Lifting in Sobolev spaces of manifold-valued maps

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- $\pi : \mathscr{E} \to \mathscr{N}$ a Riemannian covering
- $\mathscr{E}$, $\mathscr{N}$ connected Riemannian manifolds
- $\mathscr{N}$ compact
- Ω an N-dimensional simply connected smooth domain (or compact manifold)
- $s > 0$, $1 \le p < \infty$
- Lifting problem: given any $u \in W^{s,p}(\Omega; \mathscr{N})$, is it always possible to write $u = \pi \circ \varphi$, with $\varphi \in W^{s,p}(\Omega; \mathscr{E})$?
- Subsequent questions: if yes, estimates? Uniqueness? If no, "optimal" smoothness of φ? How to detect the u's that lift?
- The compactness of $\mathscr{E}$ plays a crucial role
- Compact case: $\mathscr{E}$ is compact/Non-compact case: $\mathscr{E}$ is non-compact

- $\mathscr{N} = \mathbb{RP}^2$, $\mathscr{N} = \mathbb{S}^2$, $\pi(x) = \{x, -x\}$
  Given a Sobolev non oriented vector field $u : \Omega \to \mathbb{RP}^2$, find an orientation of $\varphi : \Omega \to \mathbb{S}^2$ of u, as smooth as u
- An application. If h lifts g, then

$$\begin{gathered} \min\left\{\int_\Omega |\nabla u|^2 ; u : \Omega \to \mathbb{RP}^2, u = g \text{ on } \partial\Omega\right\} = \\ \min\left\{\int_\Omega |\nabla \varphi|^2 ; \varphi : \Omega \to \mathbb{S}^2, \varphi = h \text{ on } \partial\Omega\right\} \end{gathered} \tag{1}$$

- $\mathscr{N} = \mathbb{S}^1$, $\mathscr{E} = \mathbb{S}^1$, $\pi(z) = z^d$, $d \geq 2$. Given $u \in W^{s,p}(\Omega; \mathbb{S}^1)$, is it possible to write $u = \varphi^d$ with $\varphi \in W^{s,p}(\Omega; \mathbb{S}^1)$ (dth root problem, Bethuel and Chiron)?
- $\mathscr{N} = \mathbb{S}^1$, $\mathscr{E} = \mathbb{R}$, $\pi(t) = e^{\imath t}$. Given $u \in W^{s,p}(\Omega; \mathbb{S}^1)$, is it possible to write $u = e^{\imath\varphi}$, with $\varphi \in W^{s,p}(\Omega; \mathbb{R})$? + analogue of (1)

- The definition of $W^{s,p}(\Omega;\mathscr{N})$ and $W^{s,p}(\Omega;\mathscr{E})$?
- We embed $\mathscr{N}$ into some $\mathbb{R}^\nu$ and then

$$W^{s,p}(\Omega;\mathscr{N}) = \{u \in W^{s,p}(\Omega;\mathbb{R}^\nu);\ u(x) \in \mathscr{N} \text{ a.e.}\}$$

- Definition independent of the embedding
- $\mathscr{E}$ need not be compact: obvious definition of $W^{s,p}(\Omega;\mathscr{E})$ only when $s \le 1$. E.g., when $s < 1$

$$W^{s,p}(\Omega;\mathscr{E}) = \left\{\varphi : \Omega \to \mathscr{E};\ |\varphi|^p_{W^{s,p}} = \iint \frac{d_{\mathscr{E}}(\varphi(x),\varphi(y))^p}{|x-y|^{m+sp}}\,dxdy < \infty\right\}$$

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Theorem (Bourgain, Brezis, M 00) For the universal covering of $\mathbb{S}^1$, the lifting property holds except when

