and r_2. To see this, take, for example, $r_1 < 0$ and $r_2 > 0$. By (MI) and (5.13), $M_{r_1} \le M_0 \le M_{r_2}$, with equality iff $x_1 = x_2 = \cdots = x_n$. The other cases are treated similarly.

(2) In this case, (MI) raised to the r_2 power is nothing but (AM-GM) applied to the numbers $x_1^{r_2}, x_2^{r_2}, \ldots, x_n^{r_2}$. Equality holds iff $x_1^{r_2} = x_2^{r_2} = \cdots = x_n^{r_2}$, which amounts to $x_1 = x_2 = \cdots = x_n$.

(3) In this case, (MI) raised to the r_2 power reads, with $r \coloneqq r_2/r_1$ and $f(x) \coloneqq x^r$, $\forall\, x > 0$,

$$f\left(\sum_{j=1}^{n} \frac{1}{n} x_j^{r_1}\right) \le \sum_{j=1}^{n} \frac{1}{n} f\left(x_j^{r_1}\right), \tag{5.14}$$

which is a special case of (GJ). By strict convexity of f, equality holds in (5.14) iff $x_1^{r_2} = x_2^{r_2} = \cdots = x_n^{r_2}$, i.e., $x_1 = x_2 = \cdots = x_n$. QED

Problem 1.11. (1) Assuming that (M) holds for n, we have

$$\begin{aligned}\left(\sum_{j=1}^{n+1} |a_j + b_j|^p\right)^{1/p} &\le \left(\left[\left(\sum_{j=1}^{n} |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n} |b_j|^p\right)^{1/p}\right]^p + |a_{n+1} + b_{n+1}|^p\right)^{1/p} \\ &\le \left(\sum_{j=1}^{n+1} |a_j|^p\right)^{1/p} + \left(\sum_{j=1}^{n+1} |b_j|^p\right)^{1/p},\end{aligned}$$

where the last line uses the case $n = 2$.

(2) If $a_1 = 0$ and $b_1 = 0$, (M) is clear. Assume that $a_1 \neq 0$ or $b_1 \neq 0$. Let $t \neq 0$. Then (M) holds for $a_1, \ldots, b_2$ iff it holds for $a_1/t, \ldots, b_2/t$. Letting $t \coloneqq |a_1| + |b_1|$, we have reduced the problem to the study of the case where $|a_1| + |b_1| = 1$.

(3) Let $\lambda_1 := |a_1|, \lambda_2 := |b_1|, x_1 := \dfrac{|a_2|}{|a_1|}, x_2 := \dfrac{|b_2|}{|b_1|}$. By (J), we have

$$\begin{aligned}(|a_1|^p + |a_2|^p)^{1/p} + (|b_1|^p + |b_2|^p)^{1/p} =&\lambda_1 f(x_1) + \lambda_2 f(x_2) \\ \geq& f(\lambda_1 x_1 + \lambda_2 x_2) = (1 + (|a_2| + |b_2|)^p)^{1/p}.\end{aligned} \tag{5.15}$$

We complete the proof by noting that

$$1 + (|a_2| + |b_2|)^p = (|a_1| + |b_1|)^p + (|a_2| + |b_2|)^p \geq |a_1 + b_1|^p + |a_2 + b_2|^p.$$ QED

Problem 1.12. (1) Since $a \leq c \leq b$, there exists some $\lambda \in [0,1]$ such that $c = (1-\lambda)a + \lambda b$. On the other hand, we have

$$d = a + b - c = a + b - [(1-\lambda)a + \lambda b] = \lambda a + (1-\lambda)b.$$

Using the above and (J) (twice), we find that

$$\begin{aligned}f(c) + f(d) =& f((1-\lambda)a + \lambda b) + f(\lambda a + (1-\lambda)b) \\ \leq&(1-\lambda)f(a) + \lambda f(b) + \lambda f(a) + (1-\lambda)f(b) = f(a) + f(b).\end{aligned} \tag{5.16}$$

Assume next that f is strictly convex and that equality holds in (5.16). If $\lambda = 0$, then $a = c$ and $b = d$ (and this is clearly an equality case). Similarly, if $\lambda = 1$, we obtain the equality case $a = d$ and $b = c$. If $0 < \lambda < 1$, equality in (5.16) implies that $a = b$, and then $a = b = c = d$. To summarize, we have equality iff the sets $\{a, b\}$ and $\{c, d\}$ coincide.

(2) This existence of $x, y \in I$ such that $x + y = S$ is equivalent to $S \in J := [2\alpha, 2\beta]$. Assume that this condition is satisfied. Let $\gamma := (\alpha + \beta)/2$ be the midpoint of I.

The case where $2\alpha \leq S \leq 2\gamma$. Let $a := \alpha$ and $b := 2S - \alpha$. If the couple (x, y) is a competitor in the maximization problem, then, clearly $x, y \geq a$. Since, on the other hand, $x + y = a + b$, we have $x, y \leq b$. By item (1), we have $f(x) + f(y) \leq f(a) + f(b)$ and, by the analysis of the equality case in item (1), the unique solution (up to a permutation of points) is $x = \alpha$, $y = 2S - \alpha$.

The case where $2\gamma \leq S \leq 2\beta$. By a similar argument, the unique solution (up to a permutation of points) is $x = 2S - \beta$, $y = \beta$.

The converse is clear since, when $2\alpha \leq S < 2\gamma$, respectively $2\gamma < S \leq 2\beta$, the endpoint has to be α, respectively β. When $S = \alpha + \beta$, x and y have to be the two endpoints. All these possibilities are consistent with the solutions found above.

(3) Assume again that $S \in J$. Then, for every competing couple (x, y), we have

$$f(S) = f\left(\frac{1}{2}x + \frac{1}{2}y\right) \leq \frac{1}{2}f(x) + \frac{1}{2}f(y),$$

with equality iff $x = y$. The unique solution is therefore $x = y = S/2$. QED

Problem 1.13. "(3) ⇒ (1)" By and (1.21), (1.13), and Jensen's inequality, we have

$$\sum_{j=1}^{n} f(y_j) = \sum_{j=1}^{n} f\left(\sum_{k=1}^{n} a_{jk}x_k\right) \leq \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}f(x_k) = \sum_{k=1}^{n}\sum_{j=1}^{n} a_{jk}f(x_k) = \sum_{k=1}^{n} f(x_k),$$

where the last equality uses (1.14).

"(1) ⇒ (2)" Applying (1.14) to $f(x) \equiv x$ and to $f(x) \equiv -x$, we find that (1.19) holds.

We now prove (1.16)–(1.18). Let $1 \leq k \leq n-1$ and set $f(x) := (x - x_{k+1})^-$. Noting that $z^- \geq -z$, $\forall\, z \in \mathbb{R}$, we have

$$\sum_{j=1}^{n} f(x_j) = \sum_{j=1}^{k}(x_{k+1} - x_j) \geq \sum_{j=1}^{n} f(y_j) \geq \sum_{j=1}^{k} f(y_j) \geq \sum_{j=1}^{k}(-y_j + x_{k+1}),$$

and thus $\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j$, $\forall\, 1 \leq k \leq n-1$. QED

Problem 1.14. Assume that (1.22) and (1.16)–(1.19) hold. Let $\sigma \in S_n$ be such that $x_{\sigma(1)} \leq x_{\sigma(2)} \leq \cdots \leq x_{\sigma(n)}$. Then $x_{\sigma(1)}$ is the smallest x_j, and thus, in particular, $x_{\sigma(1)} \leq x_1 \leq y_1$. Similarly, $x_{\sigma(1)}$ and $x_{\sigma(2)}$ are the two smallest x_j's, and in particular $x_{\sigma(1)} + x_{\sigma(2)} \leq x_1 + x_2 \leq y_1 + y_2$. Etc. Finally, $x_{\sigma(1)}, \ldots x_{\sigma(n)}, y_1, \ldots, y_n$ satisfy (1.15)–(1.19). By "(1)⇒(2)" in Theorem 1.1, for each convex f we have

$$\sum_{j=1}^{n} f(y_j) \leq \sum_{j=1}^{n} f(x_{\sigma(j)}) = \sum_{j=1}^{n} f(x_j),$$

so that, as claimed, "(1)⇒(2)" holds under the assumptions (1.22) and (1.16)–(1.19).

Finally, as already noticed in Remark 1.2, the order of the x_j's and y_j's plays no role in "(3)⇒(1)". QED

Problem 1.15. (1.16) is equivalent to $g(n) = h(n)$. On the other hand, the functions g and h being affine on each of the intervals $[0, 1], [1, 2], \ldots, [n-1, n]$, we have

$$[g(t) \leq h(t),\ \forall\, t \in [0, n]] \Leftrightarrow [g(k) \leq h(k),\ \forall\, k = 0, 1, \ldots, n]. \tag{5.17}$$

Since $g(0) = h(0) = 0$ and $g(n) = h(n)$, (5.17) becomes

$$[g(t) \leq h(t),\ \forall\, t \in [0, n]] \Leftrightarrow \left[\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j,\ \forall\, k = 1, \ldots, n-1\right]$$
$$\Leftrightarrow [(1.17) - (1.19)].$$

QED

Problem 1.17. By multiplying the scalars $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m$ with the same suitable integer, we may assume that they are all integers. Let $S := \sum_{k=1}^{n} \alpha_n = \sum_{j=1}^{n} \beta_j$. Consider the ordered lists $(X_1, \ldots, X_S)$, $(Y_1, \ldots, Y_S)$ defined as follows:

$$(X_1, \ldots, X_S) := (\underbrace{x_1, \ldots, x_1}_{\alpha_1 \text{ times}}, \ldots, \underbrace{x_n, \ldots, x_n}_{\alpha_n \text{ times}}),$$

$$(Y_1, \ldots, Y_S) := (\underbrace{y_1, \ldots, y_1}_{\beta_1 \text{ times}}, \ldots, \underbrace{y_m, \ldots, y_m}_{\beta_m \text{ times}}).$$

Then, clearly, $X_1 \leq X_2 \leq \cdots \leq X_S$, $Y_1 \leq Y_2 \leq \cdots \leq Y_S$, $X_j \in I$, $Y_j \in I$, $\forall\, j$, and inequality (1.36) is equivalent to

$$\sum_{j=1}^{S} f(Y_j) \leq \sum_{j=1}^{S} f(X_j). \tag{5.18}$$

Let G and H be the auxiliary functions associated with the inequality (5.18) (as explained before the statement of Theorem 1.2). Theorem 1.1 and Problem 1.15 yield the equivalence of (5.18) with each of the following properties:

$$G(t) \leq H(t),\ \forall\, t \in [0, S], \text{ and } G(S) = H(S), \tag{5.19}$$

$$\text{there exists a DS matrix } B = (b_{jk})_{1\leq j,k\leq S} \text{ such that } Y_j = \sum_{k=1}^{S} b_{jk} X_k, \forall\, j. \tag{5.20}$$

In order to complete the proof of Theorem 1.2 in this case, it suffices to prove that "(1.39) ⇔ (5.19)", "(5.20) ⇒ (1.40) – (1.43)", and "(1.40) – (1.43) ⇒ (1.36)".

"(1.40) – (1.43) ⇒ (1.36)" We have

$$\begin{aligned}\sum_{j=1}^{m} \beta_j f(y_j) &= \sum_{j=1}^{m} \beta_j f\left(\sum_{k=1}^{n} a_{jk} x_k\right) \leq \sum_{j=1}^{m} \beta_j \sum_{k=1}^{n} a_{jk} f(x_k)\\ &= \sum_{k=1}^{n} \sum_{j=1}^{m} \beta_j a_{jk} f(x_k) = \sum_{k=1}^{n} \alpha_k f(x_k).\end{aligned}$$

"(5.20) ⇒ (1.40) – (1.43)" The first β_1 equalities in (5.20) read

$$y_1 = \sum_{\ell=1}^{\alpha_1} b_{s\ell} x_1 + \sum_{\ell=\alpha_1+1}^{\alpha_2} b_{s\ell} x_2 + \cdots + \sum_{\ell=\alpha_1+\cdots+\alpha_{n-1}^1}^{T} b_{s\ell} x_n,\ \forall\, 1 \leq s \leq \beta_1. \tag{5.21}$$

Thus y_1 is the average of the first β_1 equalities in (5.20). Similarly for the lines involving $y_2, \ldots, y_m$. In view of the above, we set (with the conventions $\alpha_0 = 0$ and $\beta_0 = 0$)

$$a_{jk} \coloneqq \frac{1}{\beta_j} \sum_{s=\beta_1+\beta_{j-1}+1}^{\beta_1+\cdots+\beta_j} \sum_{\ell=\alpha_1+\cdots+\alpha_{k-1}+1}^{\alpha_1+\cdots+\alpha_k} b_{s\ell}. \tag{5.22}$$

Property (1.43) is obvious. Property (1.41) follows from the fact that B is DS and the identity

$$\sum_{k=1}^{n} a_{jk} = \frac{1}{\beta_j} \sum_{s=\beta_1+\beta_{j-1}+1}^{\beta_1+\cdots+\beta_j} \sum_{\ell=1}^{S} b_{s\ell}.$$

Finally, property (1.42) follows from the fact that B is DS and the identity

$$\sum_{j=1}^{m} \beta_j a_{jk} = \sum_{\ell=\alpha_1+\cdots+\alpha_{j-1}+1}^{\alpha_1+\cdots+\alpha_j} \sum_{s=1}^{S} b_{s\ell}.$$

"(1.39) $\Leftrightarrow$ (5.19)" simply because $g = G$ and $h = H$. QED

Problem 1.18. Testing (1.36) with $f(x) \equiv (x - x_1)^-$, we find that $(y_1 - x_1)^- = 0$, and thus $y_1 \geq x_1$. Similarly, testing (1.36) with $f(x) \equiv (x - x_n)^+$, we obtain $(y_m - x_n)^+ = 0$, and thus $y_m \leq x_n$. QED

Problem 1.19. We consider the case where the lists are ordered $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$, the other case being similar.

Clearly, (1.50) holds for $a_1, \ldots, a_n, b_1, \ldots, b_n$ iff it holds for $a_1 - C, \ldots a_n - C, b_1 - D, \ldots, b_n - D$. We may therefore assume that $a_1 \geq 0$ and $b_1 \geq 0$.

If $a_{j-1} \leq a_j$, $\forall\, 2 \leq j \leq n$, then $x_j := a_j - a_{j-1} \geq 0$, $\forall\, 2 \leq j \leq n$. Conversely, if $x_j \geq 0$, $\forall\, j$, then

$$a_1 \leq a_1 + x_2 = a_2 \leq a_1 + x_2 + x_3 = a_3 \leq \cdots \leq a_n.$$

Set, for notational convenience, $x_1 := a_1$, and $y_1 := b_1$, so that $a_j = \sum_{k=1}^{j} x_k$, and similarly for b_j. From the above discussion, (1.50) is equivalent to

$$\sum_{j=1}^{n}\sum_{k=1}^{j} x_k \sum_{\ell=1}^{\sigma(j)} y_\ell \leq \sum_{j=1}^{n}\sum_{k=1}^{j} x_k \sum_{\ell=1}^{\tau(j)} y_\ell, \forall\, x_1, \ldots, x_n, y_1, \ldots, y_n \geq 0. \tag{5.23}$$

By counting the number of times a product $x_k y_\ell$ appears in (5.23), we find that (5.23)is equivalent to

$$\begin{aligned} \sum_{k=1}^{n}\sum_{\ell=1}^{n} x_k y_\ell \operatorname{Card} S(k,\ell) \leq \sum_{k=1}^{n}\sum_{\ell=1}^{n} x_k y_\ell \operatorname{Card} T(k,\ell), \\ \forall\, x_1, \ldots, x_n, y_1, \ldots, y_n \geq 0, \end{aligned} \tag{5.24}$$

where

$$S(k,\ell) := \{j \geq k;\ \sigma(j) \geq \ell\} \text{ and } T(k,\ell) := \{j \geq k;\ \tau(j) \geq \ell\}.$$

Next, clearly,

$$\begin{aligned} &\operatorname{Card} S(k,1) = \operatorname{Card} T(k,1) = n - k + 1, \\ &\operatorname{Card} S(1,\ell) = \operatorname{Card} T(1,\ell) = n - \ell + 1, \end{aligned}$$

and thus (5.24) amounts to

$$\begin{aligned} \sum_{k=2}^{n}\sum_{\ell=2}^{n} x_k y_\ell \operatorname{Card} S(k,\ell) \leq \sum_{k=2}^{n}\sum_{\ell=2}^{n} x_k y_\ell \operatorname{Card} T(k,\ell), \\ \forall\, x_2, \ldots, x_n, y_2, \ldots, y_n \geq 0. \end{aligned} \tag{5.25}$$

We are now in position to prove that (1.50)$\Leftrightarrow$(1.51) (which is the content of the theorem). Indeed, in view of (5.25), (1.51)$\Rightarrow$(1.50). On the other hand, if (1.50) holds, then, given $2 \leq m, p \leq n$, the choice $x_k := \delta_{km}$, $y_\ell := \delta_{\ell p}$ shows that $\operatorname{Card} S(m,p) \leq \operatorname{Card} T(m,p)$, and thus (1.51) holds. QED

Problem 1.20. Set $A_k := \{j;\ j \geq k\}$ and $B_\ell := \{j;\ \sigma(j) \geq \ell\}$. Then: (i) $\operatorname{Card} A_k = n-k+1$; (ii) $\operatorname{Card} B_\ell = n_\ell + 1$; (iii) $S(k,\ell) = A_k \cap B_\ell$.

We first note that

$$\operatorname{Card} S(k,\ell) \leq \min(\operatorname{Card} A_k, \operatorname{Card} B_\ell) = \min(n-k+1, n-\ell+1).$$

This inequality becomes equality for $\sigma = \mathrm{id}$. In view of Theorem 1.3, this implies the second inequality in (R).

We next note that, when $k + \ell \leq n+1$, we have

$$\begin{aligned}\operatorname{Card} S(k,\ell) &= \operatorname{Card} A_k + \operatorname{Card} B_\ell - \operatorname{Card}(A_k \cup B_\ell)\\ &\geq (n-k+1) + (n-\ell+1) - n = n - (k+\ell) + 2,\end{aligned}$$

while, when $k+\ell > n+1$, $\operatorname{Card} S(k,\ell) \geq 0$.

These inequalities become equalities when $\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix}$. By Theorem 1.3, this implies the first inequality in (R). QED

Problem 1.21. We define $\sigma(j)$ by induction on j. We explain how to define $\sigma(1)$ and $\sigma(2)$. The remaining part of the proof is routine.

Step 1. Choice and properties of $\sigma(1)$. Let $c_1 := \min_{1\leq j\leq n} a_j$. Consider the non-empty set $A_1 := \{k;\ a_k = c_1\}$. Let $d_1 := \min_{k\in A_1} b_k$. Consider the non-empty set $B_1 := \{\ell \in A_1;\ b_\ell = d_1\}$. To summarize, if $\ell \in B_1$, then $a_\ell = c_1 \leq a_j$, $\forall\, j$, and $b_\ell \leq b_k$, $\forall\, k \in A_1$.

We fix some $\ell \in B_1$. We claim that $b_\ell \leq b_j$, $\forall\, j$. Indeed, by definition of B_1, this is true if $j \in A_1$. If $j \notin A_1$, then $a_\ell < a_j$, and then (1.53) implies that $b_\ell \leq b_j$. Since, on the other hand, $a_\ell \leq a_j$, $\forall\, j$ (since $\ell \in A_1$), we find that, with $\sigma(1) := \ell$, we have $a_{\sigma(1)} \leq a_j$, $\forall\, j$, and $b_{\sigma(1)} \leq b_j$, $\forall\, j$.

Step 2. Choice and properties of $\sigma(2)$. Let $c_2 := \min_{j\neq\sigma(1)} a_j$, $A_2 := \{k \neq \sigma(1);\ a_k = c_2\}$, $d_2 := \min_{k\in A_2} b_k$, $B_2 := \{\ell \in A_2;\ b_\ell = d_2\}$.

Let $\ell \in B_2$ and set $\sigma(2) := \ell$. By repeating the argument in Step 1, we have $a_{\sigma(2)} \leq a_j$, $\forall\, j \neq \sigma(1)$, and $b_{\sigma(2)} \leq b_j$, $\forall\, j \neq \sigma(1)$. QED

Problem 1.22. We consider only the case of identically ordered lists; the case of oppositely ordered lists is treated similarly.

Step 1. Reduction to the case of ordered lists. Using Problem 1.21 and the facts that

$$\sum_{j=1}^n a_j b_j = \sum_{j=1}^n a_{\sigma(j)} b_{\sigma(j)},\ \sum_{j=1}^n a_j = \sum_{j=1}^n a_{\sigma(j)},\ \sum_{j=1}^n b_j = \sum_{j=1}^n b_{\sigma(j)},\ \forall\, \sigma \in S_n,$$

we may assume that $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$.

Step 2. Decomposition of the product of sums. Let

$$\sigma_1 := \mathrm{id} = \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 1 & 2 & \dots & n-1 & n \end{pmatrix}, \sigma_2 := \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 2 & 3 & \dots & n & 1 \end{pmatrix},$$
$$\sigma_3 := \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ 3 & 4 & \dots & 1 & 2 \end{pmatrix}, \dots, \sigma_n := \begin{pmatrix} 1 & 2 & \dots & n-1 & n \\ n & 1 & \dots & n-2 & n-1 \end{pmatrix}.$$

Then, for each $1 \le j \le n$, the list $\sigma_1(j), \dots, \sigma_n(j)$ is a permutation of the list $1, \dots, n$. We find that

$$\sum_{j=1}^n a_j \sum_{j=1}^n b_j = \sum_{j=1}^n \sum_{k=1}^n a_j b_k = \sum_{j=1}^n \sum_{k=1}^n a_j b_{\sigma_k(j)} = \sum_{k=1}^n \sum_{j=1}^n a_j b_{\sigma_k(j)}. \tag{5.26}$$

Step 3. Conclusion. For each k, (R) yields

$$\sum_{j=1}^n a_j b_{\sigma_k(j)} \le \sum_{j=1}^n a_j b_j. \tag{5.27}$$

We obtain (C) (and thus (GC)) from (5.26) and (5.27). QED

Problem 1.24. Clearly, the quadratic trinomial in (1.61) satisfies $f(x) \ge 0$, $\forall\, x \in \mathbb{R}$, and thus its discriminant Δ is ≤ 0. Now,

$$T(x) = \left(\sum_{j=1}^n a_j^2\right) x^2 - 2 \sum_{j=1}^n a_j \sum_{j=1}^n b_j x + \sum_{j=1}^n b_j^2,$$

and thus

$$\frac{\Delta}{4} = \left(\sum_{j=1}^n b_j\right)^2 - \left(\sum_{j=1}^n a_j^2\right)\left(\sum_{j=1}^n b_j^2\right) \le 0. \tag{5.28}$$

We have just obtained (CS). By (5.28), equality in (CS) amounts to $\Delta = 0$, which in turn amounts to the existence of some $x \in \mathbb{R}$ such that $f(x) = 0$. Finally, $f(x) = 0$ is equivalent to $a_j x = b_j$, $\forall\, j$, i.e., to the proportionality of $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$. QED

Problem 1.25. We have

$$f\left(\frac{2\alpha}{\alpha^2+1}\right) = \sum_{j=1}^n b_j^2 g(x_j), \tag{5.29}$$

where

$$g(x) := \frac{4\alpha^2}{(\alpha^2+1)^2} x^2 - \frac{4\alpha}{\alpha^2+1} x + \frac{4\alpha^2}{(\alpha^2+1)^2}, \tag{5.30}$$
$$\frac{1}{\alpha} \le x_j := \frac{a_j}{b_j} \le \alpha. \tag{5.31}$$

The roots of g are $1/\alpha$ and α, and therefore (5.30) and (5.31) imply that $g(x_j) \le 0$, $\forall\, j$. Inserting this into (5.29), we find that $f\big(2\alpha/(\alpha^2+1)\big) \le 0$. It follows that the discriminant of f is ≥ 0, which amounts to (1.62). QED

Problem 1.26. (1) By (CS), we have

$$(ab+bc+ca)^2 \le \left(a^2+b^2+c^2\right)\left(b^2+c^2+a^2\right),$$

whence (1.63).

(2) With no loss of generality, we may assume that $a \le b \le c$. Set $a_1 = b_1 := a$, $a_2 = b_2 := b$, $a_3 = b_3 := c$. Then (R) with $\sigma := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ yields

$$ab+bc+ca = a_1 b_{\sigma(1)} + a_2 b_{\sigma(2)} + a_3 b_{\sigma(3)} \le a_1 b_1 + a_2 b_2 + a_3 b_3 = a^2+b^2+c^2. \quad \text{QED}$$

Problem 1.27. *Step 1.* (1.64) holds when $3 \le n \le 6$. Denoting I_n the inequality we want to prove, we find that

$$\begin{aligned}
I_3 \Leftrightarrow & 2\left(x_1^2+x_2^2+x_3^2-x_1x_2-x_1x_3-x_2x_3\right) \ge 0 \\
\Leftrightarrow & (x_1-x_2)^2+(x_2-x_3)^2+(x_3-x_1)^2 \ge 0, \\
I_4 \Leftrightarrow & 2\left(x_1^2+x_2^2+x_3^2+x_4^2-2x_3x_1-2x_4x_2\right) \ge 0 \\
\Leftrightarrow & 2\left[(x_1-x_3)^2+(x_2-x_4)^2\right] \ge 0, \\
I_5 \Leftrightarrow & 2\sum_{j=1}^{5} x_j^2 - \sum_{1\le j<k\le 5} x_j x_k \ge 0 \\
\Leftrightarrow & \frac{1}{2}\sum_{1\le j<k\le 5} (x_j-x_k)^2 \ge 0, \\
I_6 \Leftrightarrow & 2\sum_{j=1}^{6} x_j^2 - 2\sum_{1\le j<k\le 6, 1\le |j-k|\le 2} x_j x_k + 4\sum_{1\le j\le 3} x_j x_{j+3} \ge 0 \\
\Leftrightarrow & \sum_{1\le j\le 3} (x_j - x_{j+1} + x_{j+3} - x_{j+4})^2 \ge 0.
\end{aligned}$$

When $n \ge 7$, one could provide directly a counterexample (see Step 4), but it is more instructive to get first insight from the Gauss method.

Step 2. (1.64) does not hold when $n \ge 9$. Indeed, I_n is equivalent to

$$\begin{aligned}
2x_1^2 - (n-4)\sum_{j=2,3,n-1,n} x_1 x_j + 4\sum_{j=4}^{n-2} x_1 x_j + 2\sum_{j=2}^{n} x_j^2 & \\
-(n-4)x_4x_5 + R \ge 0, &
\end{aligned} \tag{5.32}$$

where the reminder R contains none of the products $x_i x_j$ that precede it in (5.32). Using the Gauss method, we complete the first part of the expression in (5.32) to a square, and find that I_n is equivalent to

$$2\left[x_1 - \frac{n-4}{4}\sum_{j=2,3,n-1,n} x_j + \sum_{j=4}^{n-2} x_j\right]^2 - nx_4x_5 + T \ge 0, \tag{5.33}$$

where T does not contain x_1, x_4x_5, x_4^2 or x_5^2. Therefore, the validity of I_n implies that

$$2(x_1 + x_4 + x_5)^2 - nx_4x_5 \geq 0, \ \forall\, x_1, x_4, x_5 \geq 0. \tag{5.34}$$

However, (5.34) is wrong when $x_1 = 0$, $x_4 = x_5 = 1$.

Step 3. (1.64) does not hold when $n = 8$. The argument is similar to the one in the previous step. Instead of (5.32), we start from the form (recall that $n = 8$)

$$\begin{aligned} 2x_1^2 - 4 \sum_{j=2,3,7,8} x_1x_j + 4\sum_{j=4}^{6} x_1x_j + 2\sum_{j=2}^{8} x_j^2 \\ -4(x_4x_5 + x_4x_6 + x_5x_6) + \overline{R} \geq 0, \end{aligned} \tag{5.35}$$

where the reminder $\overline{R}$ contains none of the products x_ix_j that precede it in (5.35). We next rewrite (5.35) as

$$2\left[x_1 - \sum_{j=2,3,7,8} x_j + \sum_{j=4}^{6} x_j\right]^2 - 8(x_4x_5 + x_4x_6 + x_5x_6) + \overline{T} \geq 0, \tag{5.36}$$

where $\overline{T}$ does not contain $x_1, x_4x_5, x_4x_6, x_5x_6, x_4^2, x_5^2$ or x_6^2. Thus the validity of I_8 implies that

$$2(x_1 + x_4 + x_5 + x_6)^2 - 8(x_4x_5 + x_4x_6 + x_5x_6) \geq 0, \forall\, x_1, x_4, x_5, x_6 \geq 0, \tag{5.37}$$

but this is wrong when $x_1 = 0, x_4 = x_5 = x_6 = 1$.

Step 4. (1.64) does not hold when $n \geq 7$. Inspired by Step 3, let us take $x_1 = x_2 = x_3 = 1$ and $x_j = 0, \forall\, 4 \leq j \leq 7$. Then (1.64) for these values becomes $3n \leq 18$, and this does not holds for $n \geq 7$. QED

Problem 1.28. Set

$$f(a) \coloneqq \frac{a^p}{p} + \frac{b^q}{q} - ab, \ \forall\, a \geq 0.$$

Then $f'(a) = a^{p-1} - b$, and thus f has a minimum at $a_0 \coloneqq b^{1/(p-1)}$. Finally,

$$f(a_0) = \frac{b^{p/(p-1)}}{p} + \frac{b^q}{q} - b^{p/(p-1)} = b^q\left(\frac{1}{p} + \frac{1}{q} - 1\right) = 0.$$ QED

5.2 Worked inequalities

Problem 2.1. We consider, e.g., the case of identically ordered lists. The validity of (GC) when $n = 1$ is obvious. Assume that (GC) holds for $n - 1$. Let $C \coloneqq \sum_{j=1}^{n-1} a_jb_j, A \coloneqq \sum_{j=1}^{n-1} a_j, B \coloneqq \sum_{j=1}^{n-1} b_j$. We want to prove that

$$n(C + a_nb_n) - (A + a_n)(B + b_n) \geq 0,$$

knowing that

$$(n-1)C \geq AB. \tag{5.38}$$

Using (5.38), we obtain

$$\begin{aligned} n(C + a_n b_n) - (A + a_n)(B + b_n) \geq & C + n a_n b_n + AB - (A + a_n)(B + b_n) \\ = & C + (n-1) a_n b_n - a_n B - A b_n \\ = & \sum_{j=1}^{n-1} (a_j b_j + a_n b_n - a_n b_j - a_j b_n) \\ = & \sum_{j=1}^{n-1} (a_n - a_j)(b_n - b_j) \geq 0. \end{aligned}$$

QED

Problem 2.2. (2.1) is equivalent to

$$f(t) := (a^p - b^p)t^p + [a - (a-b)t]^p \leq a^p, \ \forall\, 0 \leq t \leq 1. \tag{5.39}$$

Finally, (5.39) holds since $f(0) = f(1) = a^p$ and f is convex. QED

Problem 2.3. (1) If $0 \leq y < x$, let $t \in [0,1]$ be such that $y = (1-t)(-x) + tx$. By (J),

$$\varphi(y) = \varphi((1-t)(-x) + tx) \leq (1-t)\varphi(-x) + t\varphi(x) = \varphi(x).$$

(2) Properties (2.2) and (2.3) of the norm are straightforward. We now prove that

$$\Phi_2(u + \overline{u}, v + \overline{v}) \leq \Phi_2(u,v) + \Phi_2(\overline{u}, \overline{v}), \ \forall\, u, \overline{u}, v, \overline{v} \in \mathbb{R}. \tag{5.40}$$

Using: (i) the textbook fact that a convex function on an open interval is continuous; (ii) item (1), we find that

$$\mathbb{R} \ni v \mapsto \Phi_2(u,v) \text{ is continuous}, \ \forall\, u \in \mathbb{R}, \tag{5.41}$$

$$[0,\infty) \ni u \mapsto \Phi_2(u,v) \text{ is non-decreasing}, \ \forall\, v \in \mathbb{R}. \tag{5.42}$$

In view of (5.41), it suffices to prove (5.40) when $v, \overline{v}, v + \overline{v} \neq 0$. In this case, we have (using (5.42), (2.5), and the convexity of φ):

$$\begin{aligned} \Phi_2(u + \overline{u}, v + \overline{v}) &= \Phi_2(|u + \overline{u}|, |v + \overline{v}|) \leq \Phi_2(|u| + |\overline{u}|, |v + \overline{v}|) \\ &\leq \Phi_2(|u| + |\overline{u}|, |v| + |\overline{v}|) \\ &= (|v| + |\overline{v}|)\, \varphi\left(\frac{|v|}{|v| + |\overline{v}|} \frac{|u|}{|v|} + \frac{|\overline{v}|}{|v| + |\overline{v}|} \frac{|\overline{u}|}{|\overline{v}|} \right) \\ &\leq |v|\, \varphi\left(\frac{|u|}{|v|}\right) + |\overline{v}|\, \varphi\left(\frac{|\overline{u}|}{|\overline{v}|}\right) = \Phi_2(u,v) + \Phi_2(\overline{u}, \overline{v}). \end{aligned} \tag{5.43}$$

(3) Repeat the argument in (5.43), using

$$N(x_1 + y_1, \ldots, x_{n-1} + y_{n-1}) \leq N(x_1, \ldots, x_{n-1}) + N(y_1, \ldots, y_{n-1}), \quad \forall\, x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1} \in \mathbb{R}.$$

(4) Set

$$N_n(x_1,\ldots,x_n) := \left(\sum_{j=1}^{n}|x_j|^p\right)^{1/p}, \ \forall\, x_1,\ldots,x_n \in \mathbb{R},$$
$$\varphi(t) := (1+|t|^p)^{1/p}, \ \forall\, t \in \mathbb{R}.$$

Note that φ is convex, since φ' is increasing (see (1.9) and Problem 1.11), even, satisfies (2.5) with $\ell = 1$, and that $N_n = \Phi_n$, $\forall\, n \geq 2$.

Using the above and items (2) and (3), we find that Φ_n is a norm; this implies the Minkowski inequality (M). QED

Problem 2.4. We note that item (1) is a special case of item (2). We therefore proceed directly to the proof of item (2).

"[(2.9) and (2.10)] $\Rightarrow$(2.11)" Set

$$f(x) := \sum_{j=1}^{k}\lambda_j x^{\alpha_j} - x^{\alpha}, \ \forall\, x \in (0,\infty)^n.$$

If (2.9) and (2.10) hold, then $x^0 := (1,1,\ldots,1)$ is a point of minimum of f, and thus

$$\frac{\partial f}{\partial x_\ell}(x^0) = 0, \ \forall\, 1 \leq \ell \leq n,$$

which amounts to (2.11).

"[(2.9) and (2.11)]$\Rightarrow$(2.10) " Fix $x > 0$ and set

$$g : \mathbb{R}^n \to \mathbb{R}, \ g(\beta) := x^\beta, \ \forall\, \beta \in \mathbb{R}^n.$$

Then g is clearly convex, in the sense that it satisfies (J), and thus also (GJ). Under the assumption (2.11), we have

$$g(\alpha) = g\left(\sum_{j=1}^{k}\lambda_j\alpha_j\right),$$

and thus, if (2.9) holds, (2.10) amounts to the Jensen inequality (GJ). QED

Problem 2.5. We may assume that $a < c$. Then

$$f(b) = f\left(\frac{c-b}{c-a}a + \frac{b-a}{c-a}c\right) \leq \frac{c-b}{c-a}f(a) + \frac{b-a}{c-a}f(c),$$

and this inequality is equivalent to the one in the statement. QED

Problem 2.6. Proof by induction on n, the case $n = 2$ being clear. Let $S_n(x_1, \dots, x_n)$, respectively $T_n(x_1, \dots, x_n)$, denote the left-hand side, respectively the right-hand side, of (2.13). Assume that (2.13) holds for $n-1$. Noting that

$$S_n(x_1, \dots, x_n) = S_{n-1}(x_1, \dots, x_{n-1}) - x_{n-1}f(x_1) + x_{n-1}f(x_n) + x_n f(x_1),$$
$$T_n(x_1, \dots, x_n) = T_{n-1}(x_1, \dots, x_{n-1}) - x_1 f(x_{n-1}) + x_n f(x_{n-1}) + x_1 f(x_n),$$

in view of the induction hypothesis it suffices to prove that

$$-x_{n-1}f(x_1) + x_{n-1}f(x_n) + x_n f(x_1) \geq -x_1 f(x_{n-1}) + x_n f(x_{n-1}) + x_1 f(x_n),$$

which is equivalent to

$$(x_n - x_{n-1})f(x_1) + (x_1 - x_n)f(x_{n-1}) + (x_{n-1} - x_1)f(x_n) \geq 0. \tag{5.44}$$

In turn, (5.44) follows from Problem 2.5. QED

Problem 2.7. Testing the inequality with $x_1 = \cdots = x_n = 0$, we find that $C_n \leq \dfrac{f(1) - f(0)}{n}$.