- $1 \le sp < 2 \le N = \dim \Omega$
- $0 < s < 1$, $1 \le sp < N = \dim \Omega$

- Topological obstruction to lifting: $u(z) = z/|z|$ $(z \in \mathbb{C})$
- Analytical obstruction to lifting: $\varphi \in W^{s,p}_{loc}(\Omega \setminus \{a\}) \setminus W^{s,p}(\Omega)$ such that $e^{\imath\varphi} \in W^{s,p}(\Omega)$
- It is possible to have the lifting property without estimates: in $W^{1/p,p}((0,1);\mathbb{S}^1)$, $1 < p < \infty$, every map lifts, but no norm control
- The kink: $\varphi_\varepsilon(x) = \begin{cases} 0, & \text{if } x \le 1/2 \\ 2\pi, & \text{if } x \ge 1/2 + \varepsilon \\ \text{affine}, & \text{if } 1/2 \le x \le 1/2 + \varepsilon \end{cases}$ satisfies, $\forall\, 1 < p < \infty$, $|\varphi_\varepsilon|_{W^{1/p,p}} \to \infty$, $|e^{\imath\varphi_\varepsilon}|_{W^{1/p,p}} \lesssim 1$

Theorem (Bethuel, Chiron 07) Let $\pi : \mathscr{E} \to \mathscr{N}$ be the universal covering of $\mathscr{N}$, with $\pi_1(\mathscr{N})$ infinite. Then the lifting property holds except when

- $1 \leq sp < 2 \leq N = \dim \Omega$
- $0 < s < 1$, $1 \leq sp < N = \dim \Omega$

- The topological obstruction and the kink exist without any assumption on $\pi_1(\mathscr{N})$
- The analytical obstruction is obtained from the existence of a ray in $\mathscr{E}$ (isometrically embedded real line in $\mathscr{E}$), which requires $\mathscr{E}$ non-compact
- The same result holds (essentially with the same proof) for any covering provided $\mathscr{E}$ is non-compact
- The space $W^{s,p}(\Omega; \mathscr{E})$ is defined by embedding $\mathscr{E}$
- When $\mathscr{E}$ is compact, open case: the one connected to the analytical obstruction: $0 < s < 1$, $2 \leq sp < N = \dim \Omega$

Theorem (M, Van Schaftingen 19) Assume $\mathscr{E}$ compact and π non-trivial. Then the lifting property holds except when $1 \le sp < 2 \le N = \dim \Omega$

- Thus the difference between the compact and the non-compact case occurs exactly in the range $0 < s < 1$, $2 \le sp < N = \dim \Omega$: existence in the compact case, non-existence in the non-compact case
- The result applies in particular to the dth root problem and to the orientation problem in $\mathbb{RP}^k$

Sketch of proof

- We consider only the new the case $0 < s < 1$, $2 \le sp < N = \dim \Omega$
- Strategy: establish an a priori estimate for u in a dense subset of $W^{s,p}(\Omega; \mathscr{N})$, having the lifting property: linear estimate
- The maps continuous outside some $(N - \lfloor sp \rfloor - 1)$-skeleton are dense (Brezis, M 15)
- Such maps do lift
- The a priori estimate relies on a one-dimensional reverse oscillation inequality for $f \in C^0(\mathbb{R}, \mathbb{R})$, $0 < s < 1$, $sp > 1$:

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{[\operatorname{osc}_{[x,y]} f]^p}{|y-x|^{1+sp}} \, dxdy \le C_{s,p} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(y) - f(x)|^p}{|y-x|^{1+sp}} \, dxdy$$

- When $N = \dim\Omega \geq 2$ and π is non-trivial, topological obstructions exist when $1 \leq sp < 2$
- Maps including such obstructions do not belong to the closure of smooth maps and, in general, they do not lift
- What happens in absence of topological obstructions? Other types of obstructions? Yes, in the non-compact case. In the compact case:

Theorem (M, Van Schaftingen 19) Let $0 < s < 1$, $1 < sp < 2$, $N = \dim\Omega \geq 2$. In the case of the universal covering with compact $\mathscr{E}$, the following are equivalent for $u \in W^{s,p}(\Omega;\mathscr{N})$:

(a) u is in the strong closure of $C^\infty(\overline{\Omega};\mathscr{N})$

(b) u is in the weak closure of $C^\infty(\overline{\Omega};\mathscr{N})$

(c) u lifts

- Assuming only $\mathscr{E}$ compact, we have (b) $\Longrightarrow$ (c)