We will actually prove that $\dfrac{f(1) - f(0)}{n}$ is the optimal constant, which amounts to

$$\frac{1}{\displaystyle\sum_{j=1}^{n} \frac{1}{f(x_j + 1)}} - \frac{1}{\displaystyle\sum_{j=1}^{n} \frac{1}{f(x_j)}} \geq \frac{f(1) - f(0)}{n}, \ \forall\, x_1, \dots, x_n \geq 0. \tag{5.45}$$

In turn, (5.45) is equivalent to

$$\underbrace{\sum_{j=1}^{n} \frac{f(x_j + 1) - f(x_j)}{f(x_j + 1)f(x_j)}}_{:=I} \geq \frac{f(1) - f(0)}{n} \sum_{j=1}^{n} \frac{1}{f(x_j + 1)} \sum_{j=1}^{n} \frac{1}{f(x_j)}. \tag{5.46}$$

Now, f being convex we have, by Problem 1.12,

$$f(0) + f(x_j + 1) \geq f(1) + f(x_j),$$

and thus

$$f(x_j + 1) - f(x_j) \geq f(1) - f(0) \geq 0,$$

since f is non-decreasing.

We find that

$$I \geq [f(1) - f(0)] \sum_{j=1}^{n} \frac{1}{f(x_j + 1)f(x_j)} \geq \frac{f(1) - f(0)}{n} \sum_{j=1}^{n} \frac{1}{f(x_j + 1)} \sum_{j=1}^{n} \frac{1}{f(x_j)},$$

the latter inequality following from the Chebyshev inequality (C) (using again the fact that f is non-decreasing).

Under the assumptions f non-decreasing and $f(x) > 0$, $\forall\, x \geq 0$, we find, with essentially the same proof, that

$$C_n = \frac{1}{n}\inf\{f(x+1) - f(x);\ x \geq 0\}.$$ QED

Problem 2.8. Let $\tau \in S_n$, and assume that $\mu_j = \lambda_{\tau(j)}$, $\forall\, j$. Given $\sigma \in S_n$, we have

$$z_\sigma = \sum_{j=1}^{n} \mu_j x_{\sigma(j)} = \sum_{j=1}^{n} \lambda_{\tau(j)} x_{\sigma(j)} = \sum_{k=1}^{n} \lambda_k x_{\sigma(\tau^{-1}(k))} = y_{\sigma\tau^{-1}}.$$

Equivalently, set $\Lambda := \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$ and $\mathrm{M} := \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix}$. If $\mathrm{M} = P_\tau \Lambda$, where P_τ is the permutation matrix associated with τ (see Theorem 3.1), then we have just proved that $z_\sigma = y_{\sigma\tau^{-1}}$, $\forall\, \sigma \in S_n$.

Consider now the general case. By "(2)⇒(3)" in Theorem 1.1, and by Theorem 3.1,

$$\text{there exist } c_\tau \in [0,1],\ \tau \in S_n, \text{ such that } \sum_{\tau \in S_n} c_\tau = 1 \text{ and } \Lambda = \sum_{\tau \in S_n} c_\tau P_\tau \mathrm{M}. \tag{5.47}$$

By linearity and the first part of the proof, we have

$$z_\sigma = \sum_{\tau \in S_n} c_\tau y_{\sigma\tau^{-1}} = \sum_{\tau \in S_n} c_{\tau^{-1}\sigma} y_\tau. \tag{5.48}$$

Set now $b_{\sigma,\tau} := c_{\tau^{-1}\sigma}$. Then (5.48) reads

$$z_\sigma = \sum_{\tau \in S_n} b_{\sigma,\tau} y_\tau,\ \forall\, \sigma \in S_n. \tag{5.49}$$

Using (5.47), it is straightforward that the matrix (of size $n!$) $(b_{\sigma,\tau})_{\sigma,\tau \in S_n}$ is DS. Combining this fact with (5.49) and the implication "(3)⇒(1)" in Theorem 1.1, we find that (2.15) holds. QED

Problem 2.9. Let $x_1, \ldots, x_n$ achieve the maximum in (2.16). With no loss of generality, we may assume that, for some $1 \leq \ell < n$, we have $-1 \leq x_1, \ldots, x_\ell \leq 0$, while $0 < x_{\ell+1} \leq \cdots \leq x_n \leq 1$.

We next note that, by the optimality of $x_1, \ldots, x_n$, for each $i \neq j$ we have

$$[x, y \in [-1,1],\ x + y = x_i + x_j] \Rightarrow x^3 + y^2 \leq x_i^3 + x_j^3. \tag{5.50}$$

Using (5.50), Problem (1.12), and the strict concavity of $x \mapsto x^3$ on $[-1,0]$, we find that $x_1 = \ldots = x_\ell < 0$. Using (5.50), Problem 1.12, and the strict convexity of $x \mapsto x^3$ on $[0,1]$, we find that $x_{\ell+2} = \ldots = x_n = 1$. For further use, we note that no x_i can be 0.

Now comes the main step of the proof. We claim that $x_\ell = 1$. Indeed, we argue by contradiction and assume that $0 < x_\ell < 1$. Two possibilities are to be considered.

Possibility 1. $x_1 + x_\ell \le 0$. In this case, one may check $x_1^3 + x_\ell^3 \le (x_1 + x_\ell)^3$, with equality iff $x_1 + x_\ell = 0$. By optimality, this implies that $x_1 + x_\ell = 0$, and then the numbers $0, x_2, \dots, x_\ell, 0, 1, \dots, 1$ are still optimal. This contradicts the first part of the analysis (no x_i can be 0).

Possibility 2. $x_1 + x_\ell > 0$. In particular, we have $x_1 > -1$ and $x_1^2 < x_\ell^2$. Consider the function

$$z \mapsto f(z) \coloneqq (x_1 - z)^3 + (x_\ell + z)^3 = x_1^3 + x_\ell^3 + 3z(x_\ell^2 - x_1^2) + 3z^2(x_\ell + x_1).$$

Then $f(z) > f(0), \forall\, z > 0$ and, for small z, we have $-1 < x_1 - z < x_\ell + z < 1$. This contradicts the optimality of $x_1, \dots, x_n$.

To summarize, we know that there exists $1 \le \ell < n$ such that

$$x_1 = \cdots = x_\ell = -\frac{n-\ell}{\ell} \text{ and } x_{\ell+1} = \cdots = x_n = 1.$$

Since $0 > x_1 \ge -1$, we have $\ell > n/2$. We find that the maximum in (2.16) is *exactly*

$$M \coloneqq \max\left\{(n-\ell) - \frac{(n-\ell)^3}{\ell^2};\ \frac{n}{2} < \ell < n\right\}. \tag{5.51}$$

Let now

$$g(x) \coloneqq 1 - x - \frac{(1-x)^3}{x^2},\ \forall\, x > 0,$$

so that, with $x \coloneqq \ell/n$,

$$(n-\ell) - \frac{(n-\ell)^3}{\ell^2} = ng(x).$$

The function g increases on $(0, 2/3]$ and decreases on $[2/3, \infty)$. We find that the maximum M is achieved by one of the ℓ's for which the corresponding x is closest to $2/3$. Several possibilities occur.

Case 1. $n = 3k$, with k integer. Then the optimal choice is $\ell = 2k$, leading to $x = 2/3$. We find that $M = \dfrac{3k}{4}$.

Case 2. $n = 3k + 1$, with k integer. In this case, we have two candidates: $\ell = 2k$ and $\ell = 2k+1$. After calculating $g(2k/(3k+1))$ and $g((2k+1)/(3k+1))$, we find that $\ell = 2k+1$ yields the maximum, which is $M = \dfrac{k(k+1)(3k+1)}{(2k+1)^2}$.

Case 3. $n = 3k + 2$, with k integer. Again, the candidates are $\ell = 2k$ and $\ell = 2k + 1$. The maximum corresponds to $\ell = 2k + 1$, and its value is $M = \dfrac{k(k+1)(3k+2)}{(2k+1)^2}$. QED

Problem 2.10. We have to prove that

$$\sum_{j=1}^{n} \ln\left(a_j + a_j\right) \leq \sum_{j=1}^{n} \ln\left(a_j + a_{\sigma(j)}\right) \leq \sum_{j=1}^{n} \ln\left(a_j + a_{n-j+1}\right).$$

The function ln being concave, it seems natural to try to prove the following generalization of (2.17): if $f : I = (a, b) \to \mathbb{R}$ is convex and $a_j, b_j \in I$, $\forall\, j$, satisfy $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$, then

$$\sum_{j=1}^{n} f(a_j + b_j) \geq \sum_{j=1}^{n} f(a_j + b_{\sigma(j)}) \geq \sum_{j=1}^{n} f(a_j + b_{n-j+1}). \tag{5.52}$$

Step 1. Proof of (5.52) in the special case where f is strictly convex, $a_1 < a_2 < \cdots < a_n$, and $b_1 < b_2 < \cdots < b_n$. We prove for example the first inequality; the proof of the second one is similar. Consider a permutation $\sigma \in S_n$ that achieves the maximum of the middle term in (5.52). We will prove, by contradiction, that $\sigma = \mathrm{id}$. For, otherwise, there exist $1 \leq k < \ell \leq n$ such that $\sigma(k) > \sigma(\ell)$. Then

$$a_k + b_{\sigma(\ell)} < a_\ell + b_{\sigma(\ell)}, a_k + b_{\sigma(k)} < a_\ell + b_{\sigma(k)}, \tag{5.53}$$

$$\left(a_k + b_{\sigma(\ell)}\right) + \left(a_\ell + b_{\sigma(k)}\right) = \left(a_k + b_{\sigma(k)}\right) + \left(a_\ell + b_{\sigma(\ell)}\right). \tag{5.54}$$

By (5.53)–(5.54) and the proof of the case $n = 2$ in Theorem 1.1, we see that there exists $\theta \in (0, 1)$ such that

$$\begin{pmatrix} a_k + b_{\sigma(k)} \\ a_\ell + b_{\sigma(\ell)} \end{pmatrix} = \begin{pmatrix} 1-\theta & \theta \\ \theta & 1-\theta \end{pmatrix} \begin{pmatrix} a_k + b_{\sigma(\ell)} \\ a_\ell + b_{\sigma(k)} \end{pmatrix}. \tag{5.55}$$

Using (5.55) and the strict convexity of f, we find that

$$f\left(a_k + b_{\sigma(k)}\right) + f\left(a_\ell + b_{\sigma(\ell)}\right) < f\left(a_k + b_{\sigma(\ell)}\right) + f\left(a_\ell + b_{\sigma(k)}\right).$$

Therefore, the permutation $\tau \in S_n$, defined by $\tau(j) := \begin{cases} \sigma(j), & \text{if } j \neq k, \ell \\ \sigma(\ell), & \text{if } j = k \\ \sigma(k), & \text{if } j = \ell \end{cases}$, satisfies

$$\sum_{j=1}^{n} f(a_j + b_\tau(j)) > \sum_{j=1}^{n} f\left(a_j + b_{\sigma(j)}\right),$$

contradicting the maximality of σ.

Step 2. Proof in the general case. With $\varepsilon > 0$ being sufficiently small, set $a_j^\varepsilon := a + j\varepsilon$, $b_j^\varepsilon := b_j + j\varepsilon$, $\forall\, j$, and $f^\varepsilon(x) := f(x) + \varepsilon x^2$, $\forall\, x$. Then a_j^ε, b_j^ε, and f^ε satisfy the assumptions of Step 1. We conclude by letting $\varepsilon \to 0$ and using the textbook fact that a convex function on an open interval is continuous.

Bonus (left without proof). Prove that, for every $\sigma \in S_n$, there exist DS matrices A, B such that

$$\begin{pmatrix} a_1 + b_{\sigma(1)} \\ a_2 + b_{\sigma(2)} \\ \vdots \\ a_n + b_{\sigma(n)} \end{pmatrix} = A \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix} \text{ and } \begin{pmatrix} a_1 + b_n \\ a_2 + b_{n-1} \\ \vdots \\ a_n + b_1 \end{pmatrix} = B \begin{pmatrix} a_1 + b_{\sigma(1)} \\ a_2 + b_{\sigma(2)} \\ \vdots \\ a_n + b_{\sigma(n)} \end{pmatrix}.$$

QED

Problem 2.11. The proof will be conceptually very simple, but part by intimidation.

With $f(x) := \dfrac{1}{x}$, $x \in I := (0, \infty)$, we want to prove that

$$\begin{aligned} &f(a) + f(b) + f(c) + 3f\left(\frac{a+b+c}{3}\right) \\ &\geq 2f\left(\frac{a+b}{2}\right) + 2f\left(\frac{b+c}{2}\right) + 2f\left(\frac{c+a}{2}\right), \ \forall\, a, b, c \in I. \end{aligned} \tag{5.56}$$

Since f is convex, let us try to apply Theorem 1.2 in the case where the scalars are rational. (In this approach, the fact that $a, b, c > 0$ is irrelevant.) As we saw in the proof of the theorem, this case reduces to Theorem 1.1. In order to check the conditions (1.16)–(1.19), we have to check the condition (1.15). With no loss of generality, we may assume that $a \leq b \leq c$, and then $\dfrac{a+b}{2} \leq \dfrac{a+c}{2} \leq \dfrac{b+c}{2}$. However, we do not know how $\dfrac{a+b+c}{3}$ compares to b. Therefore, we consider two cases.

Case 1. $\dfrac{a+b+c}{3} \leq b$, or, equivalently $a - 2b + c \leq 0$. We then consider, in the setting of Theorem 1.1, $n = 6$ and

$$\begin{aligned} &x_1 = a, x_2 = x_3 = x_4 = \frac{a+b+c}{3}, x_5 = b, x_6 = c, \\ &y_1 = y_2 = \frac{a+b}{2}, y_3 = y_4 = \frac{a+c}{2}, y_5 = y_6 = \frac{b+c}{2}. \end{aligned}$$

Then (1.15) is satisfied, and clearly so is (1.19). It remains to check that

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j, \ \forall\, 1 \leq k \leq 5. \tag{5.57}$$

This is the part by intimidation. It turns out that (5.57) indeed holds, and the proof for various values of k uses $a \leq b$, or $b \leq c$, or $a - 2b + c \leq 0$.

Case 2. $\dfrac{a+b+c}{3} \geq b$, that is, $a - 2b + c \geq 0$. The argument is similar. We let

$$\begin{aligned} &x_1 = a, x_2 = b, x_3 = x_4 = x_5 = \frac{a+b+c}{3}, x_6 = c, \\ &y_1 = y_2 = \frac{a+b}{2}, y_3 = y_4 = \frac{a+c}{2}, y_5 = y_6 = \frac{b+c}{2} \end{aligned}$$

and check the analogue of (5.57).

(5.56) is known as the Popoviciu inequality. QED

Problem 2.12. (2.19) looks like (GJ) applied to a concave function f. An educated guess is $f(x) := \sqrt{1-x^2}$, $|x| < 1$, which satisfies

$$f''(x) = -\frac{1}{(1-x^2)^{3/2}} < 0,$$

and therefore is indeed strictly concave. Since

$$\sqrt{a_j c_j - b_j^2} = \sqrt{a_j c_j}\, f\left(\frac{b_j}{\sqrt{a_j c_j}}\right),$$

we make the assumption

$$\sum_{j=1}^n \sqrt{a_j c_j} = 1, \tag{5.58}$$

which is compatible with the homogeneity of (2.19). Under this unrestrictive assumption, we have

$$\begin{aligned}\sum_{j=1}^n \sqrt{a_j c_j - b_j^2} &= \sum_{j=1}^n \sqrt{a_j c_j}\, f\left(\frac{b_j}{\sqrt{a_j c_j}}\right) \le f\left(\sum_{j=1}^n \sqrt{a_j c_j} \frac{b_j}{\sqrt{a_j c_j}}\right) \\ &= \sqrt{1 - \left(\sum_{j=1}^n b_j\right)^2} = \sqrt{\left(\sum_{j=1}^n \sqrt{a_j c_j}\right)^2 - \left(\sum_{j=1}^n b_j\right)^2}.\end{aligned} \tag{5.59}$$

The inequality we have obtained,

$$\sum_{j=1}^n \sqrt{a_j c_j - b_j^2} \le \sqrt{\left(\sum_{j=1}^n \sqrt{a_j c_j}\right)^2 - \left(\sum_{j=1}^n b_j\right)^2}, \tag{5.60}$$

is homogeneous and thus holds even without assuming (5.58). In addition, (5.60) is a refinement of (2.19), since, by (CS),

$$\left(\sum_{j=1}^n \sqrt{a_j c_j}\right)^2 \le \left(\sum_{j=1}^n a_j\right)\left(\sum_{j=1}^n c_j\right).$$

NB. Exactly the same proof (using, in the final step, (H) instead of (CS)) shows the following. Let $1 < p < \infty$ and α, β satisfy $\alpha + \beta = p$. Then

$$\begin{aligned}\sum_{j=1}^n \left(a_j^\alpha c_j^\beta - b_j^p\right)^{1/p} &\le \left(\left(\sum_{j=1}^n a_j^{\alpha/p} c_j^{\beta/p}\right)^p - \left(\sum_{j=1}^n b_j\right)^r\right)^{1/p} \\ &\le \left(\left(\sum_{j=1}^n a_j^\alpha\right)\left(\sum_{j=1}^n c_j^{\beta/(p-1)}\right)^{p-1} - \left(\sum_{j=1}^n b_j\right)^p\right)^{1/p}\end{aligned} \tag{5.61}$$

if $a_j > 0, b_j, c_j > 0,\ a_j^\alpha c_j^\beta > b_j^p,\ \forall\, j$.

The special choice $\alpha = 1$, $\beta = p - 1$ gives

$$\sum_{j=1}^{n} \left(a_j c_j^{p-1} - b_j^p\right)^{1/p} \le \left(\left(\sum_{j=1}^{n} a_j\right)\left(\sum_{j=1}^{n} c_j\right)^{p-1} - \left(\sum_{j=1}^{n} b_j\right)^{p}\right)^{1/p} \tag{5.62}$$

if $a_j > 0, b_j, c_j > 0,\ a_j c_j^{p-1} > b_j^p,\ \forall\, j.$

Problem 2.13. Assuming $a \le b \le c$, the couple $(a, (a+b+c)/3)$ majorizes the couple $((2a+b)/3, (2a+c)/3)$. By Theorem 1.1, we thus have (2.21).

In order to complete the proof, it suffices to show that

$$\begin{aligned} g(a) \coloneqq & f(b) + f(c) + 2f\left(\frac{a+b+c}{3}\right) - f\left(\frac{2b+a}{3}\right) \\ & - f\left(\frac{2c+a}{3}\right) - f\left(\frac{2b+c}{3}\right) - f\left(\frac{2c+b}{3}\right) \ge 0,\ \forall\, a \le b \le c. \end{aligned}$$

We have

$$g'(a) = \frac{2}{3}\left[f'\left(\frac{a+b+c}{3}\right) - \frac{1}{2}f'\left(\frac{2b+a}{3}\right) - \frac{1}{2}f'\left(\frac{2c+a}{3}\right)\right] \le 0,$$

where the inequality follows from (J) applied to the convex function f'.

Therefore, $g(a) \ge g(b)$, $\forall\, a \le b$, and in order to complet the proof it suffices to prove that $g(b) \ge 0$. This follows from (J) applied to f, since

$$g(b) = f(c) + f\left(\frac{2b+c}{3}\right) - 2f\left(\frac{2c+b}{3}\right).$$

A straightforward modification of the proof shows that (2.20) still holds if f is convex and f' is concave. (Start with $c \le b \le a$ and repeat the above argument.) QED

Problem 2.14. The statement has a majorization flavor. To make this more clear, we first note that it suffices to prove the desired inequality when $a_1 \cdots a_n = b_1 \cdots b_n$. Let $x_j \coloneqq \ln a_j, y_j \coloneqq \ln b_j, \forall\, j$. The constraints are

$$y_1 \ge y_2 \ge \cdots \ge y_n,\ x_1 \ge y_1,\ x_1 + x_2 \ge y_1 + y_2, \ldots, x_1 + \cdots + x_n = y_1 + \cdots + y_n,$$

which are the majorization conditions (1.22) and (1.16)–(1.19) stated "backwards". Thus, for every convex function f on $\mathbb{R}$, we have

$$\sum_{j=1}^{n} f(\ln a_j) \ge \sum_{j=1} f(\ln b_j).$$

In the special case where $f(x) = e^x$, we obtain the conclusion of the problem. QED

Problem 2.15. Assume that $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are identically ordered. Then we claim that

$$\sum_{j=1}^{n} a_j b_{\sigma(j)} \le \sum_{j=1}^{n} a_j b_j, \ \forall \sigma \in S_n. \tag{5.63}$$

Indeed, let $\tau \in S_n$ be such that $a_{\tau(1)} \le a_{\tau(2)} \le \cdots \le a_{\tau(n)}$ and $b_{\tau(1)} \le b_{\tau(2)} \le \cdots \le b_{\tau(n)}$ (see Problem 1.21). By (R), we have

$$\sum_{j=1}^{n} a_j b_j = \sum_{k=1}^{n} a_{\tau(k)} b_{\tau(k)} \ge \sum_{k=1}^{n} a_{\sigma^{-1}\tau(k)} b_{\tau(k)} = \sum_{j=1}^{n} a_j b_{\sigma(j)},$$

whence (5.63).

Similarly, if $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are oppositely ordered, then

$$\sum_{j=1}^{n} a_j b_{\sigma(j)} \ge \sum_{j=1}^{n} a_j b_j, \ \forall \sigma \in S_n. \tag{5.64}$$

Problem 2.16. The lists $a_1 := x_1, a_2 := \frac{x_1 x_2}{G}, \dots, a_n := \frac{x_1 \cdots x_n}{G^{n-1}}$ and $b_1 := \frac{G}{x_1}, b_2 = \frac{G^2}{x_1 x_2}, \dots, b_n := \frac{G^n}{x_1 \cdots x_n}$ are oppositely ordered (since $a_j b_j = G$, $\forall j$). By (5.64), we have

$$\begin{aligned} nG = \sum_{j=1}^{n} a_j b_j &\le a_2 b_1 + a_3 b_2 + \cdots + a_n b_{n-1} + a_1 b_n \\ &= x_2 + x_3 + \cdots + x_n + \frac{G^n}{x_1 \cdots x_{n-1}} \\ &= x_2 + x_3 + \cdots + x_n + \frac{x_1 \cdots x_n}{x_1 \cdots x_{n-1}} = \sum_{j=1}^{n} x_j = nA. \end{aligned}$$

QED

Problem 2.17. Consider the quadratic trinomial

$$T(x) := (a_1 x - b_1)^2 - \sum_{j=2}^{n} (a_j x - b_j)^2.$$

Let $x_0 := \frac{b_1}{a_1}$. Then

$$T(x_0) = -\sum_{j=2}^{n} (a_j x_0 - b_j)^2 \le 0, \tag{5.65}$$

and thus $\Delta/4 \ge 0$, which is the same as the Aczél inequality.

If we have equality, then $T(x) \ge 0$, $\forall x \in \mathbb{R}$, and in particular $T(x_0) \ge 0$. Combining this with (5.65), we find $a_j x_0 = b_j$, $\forall j = 2, \dots, n$, and this equality still holds when $j = 1$. Thus equality requires that $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are proportional. QED

Problem 2.18. (1) By homogeneity, we may assume that $a = 1/A$ and $b = 1/B$. Then

$$\frac{1}{AB} \le \frac{a_j}{b_j} \le AB, \ \forall\, j. \tag{5.66}$$

We obtain (2.24) from (1.62) with $\alpha \coloneqq AB$.

(2) Assume that (2.24) holds for some constant C instead of $\frac{4abAB}{(ab+AB)^2}$. With ℓ, m integers to be determined later, let $n \coloneqq \ell + m$, $a_1 = \cdots = a_\ell \coloneqq a$, $a_{\ell+1} = \cdots = a_n \coloneqq A$, $b_1 = \cdots = b_\ell \coloneqq B$, $b_{\ell+1} = \cdots = b_n \coloneqq b$. Then, with $x = x(\ell, m) \coloneqq \ell/m$, (2.24) reads

$$C \le \frac{(aBx + Ab)^2}{(a^2x + A^2)(B^2x + b^2)}. \tag{5.67}$$

Given any $x \ge 0$, we may choose $\ell, m \to \infty$ such that $x(\ell, m) \to x$. We find that (5.67) holds for every $x \ge 0$. We now minimize, with respect to x, the right-hand side of (5.67). This leads to the choice $x \coloneqq \frac{bA}{aB}$. For this choice, (5.67) becomes

$$C \le \frac{4abAB}{(ab+AB)^2},$$

which implies item (2). QED

Problem 2.19. By (CS) and Problem 1.27, we have

$$\sum_{j=1}^{n} \frac{x_j}{x_{j+1}+x_{j+2}} \sum_{j=1}^{n} x_j(x_{j+1}+x_{j+2}) \ge \left(\sum_{j=1}^{n} x_j\right)^2 \ge \frac{n}{2} \sum_{j=1}^{n} x_j(x_{j+1}+x_{j+2}),$$

whence the conclusion. QED

Problem 2.20. Proof by intimidation. Using the Gauss method, we find that

$$\begin{aligned} & a^2 + b^2 + c^2 + ab + bc + ca + a + b + c + \frac{3}{8} \\ = & \left(a + \frac{1}{2}b + \frac{1}{2}c + \frac{1}{2}\right)^2 + \frac{3}{4}\left(b + \frac{1}{3}c + \frac{1}{3}\right)^2 + \frac{2}{3}\left(c + \frac{1}{4}\right)^2, \end{aligned}$$

whence (2.26), with equality iff

$$a + \frac{1}{2}b + \frac{1}{2}c = -\frac{1}{2}, \ b + \frac{1}{3}c = -\frac{1}{3}, \ c = -\frac{1}{4},$$

which amounts to $a = b = c = -\frac{1}{4}$. QED

Problem 2.21. By homogeneity, we may assume that $ab = 1$. By symmetry, we may assume that $c \geq a, b$, and then $c \geq 1$. Write $a = e^{x/3}$, and then $b = e^{-x/3}$. We have to prove that

$$\frac{1}{e^x + e^{-x} + c} + \frac{1}{e^x + c^3 + c} + \frac{1}{e^{-x} + c^3 + c} \leq \frac{1}{c}, \ \forall\, c \geq 1. \tag{5.68}$$

We have $e^x + e^{-x} \geq 2$ (by (AM-GM)), and thus

$$\frac{1}{e^x + e^{-x} + c} \leq \frac{1}{2 + c}. \tag{5.69}$$

On the other hand, set $y := c^3 + c > 1$. Then the function

$$x \mapsto \frac{1}{e^x + y} + \frac{1}{e^{-x} + y}$$

has a maximum at $x = 0$, and thus

$$\frac{1}{e^x + c^3 + c} + \frac{1}{e^{-x} + c^3 + c} \leq \frac{2}{1 + c^3 + c}. \tag{5.70}$$

In view of (5.68)–(5.70), we are done if we prove that

$$\frac{1}{2 + c} + \frac{2}{1 + c^3 + c} \leq \frac{1}{c},$$

which holds even without the assumption $c \geq 1$, since it amounts to $(c-1)^2(c+1) \geq 0$.

Under the assumptions $c \geq a, b$ and $ab = 1$, equality arises iff $a = b = c = 1$. If we restore symmetry, equality arises iff $a = b = c$.

Bonus (from [5]): we have

$$a^3 + b^3 \geq a^2 b + ab^2, \tag{5.71}$$

with equality iff $a = b$. (This is an avatar of (R), but can also be seen directly, since it amounts to $(a-b)^2(a+b) \geq 0$.) Using (5.71) and its variants obtained by permutations, we find that

$$\begin{aligned}
&\frac{1}{a^3 + b^3 + abc} + \frac{1}{b^3 + c^3 + abc} + \frac{1}{c^3 + a^3 + abc} \\
&\leq \frac{1}{a^2 b + ab^2 + abc} + \frac{1}{b^2 c + c^2 b + abc} + \frac{1}{c^2 a + a^2 c + abc} \\
&= \frac{1}{ab(a+b+c)} + \frac{1}{bc(a+b+c)} + \frac{1}{ca(a+b+c)} = \frac{1}{abc},
\end{aligned}$$

with equality iff $a = b = c$. QED

Problem 2.22. Set $f(b) := (a+b+c+d)^2 - 8(ac+bd)$. Then $f'(b) < 0, \forall\, a < b < c < d$, and thus $f(b) > f(c)$. Now $f(c) = [2c - (a+d)]^2 \geq 0$. QED

Problem 2.23. The condition $\sum_{j=1}^{n} x_j^{n-1} = 1$ suggests taking the x_j^{n-1}'s as coefficients of a convex combination. With this in mind, we write

$$\sum_{j=1}^{n} \frac{x_j^{n-2}}{1 - x_j^{n-1}} = \sum_{j=1}^{n} x_j^{n-1} \frac{1}{x_j - x_j^n} = \sum_{j=1}^{n} x_j^{n-1} f(x_j),$$

where

$$f(x) \coloneqq \frac{1}{x - x^n}, \ \forall\, 0 < x < 1.$$

Noting that the minimum of f is achieved at $x_0 \coloneqq \dfrac{1}{n^{1/(n-1)}}$ and has the value $f(x_0) = \dfrac{n^{n/(n-1)}}{n-1}$, we find that

$$\sum_{j=1}^{n} x_j^{n-1} f(x_j) \ge \sum_{j=1}^{n} x_j^{n-1} \frac{n^{n/(n-1)}}{n-1} = \frac{n^{n/(n-1)}}{n-1},$$

with equality if $x_j = x_0$, $\forall\, j$. We find that $m = \dfrac{n^{n/(n-1)}}{n-1}$.

NB. Although the proof is inspired by (GJ), convexity was not used. QED

Problem 2.24. (1) Write $x = e^{-a}$, with $a > 0$. Then $f(t) = \dfrac{1 - e^{-at}}{t}$ and

$$f'(t) = \frac{ate^{-at} - 1 + e^{-at}}{t^2} = \frac{e^{-at}}{t^2}(at + 1 - e^{at}) < 0, \ \forall\, t > 0.$$

(2) Set $t_k \coloneqq (2k-1)q - 2k$, $s_k \coloneqq 2kq - 2k$, $k \ge 1$. Since $p < 2$, we have $q > 2$ and therefore $t_1 > 0$. Moreover, clearly $0 < t_k < s_k$, $\forall\, k \ge 1$.

Since

$$\frac{1}{t_k} = \frac{p-1}{2k-p}, \ \frac{1}{s_k} = \frac{p-1}{2k}, \text{ and } \frac{1}{t_k} - \frac{1}{s_k} = \frac{p-1}{2k-p} - \frac{p-1}{2k} = \frac{p(p-1)}{2k(2k-p)},$$

we find that

$$\begin{aligned} &\frac{p(p-1)}{2k(2k-p)} - \frac{p-1}{2k-p} x^{(2k-1)q-2k} + \frac{p-1}{2k} x^{2kq-2k} \\ &= \frac{1}{t_k} - \frac{1}{s_k} - \frac{1}{t_k} x^{t_k} + \frac{1}{s_k} x^{s_k} = f(t_k) - f(s_k) > 0. \end{aligned}$$

(3) (Proof due to James S. Frame) The inequality is clear for $x = 0$ and $x = 1$. Therefore, we may assume that $0 < x < 1$. Let

$$g(x) \coloneqq (1+x)^p + (1-x)^p - 2(1+x^q)^{p-1}, \ \forall\, 0 < x < 1.$$

Using (1.74) and item (2), we have, with f, t_k, s_k as in the previous items:

$$\frac{1}{2}g(x) = \sum_{k\geq 1} \frac{(p-1)\cdots(p-2k+1)}{(2k-1)!}\left[\frac{p}{2k}x^{2k} - x^{(2k-1)q} - \frac{p-2k}{2k}x^{2kq}\right]$$

$$=(2-p)x^2\left[\frac{p(p-1)}{2(2-p)} - \frac{p-1}{2-p}x^{q-2} + \frac{p-1}{2}x^{2q-2}\right]$$

$$+\sum_{k\geq 2}\frac{(2-p)(3-p)\cdots(2k-p)}{(2k-1)!}x^{2k}$$

$$\times\left[\frac{p(p-1)}{2k(2k-p)} - \frac{p-1}{2k-p}x^{(2k-1)q-2k} + \frac{p-1}{2k}x^{2kq-2k}\right]$$

QED

$$=(2-p)x^2(f(t_1)-f(s_1)) + \sum_{k\geq 2}\frac{(2-p)(3-p)\cdots(2k-p)}{(2k-1)!}x^{2k}$$

$$\times\left[f(t_k)-f(s_k)\right] > 0.$$

Problem 2.25. (1) We have

$$f(a) \leq \frac{1}{2}\left(\alpha + \frac{1}{\alpha}\right) \Leftrightarrow a^2 - \left(\alpha+\frac{1}{\alpha}\right)a + 1 \leq 0 \Leftrightarrow \left(a - \frac{1}{\alpha}\right)(a-\alpha) \leq 0.$$

This holds when $1/\alpha \leq a \leq \alpha$, with equality when $a = 1/\alpha$ or $a = \alpha$, whence (2.33).

(2) By (2.33) and homogeneity, we have

$$a_j^2 + b_j^2 \leq \left(\alpha + \frac{1}{\alpha}\right)a_j b_j,\ \forall\, j. \tag{5.72}$$

We obtain the desired result by taking, in (5.72), the sum over j. Equality holds iff $\dfrac{a_j}{b_j} \in \{1/\alpha, \alpha\}$, $\forall\, j$.

(3) Using the hint, we have, for $x > 0$,

$$g(x) - g(-x) = \frac{1}{p}\left(e^{(p-1)x} - e^{-(p-1)x}\right) - \frac{1}{q}\left(e^x - e^{-x}\right)$$

$$=2\sum_{k\geq 0}\frac{x^{2k+1}}{(2k+1)!}\left[\frac{1}{p}(p-1)^{2k+1} - \frac{1}{q}\right]$$

$$=2\sum_{k\geq 1}\frac{x^{2k+1}}{(2k+1)!}\left[\frac{1}{p}(p-1)^{2k+1} - \frac{1}{q}\right] > 0,$$

where we have used (1.4) for $k = 0$, and the fact that

$$\frac{1}{p}(p-1)^{2k+1} - \frac{1}{q} > \frac{1}{p}(p-1) - \frac{1}{q} = 0,\ \forall\, k \geq 1$$

(since $p > 2$).

(4) The case where $p = 2$ was treated above. Assume that $p > 2$. Since (using (1.4))

$$f'(a) = \frac{a^p - 1}{qa^2},\ \forall\, a > 0,$$

we find that f decreases on $(0, 1]$ and increases on $[1, \infty)$, and in particular its maximum on $[1/\alpha, \alpha]$ is achieved either for $a = 1/\alpha$ or for $a = \alpha$. By item (3), we have

$$f(1/\alpha) = g(-\ln \alpha) < g(\ln \alpha) = f(\alpha),$$

and thus

$$\frac{1}{p}a^p + \frac{1}{q} \le \frac{1}{\alpha}\left(\frac{1}{p}\alpha^p + \frac{1}{q}\right)a,\ \forall\, \frac{1}{\alpha} \le a \le \alpha. \tag{5.73}$$

By (5.73) and homogeneity, we have

$$\frac{1}{p}a_j^p + \frac{1}{q}b_j^q \le \frac{1}{\alpha}\left(\frac{1}{p}\alpha^p + \frac{1}{q}\right)a_j b_j,\ \text{if } a_j, b_j > 0 \text{ and } \frac{1}{\alpha} \le \frac{a_j}{b_j^q} \le \alpha. \tag{5.74}$$

Summing (5.74) over j, we obtain (2.34). Equality holds iff $a_j = \alpha b_j^{q-1}$, $\forall\, j$. QED

Problem 2.26. (1) We have

$$\begin{aligned}&(1 + x)^\alpha + (1 - x)^\alpha - (1 + x)^{1-\alpha} - (1 - x)^{1-\alpha}\\ &= 2\alpha(1 - \alpha)\sum_{k\ge 2} \frac{(\alpha + 1)\cdots(2k - 2 + \alpha) - (2 - \alpha)\cdots(2k - 1 - \alpha)}{(2k)!}x^{2k}.\end{aligned} \tag{5.75}$$

The assumption $1/2 < \alpha < 1$ implies that

$$\alpha > 0,\ 1 - \alpha > 0,\ \alpha + 1 > 2 - \alpha > 0, \ldots, 2k - 2 + \alpha > 2k - 1 - \alpha > 0. \tag{5.76}$$

We obtain (2.35) from (5.75) and (5.76).