Theorem (M, Van Schaftingen 19) Let $1 < p < \infty$, $N = \dim \Omega \geq 2$. Assume π non-trivial. Then there exists some $u \in W^{1/p,p}(\Omega; \mathscr{N})$ such that:

- u is in the strong closure of $C^\infty(\overline{\Omega}; \mathscr{N})$
- u does not lift

- Thus an analytical obstruction persists in the limiting case $sp = 1$
- In this case, the compactness of $\mathscr{E}$ plays no role
- u can be chosen to be smooth in $\overline{\Omega}$ except at some point $a \in \overline{\Omega}$

- The kink satisfies, $\forall\, 1 < p < \infty$, $|\varphi_\varepsilon|_{W^{1/p,p}} \to \infty$, $|e^{\imath \varphi_\varepsilon}|_{W^{1/p,p}} \lesssim 1$
- An analogue on $\mathbb{S}^1$: $M_a(z) = \dfrac{z-a}{\bar{a}z-1}$, $|a|<1$: $|M_a|_{W^{1/p,p}} \lesssim 1$ and the phase of M_a blows up in $W^{1/p,p}$ near a as $|a| \to 1$

Theorem (M 15) Let $1 < p < \infty$. Given $M > 0$, each $u \in W^{1/p,p}(\mathbb{S}^1; \mathbb{S}^1)$ with $|u|^p_{W^{1/p,p}} \le M$ can be written as

$$u = \prod_{k=1}^{K} (M_{a_k})^{\pm 1}\, e^{\imath \psi},$$

with $K \le F(M)$, $|\psi|_{W^{1/p,p}} \le G(M)$

- Proof "by induction" on M, via a bubbling analysis
- Proof by contradiction, giving a (linear) control on $F(M)$, but not on $G(M)$

Theorem (Bourgain, Brezis 03) Let $1 < p \leq 2$. Each $u \in W^{1/p,p}((0,1);\mathbb{S}^1)$ can be written as

$$u = e^{\imath(\varphi+\psi)}, \tag{1}$$

$$|\varphi|_{W^{1/p,p}} \lesssim |u|_{W^{1/p,p}}, |\psi|_{W^{1,1}} \lesssim |u|^p_{W^{1/p,p}} \tag{2}$$

Theorem (Nguyen 08) The above holds in the full range $1 < p < \infty$

- Bourgain, Brezis: proof by explicit decomposition (works in any dimension)
- Uses only $u \in L^\infty \cap W^{1/p,p}$, not the full strength of $|u| = 1$
- Nguyen: Proof by duality (works only in dimension one)
- Uses $|u| = 1$

Theorem (M 08–...) Let $1 \le p < \infty$, $s > 0$. Each $u \in W^{s,p}((0,1);\mathbb{S}^1)$ can be written as

$$u = v\, e^{\imath\varphi}, \tag{3}$$

$$|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}}, |v|_{B^{sp}_{1,1}} \lesssim |u|^p_{W^{s,p}} \tag{4}$$

- $B^{sp}_{1,1}$ is a Besov space. It coincides with the Sobolev space $W^{sp,1}$ for non integer sp. Is strictly smaller than $W^{sp,1}$ for integer sp
- Proof by explicit decomposition. Works in any dimension, with the estimate $|v|_{B^{sp}_{1,1}} \lesssim |u|^p_{W^{s,p}} + |u|^p_{W^{s,p}}$
- Uses $|u| = 1$
- "Converse" is true: if $0 < s < 1$,

$$[v \in B^{sp}_{1,1}(\Omega;\mathbb{S}^1), \varphi \in W^{s,p}(\Omega;\mathbb{R})] \implies v\, e^{\imath\varphi} \in W^{s,p}(\Omega;\mathbb{S}^1)$$

- (But $v \in W^{1,1}(\Omega;\mathbb{S}^1)$ does not imply $v \in W^{1/p,p}(\Omega;\mathbb{S}^1)$, $1 < p < \infty$)