(2) With no loss of generality, we may assume that $a + b = 2$. Set $X := a^{p/2}$, $Y := b^{p/2}$. We have to prove that

$$m := \min\left\{X^2 + Y^2 + (2^p - 2)XY;\ X \ge 0,\ Y \ge 0,\ X^{2/p} + Y^{2/p} = 2\right\} \ge 2^p. \tag{5.77}$$

Noting that (2.36) becomes an inequality when $a = b$, we actually have $m \le 2^p$, and thus (5.77) amounts to $m = 2^p$.

Let (X, Y) achieve the minimum in (5.77). If $X = 0$ or $Y = 0$, then $m = 2^p$ and we are done. Otherwise, we have $X > 0$, $Y > 0$ and there exists some $\mu \in \mathbb{R}$ such that

$$2X + (2^p - 2)Y = \mu X^{2/p-1}, \tag{5.78}$$

$$2Y + (2^p - 2)X = \mu Y^{2/p-1}, \tag{5.79}$$

$$X^{2/p} + Y^{2/p} = 2, \tag{5.80}$$

$$X^2 + Y^2 + (2^p - 2)XY = m. \tag{5.81}$$

Multiplying (5.78) by $X/2$, (5.79) by $Y/2$, adding the results and comparing the total with (5.81), we find that

$$\mu = m \le 2^p. \tag{5.82}$$

Inserting (5.82) into (5.78) and (5.79) and adding the two inequalities obtained, we obtain

$$X + Y \le X^{2/p-1} + Y^{2/p-1}. \tag{5.83}$$

Assuming, with no loss of generality, that $X \ge Y$, write $X^{2/p} = 1 + x$, $Y^{2/p} = 1 - x$, with $x \in [0, 1)$, and set $\alpha := \frac{p}{2} \in (1/2, 1)$. Then (5.83) becomes

$$(1 + x)^\alpha + (1 - x)^\alpha \le (1 + x)^{1-\alpha} + (1 - x)^{1-\alpha}. \tag{5.84}$$

By item (1), (5.84) implies that $x = 0$, and thus $X = Y = 1$ and $m = 2^p$.

(3) (2.36) *improves* (2.37). This amounts to

$$a^p + b^p + (2^p - 2)(ab)^{p/2} \le 2^{p-1}(a^p + b^p),\ \forall\, a, b > 0,$$

which in turn is equivalent to the straightforward inequality

$$a^p + b^p - 2(ab)^{p/2} = \left(a^{p/2} - b^{p/2}\right)^2 \ge 0,\ \forall\, a, b > 0.$$

(2.36) *improves* (2.30) *for x sufficiently close to* 0. If $0 \le x \le 1$, then (2.36) yields

$$(1 + x)^p + (1 - x)^p \ge 2^p - (2^p - 2)(1 - x^2)^{p/2}.$$

Therefore, the desired conclusion amounts to the existence of some $\delta > 0$ such that

$$(2^p - 2)(1 - x^2)^{p/2} + 2(1 + x^q)^{p-1} \le 2^p,\ \forall\, 0 \le x \le \delta. \tag{5.85}$$

Using (2.38) and the fact that $0 < p/2 < 1$ and $0 < p - 1 < 1$, we find that

$$(1 - x^2)^{p/2} \le 1 - \frac{p}{2}x^2,\ \forall\, x \in [0, 1], \tag{5.86}$$
$$(1 + x^q)^{p-1} \le 1 + (p - 1)x^q,\ \forall\, x \in [0, 1]. \tag{5.87}$$

Inserting (5.86)–(5.87) into (5.85), we find that

$$(2^p - 2)(1 - x^2)^{p/2} + 2(1 + x^q)^{p-1} \le 2^p - \frac{p}{2}(2^p - 2)x^2 + 2(p - 1)x^q \le 2^p, \tag{5.88}$$

the latter inequality being valid for sufficiently small $x \ge 0$ (since $q > 2$ and therefore $\frac{p}{2}(2^p - 2)x^2 > 2(p - 1)x^q$ for sufficiently small x). QED

Problem 2.27. The obvious solutions are $(x, y) = (1, 0)$ and $(x, y) = (0, 1)$. Let us prove that these are the only ones. Consider (x, y) such that $x^2 + y^2 = 1$.

If $y < 0$, then $2^x \leq 2$ and $2^y < 1$, so that (x, y) does not solve (2.39). Similarly if $x < 0$.

Next, let

$$m := \min \left\{ 2^x + 2^y;\ x, y \geq 0\, x^2 + y^2 = 1 \right\} \leq 3.$$

We claim that m is achieved only for $(x, y) = (1, 0)$ or $(0, 1)$. In view of the above discussion, this completes the proof. Indeed, argue by contradiction. If $(x, y) \in (0, 1)^2$ achieves m, then there exists some $\lambda \in \mathbb{R}$ such that

$$2^x \ln 2 = 2\lambda x, \tag{5.89}$$

$$2^y \ln 2 = 2\lambda y. \tag{5.90}$$

Set $f(x) := \dfrac{2^x}{x}$, where $0 < x \leq 1$. Then

$$f'(x) = \frac{2^x}{x^2}(x \ln 2 - 1) \leq \frac{2^x}{x^2}(\ln 2 - 1) < 0.$$

Since (5.89)–(5.90) imply that $f(x) = f(y)$, we find that $x = y$, and thus $x = y = \dfrac{1}{\sqrt{2}}$. We obtain the contradiction

$$3 \geq m = 2^{1+1/\sqrt{2}} > 2^{1,7} > 3.$$

QED

Problem 2.28. When $n = 3$, we recover the familiar inequality (1.63). Assume that $n \geq 4$. We rewrite (2.40) in the more tractable form

$$\left(\sum_{j=1}^{n} \prod_{k \neq j} a_k \right)^2 \leq \frac{1}{n^{n-3}} \left(\sum_{j=1}^{n} a_j^2 \right)^{n-1}, \ \forall\, a_1, \ldots, a_n > 0. \tag{5.91}$$

By homogeneity, we may assume that $\sum_{j=1}^{n} a_j^2 = n$. Since (5.91) becomes an equality when $a_1 = \cdots = a_n$, we find that (5.91) is equivalent to

$$\max \left\{ \sum_{j=1}^{n} \prod_{k \neq j} a_k;\ \sum_{j=1}^{n} a_j^2 = n, a_j > 0, \forall\, j \right\} = n.$$

We will actually prove the seemingly stronger assertion

$$\max \left\{ \sum_{j=1}^{n} \prod_{k \neq j} a_k;\ \sum_{j=1}^{n} a_j^2 = n, a_j \geq 0, \forall\, j \right\} = n. \tag{5.92}$$

(To see that max is achieved in (5.92), we argue as for (1.80).)

Set

$$a := (a_1 \dots, a_n), f(a) := \sum_{j=1}^{n} \prod_{k \neq j} a_k, h(a) := \sum_{j=1}^{n} a_j^2 - n, K := \{a \in [0, \infty)^n; h(a) = 0\}.$$

We claim that, if $a \in K$ and $a_j = 0$ for some j, then $f(a) < n$. Indeed, assuming, e.g., that $a_n = 0$, we have

$$\begin{aligned} f(a) &= a_1 \cdots a_{n-1} = \sqrt{a_1^2 \cdots a_{n-1}^2} \leq \left(\frac{a_1^2 + \cdots + a_{n-1}^2}{n-1}\right)^{(n-1)/2} \\ &\leq \left(1 + \frac{1}{n-1}\right)^{(n-1)/2} < \left(1 + \frac{1}{n-1}\right)^{n-1} < e^{1/2} < n. \end{aligned} \tag{5.93}$$

Therefore, at a maximum point a of f on K we have $a_j > 0$, $\forall\, j$. Since $\nabla h(a) \neq 0$ for such a, by Theorem 1.4 there exists some μ such that

$$\nabla f(a) = \mu \nabla h(a). \tag{5.94}$$

In order to have a tractable formula for ∇f, we come back to the original inequality (2.40) and write

$$f(a) = P(a) \sum_{j=1}^{n} \frac{1}{a_j}, \text{ with } P = P(a) := \prod_{j=1}^{n} a_j.$$

We find that

$$\frac{\partial f}{\partial a_\ell} = \frac{\partial P}{\partial a_\ell} \sum_{j=1}^{n} \frac{1}{a_j} - P \frac{1}{a_\ell^2} = \frac{P}{a_\ell} \sum_{j=1}^{n} \frac{1}{a_j} - P \frac{1}{a_\ell^2} = \frac{P}{a_\ell} \sum_{j \neq \ell} \frac{1}{a_j}.$$

Therefore, (5.94) amounts to the system

$$\frac{P(a)}{a_\ell} \sum_{j \neq \ell} \frac{1}{a_j} = 2\mu a_\ell, \ \forall\, \ell. \tag{5.95}$$

In particular, (5.95) implies that $\mu > 0$. Subtracting the lines k and ℓ of (5.95), with $k \neq \ell$, we find that

$$P(a) \sum_{j \neq k, \ell} \frac{1}{a_j} \left(\frac{1}{a_k} - \frac{1}{a_\ell}\right) = 2\mu(a_k - a_\ell). \tag{5.96}$$

If, say, $a_k > a_\ell$, then the left-hand side of (5.96) is negative, while the right-hand one is positive, a contradiction. Therefore, we have $a_1 = \cdots = a_n = 1$. It follows that $\max$ in (5.92) is indeed n. QED

Problem 2.29. By repeating the proof of (5.92), for every $n \geq 3$ and $\sigma \geq 1$ we have

$$\max\left\{\sum_{j=1}^{n}\prod_{k\neq j}a_k;\ \sum_{j=1}^{n}a_j^\sigma = n, a_j \geq 0, \forall\, j\right\} = n. \tag{5.97}$$

(Some care is needed in checking the validity of (5.93) when $n = 2$.) Equivalently, if $\sigma \geq 1$, we have

$$M_0^n \leq M_{-1}M_\sigma^{n-1},\ \forall\, a_1, \dots, a_n > 0. \tag{5.98}$$

Substituting, in (5.98), $a_j = b_j^r$, and letting $s \coloneqq r\sigma \geq r$, we find that

$$M_0 \leq M_{-r}M_s^{n-1},\ \forall\, 0 < r \leq s,\ \forall\, a_1, \dots, a_n > 0, \tag{5.99}$$

which is the desired generalization of (2.41). QED

Problem 2.30. Testing (2.44) with $x_j = y_j = 1$, $\forall\, j$, we find that $2^p \leq 2^n$, and thus the condition $p \leq n$ is necessary.

From now on, we assume that $0 < p \leq n$. By homogeneity, it suffices to prove that

$$\min\left\{\prod_{j=1}^{n}\left(x_j^p + y_j^p\right);\ x_j, y_j > 0,\ \prod_{j=1}^{n}x_j + \prod_{j=1}^{n}y_j = 1\right\} \coloneqq m \geq 1. \tag{5.100}$$

Assume, as suggested, that m is achieved by some configuration $(x_1, \dots, y_n)$. Set

$$S_j \coloneqq x_j^p + y_j^p,\ P \coloneqq \prod_{j=1}^{n}S_j,\ Q \coloneqq \prod_{j=1}^{n}x_j,\ R \coloneqq \prod_{j=1}^{n}y_j.$$

By Theorem 1.4, there exists some $\lambda \in \mathbb{R}$ such that

$$px_j^{p-1}\frac{P}{S_j} = \lambda\frac{Q}{x_j},\ \forall\, j,$$
$$py_j^{p-1}\frac{P}{S_j} = \lambda\frac{R}{y_j},\ \forall\, j,$$

and therefore

$$\frac{y_j^p}{x_j^p} = \frac{Q}{R},\ \forall\, j,$$

which implies that there exists some $t > 0$ such that $y_j = tx_j$, $\forall\, j$. Substituting this into (5.100), we find that (2.44) amounts to

$$(1 + t^n)^p \leq (1 + t^p)^n,\ \forall\, t > 0, \tag{5.101}$$

which is obvious for $p = n$ and we prove below for $0 < p < n$.

Set

$$f(t) \coloneqq \frac{(1+t^n)^p}{(1+t^p)^n},\ \forall\, t > 0.$$

Then

$$f'(t) = np\frac{(1+t^n)^{p-1}}{(1+t^p)^{n+1}}\left(t^{n-1} - t^{p-1}\right)\begin{cases} < 0, & \text{if } 0 < t < 1 \\ > 0, & \text{if } t > 1 \end{cases}.$$

We find that $f(t) \geq f(1) = 1$, which implies (5.101) and completes the proof.

Bonus. Sketch of the proof of the fact that m is achieved. This follows by combining two observations.

Observation 1. We have $m \leq 1$. To see this, consider the competitor $x_j = y_j = 1, \forall\, j$.

Observation 2. We may replace any competitor with another competitor such that $S_1 = S_2 = \cdots = S_n$. Indeed, let

$$g(x_1, \ldots, y_n) \coloneqq \prod_{j=1}^{n}\left(x_j^p + y_j^p\right).$$

Given any competitor $(x_1, \ldots, y_n)$ in (5.100), consider the positive numbers $t_1, \ldots, t_n$ such that

$$t_1 \cdots t_n = 1,$$
$$(t_1x_1)^p + (t_1y_1)^p = (t_2x_2)^p + (t_2y_2)^p = \cdots = (t_nx_n)^p + (t_ny_n)^p.$$

Then $(t_1x_1, \ldots, t_ny_n)$ is still a competitor in (5.100) such that

$$g(t_1x_1, \ldots, t_ny_n) = g(x_1, \ldots, y_n)$$

and, in addition,

$$\left(x_j^p + y_j^p\right) = [g(x_1, \ldots, y_n)]^{1/n},\ \forall\, j.$$

Combining the two observations, we find that

$$m = \min\{g(x_1 \ldots, y_n);\ 0 \leq x_j, y_j \leq 1,\ \prod_{j=1}^{n} x_j + \prod_{j=1}^{n} y_j = 1\},$$

and then the fact that m is achieved amounts to the textbook argument "a continuous function on a compact set has a maximum point and a minimum point". QED

Problem 2.31. By symmetry, we may assume that $a \le b \le c$. We consider the case where f is non-decreasing, the other case being similar. Since f is non-decreasing and non-negative, the numbers $x \coloneqq f(a)$, $y \coloneqq f(b) - f(a)$ and $z \coloneqq f(c) - f(b)$ are non-negative and satisfy $f(a) = x$, $f(b) = x + y$, $f(c) = x + y + z$. Then

$$\begin{aligned}(a-b)(a-c)f(a) &+ (b-c)(b-a)f(b) + (c-a)(c-b)f(c) \\ =&[(a-b)(a-c) + (b-c)(b-a) + (c-a)(c-b)]\,x \\ &+ [(b-c)(b-a) + (c-a)(c-b)]\,y + (c-a)(c-b)z \\ =&\big[(b-a)^2 + (c-a)(c-b)\big]\,x + (c-b)^2 y + (c-a)(c-b)z \ge 0.\end{aligned}$$

QED

Problem 2.32. Recall that $a, b, c > 0$ are the side lengths of a triangle iff $a < b + c, b < c + a, c < a + b$. This yields immediately the implication "$\Leftarrow$".

Conversely, set

$$u \coloneqq \frac{a+c-b}{2} > 0,\ v \coloneqq \frac{b+a-c}{2} > 0,\ w \coloneqq \frac{c+b-a}{2} > 0.$$

Then, clearly, $a = u + v, b = v + w, c = w + u$. QED

Problem 2.33. Substituting $a = u + v$, etc., (2.47) is equivalent to

$$\begin{aligned}(u+v)^3 + (v+w)^3 &+ (w+u)^3 \\ &\ge (u+v)(v-u)^2 + (v+w)(w-v)^2 + (w+u)(u-w)^2 \\ &\quad + 3(u+v)(v+w)(w+u),\end{aligned}$$

which can be rewritten, using the identities $(u+v)^3 - (u+v)(v-u)^2 = 4uv(u+v)$, etc., as

$$4uv(u+v) + 4vw(v+w) + 4wu(w+u) \ge 3(u+v)(v+w)(w+u),$$

and, after simplifications, as

$$6uvw \le u^2v + uv^2 + v^2w + vw^2 + w^2u + wu^2. \tag{5.102}$$

In turn, (5.102) is a special case of (2.9)–(2.10), with

$$\lambda_j = 1/6, \alpha = (1,1,1), \alpha_1 = (2,1,0), \alpha_2 = (1,2,0), \alpha_3 = (0,2,1), \text{ etc.}$$

QED

Problem 2.34. (1) Noting that

$$\begin{aligned}\left(a^2+b^2-c^2\right)\left(b^2+c^2-a^2\right) &= \left(b^2 + \left(a^2-c^2\right)\right)\left(b^2 - \left(a^2-c^2\right)\right) \\ &= b^4 - \left(a^2-c^2\right)^2 \\ (a+b-c)(b+c-a) = (b+(a-c))(b-(a-c)) &= b^2 - (a-c)^2,\end{aligned}$$

and developing the expressions in (2.48), we find that (2.48) amounts to

$$2b^2(c-a)^2\left(c^2+a^2-b^2\right) \ge 0,$$

which holds since, by assumption, $c^2 + a^2 > b^2$.

(2) If a, b, c are the lengths of an acute triangle, then it suffices to use item (1) and multiply the three inequalities of the type (2.48) (obtained by circular permutation).

Otherwise, we have, for example, $a^2 \geq b^2 + c^2$. But then $a^2 + b^2 > c^2$ and $a^2 + c^2 > b^2$, so that the left-hand side of (2.48) is non-positive, and thus (2.49) holds. QED

Problem 2.36. Set, for each $j \geq 2$ for which this makes sense, $\alpha_j \coloneqq a_j - a_{j-1}$. Then convexity is equivalent to $\alpha_j \leq \alpha_{j+1}$, whenever this inequality makes sense. By Problem 2.35, this is thus equivalent to the existence of non-negative numbers $x_3, \ldots,$ such that:

$$a_2 = a_1 + \alpha_2,\ a_3 = a_2 + \alpha_3 = a_1 + \alpha_2 + x_3,\ a_4 = a_3 + \alpha_4 = a_1 + \alpha_2 + x_3 + x_4, \ldots,$$

that is, when $j \geq 3$, $a_j = a_1 + \alpha_2 + \sum_{\ell=3}^{j} x_\ell$. QED

Problem 2.37. *Step 1.* Reformulation of (5.24). Let x_j be as in the solution of Problem 2.36. Set $x_1 \coloneqq a_1$, $x_2 \coloneqq \alpha_2 = a_2 - a_1$. Define similarly y_j. With the notation in the solution of Problem 1.19, and by repeating the argument leading to (5.24), item (1) is equivalent to

$$\begin{gathered}\sum_{k=1}^{n}\sum_{\ell=1}^{n} x_k y_\ell \operatorname{Card} S(k,\ell) \leq \sum_{k=1}^{n}\sum_{\ell=1}^{n} x_k y_\ell \operatorname{Card} T(k,\ell),\\ \forall\, x_1, x_2, y_1, y_2,\ \forall\, x_3, \ldots, x_n, y_3, \ldots, y_n \geq 0.\end{gathered} \tag{5.103}$$

As noticed there, $\operatorname{Card} S(k,1) = \operatorname{Card} T(k,1)$, $\forall\, k$, and $\operatorname{Card} S(1,\ell) = \operatorname{Card} T(1,\ell)$, $\forall\, \ell$, and thus (5.103) amounts to

$$\begin{gathered}\sum_{k=2}^{n}\sum_{\ell=2}^{n} x_k y_\ell \operatorname{Card} S(k,\ell) \leq \sum_{k=2}^{n}\sum_{\ell=2}^{n} x_k y_\ell \operatorname{Card} T(k,\ell),\\ \forall\, x_2, y_2,\ \forall\, x_3, \ldots, x_n, y_3, \ldots, y_n \geq 0.\end{gathered} \tag{5.104}$$

Step 2. Simple formulas for $\operatorname{Card} S(k,2)$ and $\operatorname{Card} S(2,\ell)$, and consequences. Since

$$S(k,2) = \{k, k+1, \ldots, n\} \setminus \{\sigma^{-1}(1)\},\ \forall\, k,$$

we have

$$\operatorname{Card} S(k,2) = \begin{cases} n-k+1, & \text{if } k > \sigma^{-1}(1) \\ n-k, & \text{if } k \leq \sigma^{-1}(1), \end{cases},\ \forall\, k,$$

and, in particular, $\operatorname{Card} S(k,2)$ takes twice *only* the value $n - \sigma^{-1}(1)$.

Similarly, we have

$$S(2,\ell) = \{\sigma^{-1}(\ell), \sigma^{-1}(\ell+1), \ldots, \sigma^{-1}(n)\} \setminus \{1\}, \forall\, \ell,$$

and

$$\operatorname{Card} S(2,\ell) = \begin{cases} n-\ell+1, & \text{if } \ell > \sigma(1) \\ n-\ell, & \text{if } \ell \le \sigma(1), \end{cases}, \ \forall\, \ell.$$

In particular, $\operatorname{Card} S(2,\ell)$ takes twice *only* the value $n-\sigma(1)$.

From the above, we derive the following consequences:

$$\begin{aligned} &[\operatorname{Card} S(k,2) = \operatorname{Card} T(k,2),\ \forall\, k, \text{ and } \operatorname{Card} S(2,\ell) = \operatorname{Card} T(2,\ell),\ \forall\, \ell] \\ &\qquad\qquad\qquad \Leftrightarrow \left[\sigma(1) = \tau(1) \text{ and } \sigma^{-1}(1) = \tau^{-1}(1)\right]. \end{aligned} \tag{5.105}$$

Step 3. "(2)$\Rightarrow$ (1)" By "$\Leftarrow$" in (5.105), we have $\operatorname{Card} S(k,2) = \operatorname{Card} T(k,2)$, $\forall\, k$, and $\operatorname{Card} S(2,\ell) = \operatorname{Card} T(2,\ell)$, $\forall\, \ell$. Therefore, (5.104) amounts to

$$\begin{aligned} &\sum_{k=3}^{n}\sum_{\ell=3}^{n} x_k y_\ell \operatorname{Card} S(k,\ell) \le \sum_{k=3}^{n}\sum_{\ell=3}^{n} x_k y_\ell \operatorname{Card} T(k,\ell), \\ &\qquad\qquad\qquad \forall\, x_3,\dots,x_n,y_3,\dots,y_n \ge 0. \end{aligned} \tag{5.106}$$

In turn, (5.106) holds, thanks to the assumption (2.51) (which amounts to $\operatorname{Card} S(k,\ell) \le \operatorname{Card} T(k,\ell)$, $\forall\, k,\ell \ge 3$).

Step 4. "(1)$\Rightarrow$ (2)" *Step 1.* $\sigma(1) = \tau(1)$ and $\sigma^{-1}(1) = \tau^{-1}(1)$. Fix $m \ge 1$ and let $x_k \coloneqq \delta_{km}$. Letting first $y_2 = 1$, then $y_2 = -1$ in (5.103), we find that $\operatorname{Card} S(m,2) = \operatorname{Card} T(m,2)$, $\forall\, m$, and similarly $\operatorname{Card} S(2,p) = \operatorname{Card} T(2,p)$, $\forall\, p$. By "$\Rightarrow$" in (5.105), this implies that $\sigma(1) = \tau(1)$ and $\sigma^{-1}(1) = \tau^{-1}(1)$.

Finally, let $3 \le m, p \le n$. By letting $x_k \coloneqq \delta_{km}$ and $y_\ell \coloneqq \delta_{\ell p}$, we find that $\operatorname{Card} S(m,p) \le \operatorname{Card} T(m,p)$, i.e., (2.51) holds. QED

Problem 2.38. (1) Set $S \coloneqq a+b$ and $P \coloneqq ab$. Then (2.52) amounts to

$$S^2P - 3P^2 - 2P + 1 \ge 0.$$

Noting that $S^2 \ge 4P$, we find that

$$S^2P - 3P^2 - 2P + 1 \ge (P-1)^2 \ge 0,$$

whence the conclusion. Equality occurs iff $P = 1$ and $S = 4$, i.e., $a = b = 1$.
(2) Set $Q \coloneqq \sqrt{P} \ge \frac{1}{2}$. Since $S \ge 2Q$, we may write $S = 2Q + x$, with $x \ge 0$. (2.53) is equivalent to

$$\underbrace{\left[(1+Q)^2 - 2\right]}_{\ge 1/4} x^2 + 2(1+Q)^2 \underbrace{(2Q-1)}_{\ge 0} x \ge 0.$$

Equality holds iff $x = 0$, i.e., $a = b$.

Remark. By carefully choosing $x > 0$ in the above, one may see that, under the assumption $ab < \frac{1}{4}$, inequality (2.53) is, in general, *wrong*. QED

Problem 2.39. Write $a = 1 + x$, $b = 1 + y$, $c = 1 + z$, where $x, y, z \geq 0$. With these substitutions, (2.54) becomes $x^3 + y^3 + z^3 \geq 3xyz$, which follows from (AM-GM). QED

Problem 2.40. It is natural to set, as for a triangle,

$$-a + b + c + d = 4x > 0, \ a - b + c + d = 4y > 0, \text{ etc.},$$

so that

$$a + b + c + d = 2(x + y + z + t), \ a = -x + y + z + t, \text{ etc.}$$

With these substitutions, (2.55) becomes

$$4^4 xyzt \leq 2^4 (z + t)(t + x)(x + y)(y + z),$$

which follows from $z + t \geq 2\sqrt{zt}$, etc. QED

Problem 2.41. The last constraint, $2d > a$, seems superfluous, since, if all the other constraints except this one are satisfied, then the inequality to prove is clear. Therefore, we keep only the first three constraints, and write, with $x, y, z > 0$,

$$a = \frac{b}{2} + 4x, b = \frac{c}{2} + 4y, \ c = \frac{d}{2} + 4z,$$

leading to

$$a = \frac{d}{8} + 4x + 2y + z, \ b = \frac{d}{4} + 4y + 2z, \ c = \frac{d}{2} + 4z. \tag{5.107}$$

By homogeneity, we may choose $d = 8$, and then, inserting (5.107) into (2.56), we have to prove that

$$8^3 xyz(16 - (1 + 4x + 2y + z)) \leq (1 + 4x + 2y + z)(2 + 4y + 2z)(4 + 4z)8,$$

which is equivalent to

$$128xyz \leq (1 + 4x + 2y + z)(1 + 2y + z)(1 + z) + 8xyz(1 + 4x + 2y + z) =: R.$$

This inequality could be proved by majorization, but a simpler approach consists of applying (AM-GM) to each expression above, by writing, e.g.,

$$1 + 4x + 2y + z = 1 + x + x + x + x + y + y + z \geq 8\big(1 \cdot x^4 \cdot y^2 \cdot z\big)^{1/8}.$$

We then get

$$\begin{aligned} R \geq & 8\big(1 \cdot x^4 \cdot y^2 \cdot z\big)^{1/8} \cdot 4\big(1 \cdot y^2 \cdot z\big)^{1/4} \cdot 2(1 \cdot z)^{1/2} + 8xyz \cdot 8\big(1 \cdot x^4 \cdot y^2 \cdot z\big)^{1/8} \\ = & 64x^{1/2}y^{1/4}z^{1/8}\left(y^{1/2}z^{3/4} + xyz\right) \geq 128x^{1/2}y^{1/4}z^{1/8}\left(y^{1/2}z^{3/4} \cdot xyz\right)^{1/2} \\ = & 128xyz. \end{aligned}$$

QED

5.3 More on inequalities

Problem 3.1. If $A = (a_{jk})_{1\le j,k\le n}$, then, clearly, $a_{jk} \ge 0$ and (with δ standing for the Kronecker symbol)

$$a_{jk} = \sum_{\sigma\in S_n} \lambda_\sigma \delta_{k\sigma(j)}.$$

We have

$$\sum_{k=1}^n a_{jk} = \sum_{k=1}^n \sum_{\sigma\in S_n} \lambda_\sigma \delta_{k\sigma(j)} = \sum_{\sigma\in S_n} \sum_{k=1}^n \lambda_\sigma \delta_{k\sigma(j)} = \sum_{\sigma\in S_n} \lambda_\sigma = 1,\ \forall\, j,$$

and similarly $\sum_{j=1}^n a_{jk} = 1, \forall\, k$. QED

Problem 3.2. (1) Each line (and each column) must contain a non-zero entry.

(2) By the above, on each line and on each column of A there is exactly one non-zero entry, which has to be one. If $a_{jf(j)}$ denotes non-zero entry on line j, then f is injective, (for otherwise the column $f(j)$ would contain two non-zero entries), and thus f is a permutation and $A = P_f$. QED

Problem 3.3. Since A is not a permutation matrix, it contains some entry $a_{j_1k_1} \in (0,1)$. Since the sum of the entries on the line j_1 is 1, there has to be another entry on this line such that $a_{j_1k_2} \in (0,1)$. Similarly, there has to be an entry on the column k_2, different from $a_{j_1k_2} \in (0,1)$ and such that $a_{j_2k_2} \in (0,1)$. We continue as above until the (finite) step where we obtain again one of the entries considered before. This yields m different entries, with $m \ge 4$. Consider now the *smallest* m that occurs in such a process. We claim that m is *even*, whence the conclusion of the problem. Argue by contradiction and assume that m is odd. Then the first, the second, and the last entry in the chain are on the same line. This means that we can: (i) remove the first and the last entry in the chain; (ii) replace the first entry with the last one, and obtain a chain of $(m-1)$ entries, contradicting thus the minimality of m. QED

Problem 3.4. (1) Let

$$\alpha \coloneqq \min\{a_{j_1k_1}, a_{j_2k_2}, \ldots\} > 0,$$

respectively

$$\beta \coloneqq \min\{a_{j_1k_2}, a_{j_2k_3}, \ldots\} > 0.$$

Let B, respectively C, be the matrix obtained by replacing, in A, the cycle $a_{j_1k_1}, \ldots, a_{j_\ell k_1}$ with $a_{j_1k_1} - \alpha, a_{j_1k_2} + \alpha, a_{j_2k_2} - \alpha, a_{j_2k_3} + \alpha, \ldots, a_{j_\ell k_1} + \alpha$, respectively $a_{j_1k_1} + \beta, a_{j_1k_2} - \beta, a_{j_2k_2} + \beta, a_{j_2k_3} - \beta, {}^{\prime}s, a_{j_\ell k_1} - \beta$.

By definition of α, respectively β, B, respectively C, have at least one zero entry more than A. B and C are also clearly DS. Finally, if we set $t \coloneqq \dfrac{\alpha}{\alpha+\beta} \in (0,1)$, then

$(1-t)C + tD = A$. (This can be checked separately on the entries of the form $a_{j_\ell k_\ell}$, $a_{j_\ell k_{\ell+1}}$, and on the entries that are not in the cycle.)

(2) By Problem 3.2, iterating the process in item (1), we express, in at most $n^2 - n$ steps, A as a convex combination of permutation matrices. QED

Problem 3.5. The square matrices of size n form a vector space of dimension n^2. QED

Problem 3.6. If A is of size n, set $B \coloneqq A - \mathrm{Id}$, $B = (b_{jk})_{1\le j,k\le n}$. Consider the application

$$\Phi(A) \coloneqq (b_{jk})_{1\le j,k\le n-1}.$$

When A is DS, B (and thus A) is completely determined by $\Phi(A)$, since

$$b_{jn} = -\sum_{k=1}^{n-1} b_{jk},\ \forall\, 1 \le j \le n-1,\ b_{nk} = -\sum_{j=1}^{n-1},\ \forall\, 1 \le k \le n.$$

Thus Φ is one-to-one on the set of DS matrices, which can therefore be identified with a set of matrices of size $(n-1)$, and thus with a part of a space of dimension $(n-1)^2$.

Les A be DS. Write, as in Theorem 3.1, $A = \sum_{\sigma\in S_n} \lambda_\sigma P_\sigma$. Clearly, $\Phi(A) = \sum_{\sigma\in S_n} \lambda_\sigma \Phi(P_\sigma)$. By the Steinitz lemma, we may assume that at most $(n-1)^2$ of the λ_σ's are non-zero. Finally, $A = \sum_{\sigma\in S_n} \lambda_\sigma P_\sigma$, where at most $(n-1)^2$ of the λ_σ's are non-zero. QED

Problem 3.7. We have to consider the case where g' is (strictly) increasing. The proof is essentially the same as the one of Theorem 3.3. Using the same notation, it is convenient to write, this time,

$$K(x,y,z) = r\left(\frac{\gamma-\beta}{\gamma-\alpha}\alpha + \frac{\beta-\alpha}{\gamma-\alpha}\gamma\right) - \frac{\gamma-\beta}{\gamma-\alpha}r(\alpha) - \frac{\beta-\alpha}{\gamma-\alpha}r(\gamma).$$

If $x_j < x_{j+1}$, then $\gamma(x_{j+1}) > \beta(x_{j+1}) > \alpha(x_{j+1})$, and we find that $F'(x_{j+1}) > 0$. If $x_j = x_{j+1}$, then, as in the proof of Theorem 3.3, for $y > x_{j+1}$ close to x_{j+1} we have $F'(y) > 0$. In both cases, we obtain a contradiction. QED

Problem 3.9. (1) (3.19) is equivalent to

$$4(x+y)\sqrt{xy} \le (x+y)^2 + 4xy,\ \forall\, x,y > 0,$$

which in turn amounts to $\left(\sqrt{x} - \sqrt{y}\right)^4 \ge 0$.

(2) Assume that (3.20) holds. Then

$$\theta \ge \frac{G-H}{A-H},\ \forall\, x,y>0,\ x\ne y,$$

and in particular, by taking $x = t$, $y = \dfrac{1}{t}$, with $t \ne 1$, we find that

$$\theta \ge \lim_{t\to 1}\frac{G-H}{A-H} = \lim_{t\to 1}\frac{2t}{(t+1)^2} = \frac{1}{2}.$$

Finally, when $\theta \geq \frac{1}{2}$, (3.20) follows from (3.19), since

$$\theta A + (1-\theta)H = \frac{1}{2}A + \frac{1}{2}H + \left(\theta - \frac{1}{2}\right)(A - H) \geq \frac{1}{2}A + \frac{1}{2}H.$$ QED

Problem 3.10. Testing (3.21) with $x_1 = \varepsilon > 0$ and $x_2 = \cdots = x_n = 1$ and letting $\varepsilon \to 0$, we find that $\theta \geq \left(\frac{n-1}{n}\right)^{n-1}$. Same proof for (3.22). QED

Problem 3.11. By homogeneity, we may assume that $xy = 1$ and thus $y = 1/x$. In this case, we have $M_{-r} = 1/M_r$ and $M_0 = 1$, so that (3.23) is proved as follows:

$$\frac{1}{2}M_r^q + \frac{1}{2}M_{-r}^q \geq (M_r M_{-r})^{q/2} = 1 = M_0^q.$$

The optimality of (3.23) amounts to the fact that, if

$$\theta M_r^q + (1-\theta)M_{-r}^q \geq 1, \ \forall\, x, y > 0, \tag{5.108}$$

then $\theta \geq 1/2$. Let $y = 1/x$, so that

$$M_r^q = \left(\frac{x^r + x^{-r}}{2}\right)^{q/r}, M_{-r}^q = \frac{1}{M_r^q}.$$

Since the image of the function

$$[1, \infty) \ni x \mapsto M_r^q$$

is $[1, \infty)$, the validity of (5.108) for this choice of y amounts to

$$f(t) := \theta t + (1-\theta)\frac{1}{t} \geq 1, \ \forall\, t \geq 1. \tag{5.109}$$

If $\theta < 1/2$, then f has a minimum equal to $2\sqrt{\theta(1-\theta)} < 2$ at $t = \sqrt{\frac{1-\theta}{\theta}}$. Therefore, (5.109) implies that $\theta \geq 1/2$.

Alternatively, since $f(1) = 1$, if (5.109) holds then $f'(1) \geq 0$, i.e., $\theta \geq 1/2$. QED

Problem 3.12. *Step 1.* We follow the first hint. By homogeneity and symmetry, we may assume that $xy = 1$ and $x \geq 1$. Since (3.24) is clear when $x = y = 1$, we may further assume that $x > 1$, so that we may write $x = e^t$, with $t > 0$. We have reduced the problem to

$$f(t) := \frac{e^{rt} + e^{-rt}}{2} + r\left(\frac{2}{e^t + e^{-t}}\right)^r > r + 1, \ \forall\, t > 0. \tag{5.110}$$

Step 2. We follow the second hint. Since $f(0) = r+1$, it thus suffices to prove that $f'(t) > 0, \forall\, t > 0$. Writing f in the more compact form $f(t) = \cosh(rt) + r(\cosh t)^{-r}$ and using the fact that $\cosh' = \sinh$, we have to prove that

$$\frac{\sinh(rt)}{r}(\cosh t)^{r+1} > \sinh t,\ \forall\, t > 0. \tag{5.111}$$

Step 3. We follow the third hint. In order to prove (5.111), we rely on (3.25)–(3.26) and find that (5.111) amounts to

$$\left(t + \frac{r^2 t^3}{3!} + \frac{r^4 t^5}{5!} + \cdots\right)\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)^{r+1} > t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots,$$

which clearly holds, since

$$\begin{aligned}\left(t + \frac{r^2 t^3}{3!} + \frac{r^4 t^5}{5!} + \cdots\right)\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)^{r+1} &> t\left(1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots\right)\\ &> t + \frac{t^3}{3!} + \frac{t^5}{5!} + \cdots\end{aligned}$$

Step 4. The optimality of (3.24). Let $x = e^t$, $y = e^{-t}$, with $t \geq 0$. Set

$$g(t) \coloneqq \theta M_r^r + (1-\theta)M_{-1}^r = \theta\cosh(rt) + (1-\theta)(\cosh t)^{-r}.$$

It suffices to prove that, if $g(t) \geq 1, \forall\, t \geq 0$, then $\theta \geq \dfrac{1}{r+1}$. For this purpose, we note that

$$\begin{aligned}g'(t) &= \theta r\sinh(rt) - (1-\theta)r(\sinh t)(\cosh t)^{-r-1},\\ g''(t) &= \theta r^2\cosh(rt) + (1-\theta)r(r+1)(\sinh t)^2(\cosh t)^{-r-2} - (1-\theta)r(\cosh t)^{-r}.\end{aligned}$$

We have $g(0) = 1$ and $g'(0) = 0$. Therefore, if $g(t) \geq 1, \forall\, t$, then $g''(0) \geq 0$. Since $g''(0) = \theta r^2 - (1-\theta)r$, we find that $\theta \geq \dfrac{1}{r+1}$. QED

Problem 3.13. *Step 1.* Initial reductions. The case where $r = 1$ was settled in Problem 3.11. We therefore assume that $0 < r < 1$. The reduction to the case where $y = 1/x$ and $x > 1$ is obtained as in the previous problem.

Step 2. Identification of the limiting case. The derivative of the right-hand side of (3.27) is

$$\frac{q}{r+1}\left(x^{r-1} - x^{-r-1}\right)\left(\frac{x^r + x^{-r}}{2}\right)^{q/r-1} - \frac{qr}{r+1}\left(1 - x^{-2}\right)\left(\frac{x + x^{-1}}{2}\right)^{-q-1},$$

and thus it suffices to prove that

$$\left(x^{r-1} - x^{-r-1}\right)\left(\frac{x^r + x^{-r}}{2}\right)^{q/r-1} > r\left(1 - x^{-2}\right)\left(\frac{x + x^{-1}}{2}\right)^{-q-1}, \quad \forall\, 0 < r < 1,\ \forall\, q > 0,\ \forall\, x > 1. \tag{5.112}$$

For fixed x, the left-hand side of (5.112) increases with q, while the right-hand side decreases with q. Therefore, it suffices to prove the counterpart of (5.112) when $q = 0$, namely,

$$\left(x^{r-1} - x^{-r-1}\right)\left(\frac{x^r + x^{-r}}{2}\right)^{-1} > r\left(1 - x^{-2}\right)\left(\frac{x + x^{-1}}{2}\right)^{-1}, \quad \forall\, 0 < r < 1,\ \forall\, x > 1. \tag{5.113}$$

Step 3. Proof of (5.113). Multiplying (5.113) with $\frac{x^{r+2}}{2}\left(x^r + x^{-r}\right)\left(x + x^{-1}\right)$, we see that we have to prove that $f(x) > 0$, $\forall\, x > 1$, where

$$\begin{aligned} f(x) \coloneqq &\left(x^{2r} - 1\right)\left(x^2 + 1\right) - r\left(x^2 - 1\right)\left(x^{2r} + 1\right) \\ = &(1 - r)x^{2r+2} - (1 + r)x^2 + (1 + r)x^{2r} - (1 - r),\ \forall\, x > 0. \end{aligned}$$

By intimidation, we have

$$\begin{aligned} f'(x) &= 2\left(1 - r^2\right)x^{2r+1} - 2(1 + r)x + 2r(1 + r)x^{2r-1}, \\ f''(x) &= 2\left(1 - r^2\right)(2r + 1)x^{2r} - 2(1 + r) + 2r(1 + r)(2r - 1)x^{2r-2}, \\ f'''(x) &= 4r\left(1 - r^2\right)(2r + 1)x^{2r-1} - 4r\left(1 - r^2\right)(2r - 1)x^{2r-3}, \end{aligned}$$

so that $f(1) = 0$, $f'(1) = 0$, $f''(1) = 0$, and

$$f'''(x) = 4r\left(1 - r^2\right)x^{2r-3}\left[(2r + 1)\left(x^2 - 1\right) + 2\right] > 0,\ \forall\, 0 < r < 1,\ \forall\, x > 1.$$

We find that, indeed, $f(x) > 0$, $\forall\, x > 1$, and we are done.

Step 4. Optimality. Assume that θ is such that $M_0^q \leq \theta M_r^q + (1 - \theta)M_{-1}^q$. With $y = 1/x$, set

$$g(x) \coloneqq \theta M_r^q + (1 - \theta)M_{-1}^q.$$

Again by intimidation, we have

$$\begin{aligned} g'(x) = &\,\theta q\left(x^{r-1} - x^{-r-1}\right)M_r^{q-1} - (1 - \theta)\left(1 - x^{-2}\right)M_{-1}^{q-1}, \\ g''(x) = &\,\theta q(q - 1)\left(x^{r-1} - x^{-r-1}\right)^2 M_r^{q-2} + \theta q\left((r - 1)x^{r-2} + (r + 1)x^{-r-2}\right)M_r^{q-1} \\ &+ (1 - \theta)q(q - 1)\left(1 - x^{-2}\right)^2 M_{-1}^{q-2} - 2(1 - \theta)qx^{-3}M_{-1}^{q-1}. \end{aligned}$$

Since $g(1) = 1$ and $g'(1) = 0$, if $g(x) \geq 1$, $\forall\, x > 0$, then $g''(1) \geq 0$. This amounts to $\theta \geq \frac{1}{r + 1}$.

Step 5. Bonus. (Proof by intimidation.) We use notation similar to the one in Step 4 in Problem 3.12. Write $x = e^t$, $y = e^{-t}$, and set

$$g(t) \coloneqq \frac{1}{r + 1}M_r^q + \frac{r}{r + 1}M_{-1}^q = \frac{1}{r + 1}[\cosh(rt)]^{q/r} + \frac{r}{r + 1}[\cosh t]^{-q}.$$

By intimidation, we have $g(0) = 1$, $g'(0) = 0$, $g''(0) = 0$, $g'''(0) = 0$, and

$$g^{(4)}(0) = \frac{qr}{r+1}\big(2 - 2r^2 + 3q + 3qr\big).$$

Since $r > 1$, for sufficiently small $q > 0$ we have $g^{(4)}(0) < 0$. For such q, for sufficiently small $t > 0$, we have $g(t) < 1$, and therefore (3.27) does not hold. QED

Problem 3.14. *Step 1.* Initial reductions. Letting $y = 1/x$, $x = e^t$, with $t > 0$, it suffices to prove that

$$\frac{1}{r+1}[\cosh(rt)]^{q/r} + \frac{r}{r+1}[\cosh t]^{-q} > 1, \ \forall\, t > 0, \tag{5.114}$$

and therefore it suffices to prove that the derivative of the left-hand side of (5.114) is > 0 when $t > 0$. This amounts to

$$\sinh(rt)[\cosh(rt)]^{q/r-1} > r\sinh(rt)[\cosh t]^{-q-1}, \ \forall\, t > 0. \tag{5.115}$$

Since the left-hand side of (5.115) increases with q, while the right-hand decreases with q, it suffices to prove (5.115) when $q = 1$. This amounts to

$$\frac{\sinh(rt)}{r\sinh t}[\cosh t]^2 > [\cosh(rt)]^{1-1/r}. \tag{5.116}$$

Step 2. We use the hint (ii): since $r > 1$ and $t > 0$, we have

$$\frac{\sinh(rt)}{r\sinh t} = \frac{\dfrac{rt}{1!} + \dfrac{r^3t^3}{3!} + \dfrac{r^5t^5}{5!} + \cdots}{\dfrac{rt}{1!} + \dfrac{rt^3}{3!} + \dfrac{rt^5}{5!} + \cdots} > 1. \tag{5.117}$$

Step 3. We use the hint (iii): when $0 < r \le 2$, we have

$$\begin{aligned}[\cosh(rt)]^{1-1/r} =& \big[2\cosh^2(rt/2) - 1\big]^{1-1/r} \le \big[2\cosh^2 t - 1\big]^{1-1/r} \\ \le& \big[2\cosh^2 t - 1\big]^{1/2}.\end{aligned} \tag{5.118}$$

In view of (5.117) and (5.118), we are done if

$$\cosh^2 t \ge \big[2\cosh^2 t - 1\big]^{1/2}.$$

But this reduces to $\big[\cosh^2 t - 1\big]^2 \ge 0$.

Step 4. We prove that (3.28) does not hold when $r = 3$ and $q = 1$. Following the hint, we let $y = 1/x$ and set

$$z \coloneqq x + \frac{1}{x} \ge 2.$$

Since

$$x^3 + \frac{1}{x^3} = z^3 - 3z,$$

and, for any $z \geq 2$, one can find some $x > 0$ such that $z = x + 1/x$, the inequality we want to disprove reads

$$1 \leq \frac{1}{4}\sqrt[3]{\frac{z^3 - 3z}{2}} + \frac{3}{4}\frac{2}{z}, \forall\, z \geq 2.$$

Equivalently, we want to disprove the inequality

$$z^3\big(z^3 - 3z\big) - 16(2z - 2)^3 \geq 0, \forall\, z \geq 2,$$

which is in turn equivalent to $f(z) \geq 0$, $\forall\, z \geq 2$, where, by intimidation,

$$\begin{aligned} f(z) :=& z^6 - 3z^4 - 128z^3 + 576z^2 - 864z + 432 \\ =& (z-2)^2\big(z^4 + 4z^3 + 9z^2 - 108z + 108\big). \end{aligned}$$

Since the expression in the last bracket is < 0 when $z = 2$, we find that, indeed, (3.28) does not hold when $r = 3$ and $q = 1$. QED

Problem 3.16. Similarly to the solution of Problem 3.15, we have to prove that $f(t) \geq 2r$, $\forall\, 0 \leq t \leq 1$, where

$$f(t) := (1+t)^r + (1-t)^r + \alpha P(t),\ \alpha := 2r - 2,\ P(t) := \big(1 - t^2\big)^{r/2}.$$

Arguing as in Step 2 in the solution of Problem 3.16, we are done if we prove that

$$g(t) := (1+t)^r - (1-t)^r - 2rt > 0,\ \forall\, 0 < t \leq 1. \tag{5.119}$$

Since $g(0) = 0$, (5.119) follows if $g'(t) > 0$, $\forall\, 0 < t \leq 1$. In turn, this follows from the strict convexity of $[0, \infty) \ni x \mapsto h(x) := x^{r-1}$, which implies that

$$g'(t) = 2r\left[\frac{1}{2}h(1+t) + \frac{1}{2}h(1-t) - h\left(\frac{1}{2}(1+t) + \frac{1}{2}(1-t)\right)\right] > 0,\ \forall\, 0 < t \leq 1.$$

For the optimality part, assume that θ is such that

$$M_1^r \leq \theta M_r^r + (1-\theta)M_0^r,\ \forall\, x, y > 0. \tag{5.120}$$

Testing (5.120) with $x = 1$ and $y = 1 + 2t$, $t \geq 0$, we find that

$$f(t) := \frac{\theta}{2}[1 + (1+2t)^r] + (1-\theta)(1+2t)^{r/2} - (1+t)^r \geq 0,\ \forall\, t \geq 0.$$

By intimidation, we have $f(0) = 1$, $f'(0) = 0$,

$$f''(0) = 2\theta r(r-1) + (1-\theta)r(r-2) - r(r-1).$$

If (5.120) holds, then $f''(0) \geq 0$, which is equivalent to $\theta \geq 1/r$. QED

Problem 3.17. With no loss of generality, we may assume that $q < r$ and $x + y = 2$. Then $M_1 = 1$, $0 \le M_0 \le 1$, and (3.29) reads

$$\begin{aligned}(M_r^q)^p = M_r^r \ge& 2^{r-1} - \left(2^{r-1} - 1\right) M_0^r \\ =& 2^{(q-q/r)p} - \left(2^{(q-q/r)p} - 1^p\right)(M_0^q)^p.\end{aligned} \tag{5.121}$$

Using (5.121) and the inequality (2.1) in Problem 2.2, we find that

$$\begin{aligned}M_r^q \ge& \left[2^{(q-q/r)p} - \left(2^{(q-q/r)p} - 1^p\right)(M_0^q)^p\right]^{1/p} \\ \ge& 2^{q-q/r} - \left(2^{q-q/r} - 1\right) M_0^q = 2^{q-q/r} M_1^q - \left(2^{q-q/r} - 1\right) M_0^q,\end{aligned}$$

which is equivalent to (3.33).

The optimality of $\theta = \dfrac{1}{2^{q-q/r}}$ when

$$M_1^q \le \theta M_r^q + (1-\theta) M_0^q, \ \forall\, x, y > 0, \tag{5.122}$$

holds is obtained by testing (5.122) with $x = 2$ and $y = 0$. QED

Problem 3.20. Proof by contradiction. Take $x_1 = t > 0$ and $x_2 = \cdots = x_n = 1$, and set

$$F(t) := \theta M_r^r + (1-\theta) M_0^r - M_1^r = \theta \frac{(n-1) + t^r}{n} + (1-\theta) t^{r/n} - \left(\frac{n-1+t}{n}\right)^r.$$

For θ as in the statement, we have $F(0) = 0$. On the other hand, when $r > n$ we have

$$F'(0) = -\frac{r}{n}\left(\frac{n-1}{n}\right)^{r-1} < 0,$$

and thus, for small $t > 0$, we have $F(t) < 0$, implying that (3.40) does not hold. QED

Problem 3.21. For the validity of (3.47), repeat the solution of Problem 3.17, assuming, e.g., that $\sum x_j = n$. For the optimality, test (3.47) with $x_1 = \varepsilon, x_2 = \cdots = x_n = 1$, and let $\varepsilon \to 0$. QED

Problem 3.22. Fix $x_1, \ldots, x_n$. For $1 < r \le n$ and $q = 1$ (3.33) (when $n = 2$), respectively (3.47) (when $n \ge 3$) are equivalent to $F(r) \ge 0$, where

$$F(r) := (n-1)^{1-1/r}\left(\sum_{j=1}^n x_j^r\right)^{1/r} + \left(n - n^{1/r}(n-1)^{1-1/r}\right)\prod_{j=1}^n x_j^{1/n} - \sum_{j=1}^n x_j.$$

Since, clearly, $F(1) = 0$, we find that $F'(1) \ge 0$. Next, with

$$S = S(r) := \sum_{j=1}^n x_j^r,$$

we have, by intimidation,

$$F'(r) = \frac{1}{r^2}(n-1)^{1-1/r}S^{1/r}\ln(n-1)$$
$$+ (n-1)^{1-1/r}S^{1/r}\left[-\frac{1}{r^2}\ln S + \frac{1}{r}\frac{\sum_{j=1}^{n} x_j^r \ln x_j}{S}\right]$$
$$+ \frac{1}{r^2}n^{1/r}(n-1)^{1-1/r}\ln\left(\frac{n}{n-1}\right)\prod_{j=1}^{n} x_j^{1/n}.$$

In particular, we have

$$F'(1) = S(1)\ln(n-1) - S(1)\ln S(1) + \sum_{j=1}^{n} x_j \ln x_j + n\ln\left(\frac{n}{n-1}\right)\prod_{j=1}^{n} x_j^{1/n},$$

so that, by intimidation, the condition $F'(1) \geq 0$ is precisely (3.49). QED

5.4 Problems on inequalities

Problem 4.1. With no loss of generality, we may assume that $x_n \leq x_j$, $\forall\, j$, and in particular $x_n \leq x_2$. Then

$$\frac{x_1}{x_2 + x_1} + \frac{x_n}{x_n + x_1} \leq \frac{x_1}{x_n + x_1} + \frac{x_n}{x_n + x_1} = 1. \tag{5.123}$$

Since, on the other hand, we clearly have

$$\frac{x_j}{x_j + x_{j+1}} < 1, \forall\, 2 \leq j \leq n-1, \tag{5.124}$$

we obtain (4.1) from (5.123) and (5.124).

In order to prove the optimality of (4.1), assume that it holds for some C (in place of $n-1$). Letting $x_n := 1, x_{n-1} := t > 0, x_{n-2} := t^2, \ldots, x_1 := t^{n-1}$, we find that

$$C \geq \lim_{t\to\infty}\left(\frac{t^{n-1}}{t^{n-1} + t^{n-2}} + \cdots + \frac{t}{t+1} + \frac{1}{1+t^n}\right) = n-1.$$ QED

Problem 4.2. We will establish the inequality

$$\frac{a_1 + \cdots + a_n}{n} - \sqrt[n]{a_1 \cdots a_n} \geq \frac{1}{n}\left(\sqrt{a_1} - \sqrt{a_2}\right)^2, \ \forall\, n \geq 2, \ \forall\, a_1, \ldots, a_n > 0. \tag{5.125}$$

Granted (5.125), we have, by symmetry,

$$\frac{a_1 + \cdots + a_n}{n} - \sqrt[n]{a_1 \cdots a_n} \geq \frac{1}{n}\left(\sqrt{a_j} - \sqrt{a_k}\right)^2, \quad \forall\, n \geq 2, \ \forall\, 1 \leq j < k \leq n, \ \forall\, a_1, \ldots, a_n > 0, \tag{5.126}$$

and summing up all the inequalities (5.126) we obtain (4.2). Incidentally, (4.2) implies that equality in (AM-GM) requires $a_1 = \cdots = a_n$.

We now check the validity of (5.125). The case $n = 2$ is clear (with equality). Let $f_n(a_1, \ldots, a_n)$ denote the left-hand side of (5.126). We will prove the inequality

$$f_n(a_1, \ldots, a_n) \geq \frac{n-1}{n} f_{n-1}(a_1, \ldots, a_{n-1}), \quad \forall\, n \geq 3,\ \forall\, a_1, \ldots, a_n > 0. \tag{5.127}$$

(Clearly, (5.127) implies (5.125).)

Fix $a_1, \ldots, a_{n-1}$. Then $a \mapsto f_n(a_1, \ldots, a_{n-1}, a)$ attains its minimum for

$$a \coloneqq \sqrt[n-1]{a_1 \cdots a_{n-1}},$$

and for this a we have

$$f_n(a_1, \ldots, a_{n-1}, a) = \frac{n-1}{n} f_{n-1}(a_1, \ldots, a_{n-1}),$$

whence the conclusion. QED

Problem 4.3. The function

$$(-\infty, T) \ni x \mapsto f(x) \coloneqq \frac{x}{T-x} = -1 + \frac{T}{T-x}$$

is strictly convex. (This can be checked either directly *via* (J), or by noting that $f''(x) = 2T/(T-x)^3 > 0$.) (GJ) yields

$$f\left(\frac{S}{n}\right) \leq \sum_{j=1}^{n} \frac{1}{n} f(a_j),$$

which amounts to (4.3). Moreover, equality occurs in (4.3) iff $a_1 = \ldots = a_n$. QED

Problem 4.4. With a, c fixed, we study the function

$$(c, \infty) \ni b \mapsto f(b) \coloneqq \sqrt{ab} - \sqrt{c(a-c)} - \sqrt{c(b-c)}.$$

We have

$$f'(b) = ((a-c)b - ac)g(b),$$

with $g(b) > 0$. Thus f is decreasing on $(c, , b_0]$ and increasing on $[b_0, \infty)$, where $b_0 \coloneqq \dfrac{ac}{a-c} > c$. Since $f(b_0) = 0$, we have $f(b) \geq 0, \forall\, b > c$. QED

Problem 4.5. Setting $f(\Lambda) \coloneqq x^\Lambda$, $\Lambda \in \mathbb{R}^3$, inequality (4.5) is equivalent to

$$\begin{aligned} &f(1, 0, -1) + f(-1, 1, 0) + f(0, -1, 1) \\ &\leq f(\alpha, -\alpha, 0) + f(0, \alpha, -\alpha) + f(-\alpha, 0, \alpha). \end{aligned} \tag{5.128}$$

Using Problem 2.4 and the fact that $\alpha \geq 2$, we have

$$f(1,0,-1) = f\left(\frac{\alpha+3}{3\alpha}(\alpha,-\alpha,0) + \frac{\alpha+3}{3\alpha}(0,\alpha,-\alpha) + \frac{\alpha-2}{3\alpha}(-\alpha,0,\alpha)\right)$$
$$\leq \frac{\alpha+3}{3\alpha}f(\alpha,-\alpha,0) + \frac{\alpha+3}{3\alpha}f(0,\alpha,-\alpha) + \frac{\alpha-2}{3\alpha}f(-\alpha,0,\alpha),$$

and two other similar inequalities. We obtain (5.128) by taking the sum of these three inequalities. QED

Problem 4.6. Possibly by interchanging a and b, we may assume that $r \geq s$. Let $t \coloneqq s/r \leq 1$. Denoting $x \coloneqq a^r, y \coloneqq b^r, y \coloneqq c^s$, we are looking for

$$\max\left\{x^{t+1} + y^{t+1} + z^{t+1} - xy^t - yz^t - zx^t;\ \alpha^r \leq x, y, z \leq \beta^r\right\}. \tag{5.129}$$

The maximum in (5.129) is achieved ("continuous function on a compact"). Fix y and z and consider the function

$$(0,\infty) \ni x \mapsto f(x) \coloneqq x^{t+1} + y^{t+1} + z^{t+1} - xy^t - yz^t - zx^t.$$

Since $0 < t \leq 1$, have

$$f''(x) = t(t+1)x^{t-1} - t(t-1)zx^{t-2} > 0,\ \forall\, t > 0,$$

and thus f is convex. We find that $f(x) \leq \max\left(f(\alpha^r), f(\beta^r)\right)$. The same argument holds in the variables y and z. We find that the expression considered in (5.129) has a maximum point $(x,y,z) \in \{\alpha^r, \beta^r\}^3$. Calculating the expression at the 8 possible triples, we find that $\max = (\beta^r - \alpha^r)(\beta^s - \alpha^s)$. QED

Problem 4.7. The function $(0,\infty) \ni x \mapsto 1/x^2$ being convex, we try to use Theorem 1.1. The equation

$$2x + 2y + z = \alpha(4x+y) + \beta(4y+z) + (4z+x)$$

(with α, β, γ independent of x, y, z) has, by identification of coefficients, the solution $\alpha = 6/13$, $\beta = 5/13$, $\gamma = 2/13$. We find that

$$\begin{pmatrix} 2x+2y+z \\ 2y+2z+x \\ 2z+2z+y \end{pmatrix} = A \begin{pmatrix} 4x+y \\ 4y+z \\ 4z+x \end{pmatrix}, \text{ where } A = \begin{pmatrix} \alpha & \beta & \gamma \\ \gamma & \alpha & \beta \\ \beta & \gamma & \alpha \end{pmatrix}.$$

A being DS, we conclude *via* Theorem 1.1. QED

Problem 4.8. Let $f(a,b,c)$ denote the left-hand side of (4.8). The form of the denominators suggests writing

$$\frac{a^3}{a^2+ab+b^2} = \frac{a^3-b^3}{a^2+ab+b^2} + \frac{b^3}{a^2+ab+b^2} = a - b + \frac{b^3}{a^2+ab+b^2}. \tag{5.130}$$

Summing (5.130) and its permutations, we find that $f(a,b,c) = f(b,a,c)$, and thus

$$\begin{aligned}2f(a,b,c) =&f(a,b,c) + f(b,a,c)\\ =&\frac{a^3+b^3}{a^2+ab+b^2} + \frac{b^3+c^3}{b^2+bc+c^2} + \frac{c^3+a^3}{c^2+ca+a^2}\\ =&\frac{(a+b)\big(a^2-ab+b^2\big)}{a^2+ab+b^2} + \frac{(b+c)\big(b^2-bc+c^2\big)}{b^2+bc+c^2} + \frac{(c+a)\big(c^2-ca+a^2\big)}{c^2+ca+a^2}\\ =&2(a+b+c) - 2\frac{ab(a+b)}{a^2+ab+b^2} - 2\frac{bc(b+c)}{b^2+bc+c^2} - 2\frac{ca(c+a)}{c^2+ca+a^2}.\end{aligned}$$

Therefore, (4.8) amounts to

$$\frac{ab(a+b)}{a^2+ab+b^2} + \frac{bc(b+c)}{b^2+bc+c^2} + \frac{ca(c+a)}{c^2+ca+a^2} \le \frac{2}{3}(a+b+c). \tag{5.131}$$

Since $3ab \le a^2+ab+b^2$, we find that

$$\frac{ab(a+b)}{a^2+ab+b^2} \le \frac{a+b}{3}. \tag{5.132}$$

(5.131) is obtained by summing (5.132) and its permutations.

Alternatively, one can see that we are in the special case $f(x) = x^3$, $x > 0$, of (2.20). QED

Problem 4.9. With $f(t) \coloneqq \dfrac{a^t+b^t}{a^{t/2}b^{t/2}}$, the conclusion amounts to $f(2r+1) > f(1)$, $\forall\, r > 0$. It thus suffices to prove that f is increasing on $(0,\infty)$. Assuming, e.g., that $a > b$, write $a/b = e^{2x}$, with $x > 0$. Then $f(t) = e^{xt} + e^{-xt}$ and therefore

$$f'(t) = x\big(e^{xt} - e^{-xt}\big) > 0,\ \forall\, t > 0.$$ QED

Problem 4.10. With no loss of generality, we may assume that $a \ge b \ge c$. (4.9) may be rewritten as

$$f(a) \coloneqq \big(a^3+b^3+c^3+3abc\big) - \big(a^2b+ab^2+b^2c+bc^2+c^2a+ca^2\big) \ge 0.$$

We have

$$\begin{aligned}&f'(a) = \big(3a^2+3bc\big) - \big(2ab+b^2+c^2+2ac\big),\\ &f''(a) = 6a - 2(b+c) \ge 0,\ \forall\, a \ge b \ge c.\end{aligned}$$

Therefore,

$$f'(a) \ge f'(b) = c(b-c) \ge 0$$

and thus

$$f(a) \ge f(b) = c^3 + b^2c - 2bc^2 = c(b-c)^2 \ge 0.$$ QED

Problem 4.11. Denote $f(a,b,c,d)$ the difference between the left-hand side and the right-hand side of (4.10), so that (4.10) amounts to $f(a,b,c,d) \geq 0$. With no loss of generality, we may assume that $a \geq b \geq c \geq d \geq 0$. Set, with b,c,d fixed, $g(a) \coloneqq f(a,b,c,d)$. Then

$$g''(a) = 12a^2 - 2\big(b^2 + c^2 + d^2\big) \geq 0, \ \forall\, a \geq b,$$

and thus

$$g'(a) \geq g'(b) = 2b\big(b^2 + cd - c^2 - d^2\big) \geq 2b\big(c^2 + cd - c^2 - d^2\big) = 2bd(c-d) \geq 0.$$

We find that $f(a,b,c,d) \geq f(b,b,c,d)$, and thus it suffices to prove that $f(b,b,c,d) \geq 0$ if $b \geq c \geq d \geq 0$.

Setting, with c,d fixed, $h(b) \coloneqq f(b,b,c,d)$, we have

$$h'(b) = 4b\big(b^2 + cd - c^2 - d^2\big) \geq 4b\big(c^2 + cd - c^2 - d^2\big) = 4bd(c-d) \geq 0,$$

and thus $h(b) \geq h(c)$.

Finally,

$$f(a,b,c,d) \geq f(b,b,c,d) \geq f(c,c,c,d) = d(c-d)\big(2c^2 - cd - d^2\big) \geq 0. \qquad \text{QED}$$

Problem 4.12. We may assume that $a \geq b \geq c$. Let $f(a)$ denote the left-hand side of (4.11). Then

$$\begin{aligned} f'(a) =& \frac{1}{1+bc} - \frac{bc}{(1+ca)^2} - \frac{bc}{(1+ab)^2} \geq \frac{1}{1+bc} - \frac{bc}{(1+bc)^2} - \frac{bc}{(1+bc)^2} \\ =& \frac{1-bc}{(1+bc)^2} \geq 0 \text{ (since } bc \leq 1). \end{aligned}$$

Therefore,

$$f(a) \leq f(1) = \frac{1}{1+bc} + \frac{b}{1+c} + \frac{c}{1+b} =: g(b).$$

We have

$$\begin{aligned} g'(b) =& -\frac{c}{(1+bc)^2} + \frac{1}{1+c} - \frac{c}{(1+b)^2} \geq -\frac{c}{(1+c)^2} + \frac{1}{1+c} - \frac{1}{(1+b)^2} \\ =& \frac{1}{(1+c)^2} - \frac{1}{(1+b)^2} \geq 0, \end{aligned}$$

and thus

$$g(b) \leq g(1) = \frac{1}{1+c} + \frac{1}{1+c} + \frac{c}{2} = \frac{4+c+c^2}{2+2c} \leq \frac{4+c+c}{2+2c} \leq 2. \qquad \text{QED}$$

Problem 4.13. This is a special case of (2.10) (with $f = \ln$). QED

Problem 4.14. Setting $a_j = e^{x_j}$, with $x_j < 0$, $\forall\, j$, (4.13) amounts to

$$f\left(\sum_{j=1}^{n} \frac{1}{n} x_j\right) \geq \sum_{j=1}^{n} \frac{1}{n} f(x_j), \tag{5.133}$$

where $f(x) \coloneqq \ln(1 - e^x)$, $\forall\, x < 0$. Since

$$f''(x) = -\frac{e^x}{(1-e^x)^2} < 0, \ \forall\, x < 0,$$

f is concave, and (5.133) follows from (GJ).

Incidentally, f is strictly concave, and equality in (5.133) occurs iff $x_1 = \cdots = x_n$, so that equality in (4.13) occurs iff $a_1 = \cdots = a_n$. QED

Problem 4.15. By symmetry, it suffices to find the maximal value in (4.14) under the additional assumption $a \geq b \geq c$. Set $f(x) \coloneqq x + \dfrac{1}{x}$, $\forall\, x > 0$. Then the quantity under investigation is

$$g(a,b,c) \coloneqq f\left(\sqrt{a/b}\right) f\left(\sqrt{b/c}\right) f\left(\sqrt{c/a}\right).$$

Noting that f is decreasing on $(0,1]$ and increasing on $[1,\infty)$, we find that

$$\begin{aligned} g(a,b,c) \leq & f\left(\sqrt{\alpha/b}\right) f\left(\sqrt{b/c}\right) f\left(\sqrt{c/\alpha}\right) \leq f\left(\sqrt{\alpha/b}\right) f\left(\sqrt{\alpha b}\right) f\left(\sqrt{1/\alpha^2}\right) \\ = & \frac{\alpha^2+1}{\alpha^2}\left(\alpha f(b) + (\alpha^2+1)\right) \leq \frac{\alpha^2+1}{\alpha^2}\left(\alpha f(1) + (\alpha^2+1)\right) \\ = & \frac{(\alpha^2+1)(\alpha+1)^2}{\alpha^2} \eqqcolon M. \end{aligned}$$

Since, in the above, equality is achieved when $a = \alpha, c = 1/\alpha, b = 1$, we find that the value M is max in (4.14). QED

Problem 4.16. With $f(x) =\coloneqq \dfrac{1}{x}$, $x > 0$, (4.15) becomes

$$\begin{aligned} &\sum_{j=1}^{n-1} (x_j - x_{j+1}) f\left(x_j^k + x_j^{k-1} x_{j+1} + \cdots + x_j x_{j+1}^{k-1} + x_{j+1}^k\right) \\ &\geq (x_1 - x_n) f\left(x_1^k + x_1^{k-1} x_n + \cdots + x_1 x_n^{k-1} + x_n^k\right). \end{aligned} \tag{5.134}$$

By homogeneity, we may assume that $x_1 - x_n = 1$, and then (5.134) follows from (GJ), using the convexity of f and the identities

$$\begin{aligned} &\sum_{j=1}^{n-1} (x_j - x_{j+1})\left(x_j^k + x_j^{k-1} x_{j+1} + \cdots + x_j x_{j+1}^{k-1} + x_{j+1}^k\right) = \sum_{j=1}^{n-1} \left(x_j^{k+1} - x_{j+1}^{k+1}\right) \\ &= (x_1 - x_n)\left(x_1^k + x_1^{k-1} x_n + \cdots + x_1 x_n^{k-1} + x_n^k\right). \end{aligned}$$

QED

Problem 4.17. This is a generalization of Problem 4.8 (which corresponds to $k = 2$). However, when k is odd, we cannot repeat the proof of (4.8). We present a similar, but different approach, which holds for every k, even or odd.

For notational simplicity, set

$$f(x,y) := \frac{x^{k+1}}{x^k + x^{k-1}y + \cdots + xy^{k-1} + y^k}, \ \forall\, x, y > 0.$$

As in Problem 4.8, we have $f(x,y) = x - y + f(y,x)$, and thus

$$\begin{aligned} \sum_{j=1}^{n} f(x_j, x_{j+1}) &= \sum_{j=1}^{n} f(x_{j+1}, x_j) = \frac{1}{2}\sum_{j=1}^{n} [f(x_j, x_{j+1}) + f(x_{j+1}, x_j)] \\ &= \frac{1}{2}\sum_{j=1}^{n} \frac{x_j^{k+1} + x_{j+1}^{k+1}}{x_j^k + x_j^{k-1}x_{j+1} + \cdots + x_j x_{j+1}^{k-1} + x_{j+1}^k}. \end{aligned} \tag{5.135}$$

In view of (5.135), order to complete the proof of (4.16) it suffices to prove that

$$(k+1)\left(x^{k+1} + y^{k+1}\right) \geq (x+y)\sum_{\ell=0}^{k} x^\ell y^{k-\ell}, \ \forall\, x, y > 0. \tag{5.136}$$

In turn, (5.136) can be obtained either by majorization, or by using (GMI) as follows. For $1 \leq \ell \leq k+1$, we have

$$\begin{aligned} x^{k+1} + y^{k+1} =& \left(\frac{\ell}{k+1}x^{k+1} + \frac{k+1-\ell}{k+1}y^{k+1}\right) \\ &+ \left(\frac{k+1-\ell}{k+1}x^{k+1} + \frac{\ell}{k+1}y^{k+1}\right) \geq x^\ell y^{k+1-\ell} + x^{k+1-\ell}y^\ell. \end{aligned} \tag{5.137}$$

Adding the inequalities (5.137) with $\ell = 1, \ldots, k+1$, we obtain (5.136). QED

Problem 4.18. If we have

$$\frac{1}{a^\alpha + b^\alpha + 2\alpha - 2} \leq \frac{1}{4\alpha}\left(\frac{1}{a} + \frac{1}{b}\right), \ \forall\, a, b > 0, \tag{5.138}$$

then we are done. In turn, (5.138) is equivalent to

$$(a+b)(a^\alpha + b^\alpha + 2\alpha - 2) \geq 4\alpha ab,$$

which follows from (AM-GM) and (GMI):

$$(a+b)\left(\frac{1}{2\alpha}a^\alpha + \frac{1}{2\alpha}b^\alpha + \frac{2\alpha-2}{2\alpha}\right) \geq 2\sqrt{ab}\cdot a^{\alpha/(2\alpha)}\cdot b^{\alpha/(2\alpha)}\cdot 1^{(2\alpha-2)/2\alpha} = 2ab. \quad \text{QED}$$

Problem 4.19. Set $S_j \coloneqq b_1 + \cdots b_j$, $\forall\, 1 \le j \le n$, and $S_0 \coloneqq 0$. Then $S_j \ge 0$ and $b_j = S_j - S_{j-1}$, $\forall\, 1 \le j \le n$, so that

$$\sum_{j=1}^{n} a_j b_j = \sum_{j=1}^{n} a_j (S_j - S_{j-1}) = \sum_{j=1}^{n} a_j S_j - \sum_{j=1}^{n} a_j S_{j-1} = \sum_{j=1}^{n} a_j S_j - \sum_{j=0}^{n-1} a_{j+1} S_j$$
$$= \sum_{j=1}^{n} a_j S_j - \sum_{j=1}^{n-1} a_{j+1} S_j = \sum_{j=1}^{n-1} (a_j - a_{j+1}) S_j + a_n S_n,$$

from which the conclusion follows easily.

NB. The identity

$$\sum_{j=1}^{n} a_j b_j = \sum_{j=1}^{n-1} (a_j - a_{j+1}) S_j + a_n S_n$$

is known as the Abel summation formula. QED

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