A muggle's approach to Bernstein's approximation theorem

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Captatio benevolentiæ. Bernstein's approximation theorem provides an explicit approximating sequence for continuous functions on $[0,1]$. More specifically, it asserts that, if $f \in C([0,1])$ and we set

$$T_n f(x) = \sum_{k=0}^{n} \binom{n}{k} f\left(\frac{k}{n}\right) x^k (1-x)^{n-k}, \ \forall x \in [0,1], \tag{1}$$

then

$$T_n f \to f \text{ uniformly on } [0,1] \text{ as } n \to \infty. \tag{2}$$

We propose here a muggle's proof of the theorem, involving essentially no trick. We next implement a slightly less natural approach, relying on easier calculations. *The emphasis here is about the non-magical proof.* It is definitely longer than the shortest available ones, but it is straightforward.

Proof of Bernstein's theorem. The starting point is standard. The operators $T_n : C([0,1]) \to C([0,1])$ are linear, continuous, and satisfy the uniform bound $\|T_n\| \le 1$, $\forall n$. Therefore,

$$\text{it suffices to establish (2) for } f \in A, \text{ with } A \subset C([0,1]), \ \overline{\operatorname{span}(A)} = C([0,1]). \tag{3}$$

The standard choice is to consider $A = \{x^\ell;\, \ell \in \mathbb{N}\}$. We propose here a different choice, for which $T_n f$ is *easily and explicitly computable when* $f \in A$. Let

$$f_a(x) := \exp(ax), \ a \in \mathbb{C}. \tag{4}$$

Clearly,

$$\begin{aligned} T_n f_a(x) &= \sum_{k=0}^{n} \binom{n}{k} \exp(ak/n) x^k (1-x)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (x \exp(a/n))^k (1-x)^{n-k} = (x\exp(a/n) + 1 - x)^n, \end{aligned} \tag{5}$$

and thus proving the theorem for f_a amounts to

$$(x\exp(a/n) + 1 - x)^n \to \exp(ax) \text{ uniformly on } [0,1] \text{ as } n \to \infty. \tag{6}$$

Note that, at least for the pointwise convergence, (6) is clear, since

$$\begin{aligned} (x\exp(a/n) + 1 - x)^n &= (x(1 + a/n + O(a^2/n^2)) + 1 - x)^n = (1 + ax/n + O(a^2 x/n^2))^n \\ &\to \exp(ax) \text{ as } n \to \infty. \end{aligned} \tag{7}$$

We next present two possible choices of exponentials.

Choice 1. Consider $A_1 := \{f_a; a \in 2\imath\pi\mathbb{Z}\}$. The space $\overline{\mathrm{span}(A_1)}$ consists of all the 1-periodic continuous functions. Since every continuous function on $[0,1]$ is the sum of a continuous 1-periodic function and of an appropriate multiple of the identity, we are led to the choice $A := \{\mathrm{id}\} \cup A_1$. Noting that, for $n \geq 1$, we have

$$
\begin{aligned}
T_n \mathrm{id}(x) = &\sum_{k=0}^{n} \binom{n}{k} \frac{k}{n} x^k (1-x)^{n-k} = \sum_{k=1}^{n} \binom{n-1}{k-1} x^k (1-x)^{n-k} \\
=&x \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} x^\ell (1-x)^{n-1-\ell} = x,
\end{aligned}
$$

we find (2) holds for $f = \mathrm{id}$, and therefore it suffices to check (2) for $f \in A_1$. This requires writing estimates for *complex* exponentials, and may be tricky to implement in class.

Therefore, we rather go for a slightly less natural

Choice 2. Consider $A := \{f_a; a \in \mathbb{R}_+\}$. To start with, it is not clear that $\mathrm{span}(A)$ is dense in $C([0,1])$. This is indeed the case, but requires a proof. We present below a proof of (3) for this A, relying on very elementary estimates, in principle well-known to students.

Step 0. Elementary inequalities. Using the second order Taylor formula

$$
f(x) = f(0) + x f'(0) + \frac{x^2}{2} f''(c) \text{ for some } c \in [0,x],
$$

we find that

$$
1 + x \leq \exp(x) \leq 1 + x + x^2 \text{ if } x \leq \ln 2, \tag{8}
$$

$$
x - x^2/2 \leq \ln(1+x) \leq x \text{ if } x \geq 0. \tag{9}
$$

Step 1. (6) *holds for* f_a, $a \in \mathbb{R}_+$. On the one hand, we have

$$
\begin{aligned}
(x\exp(a/n) + 1 - x)^n \geq &(x(1+a/n) + 1 - x)^n = (1 + ax/n)^n \\
&= \exp(n \ln(1 + ax/n)) \geq \exp(n(ax/n - a^2x^2/2n^2)) \\
&= \exp(ax)\exp(-a^2x^2/2n) \geq \exp(-a^2/2n)\exp(ax), \ \forall x \in [0,1].
\end{aligned} \tag{10}
$$

On the other hand, if $n \geq a/\ln 2$, we have

$$
\begin{aligned}
(x\exp(a/n) + 1 - x)^n \leq &(x(1 + a/n + a^2/n^2) + 1 - x)^n = (1 + ax/n + a^2x/n^2)^n \\
&= \exp(n \ln(1 + ax/n + a^2x/n^2)) \leq \exp(n(ax/n + a^2x/n^2)) \\
&= \exp(ax)\exp(a^2x/n) \leq \exp(a^2/n)\exp(ax), \ \forall x \in [0,1].
\end{aligned} \tag{11}
$$

Combining (10)–(11), we find, for $n \geq a/\ln 2$,

$$
\begin{aligned}
\|T_n f_a - f_a\|_\infty \leq &\exp(a) \max\left\{1 - \exp(-a^2/2n), \exp(a^2/n) - 1\right\} \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}
$$

Step 2. span(A) *is dense in* $C([0,1])$. Since polynomials are dense in $C([0,1])$, it suffices to prove that the closure of span(A) contains all the monomials. At least intuitively, this follows from

$$\lim_{\varepsilon\to 0+} \underbrace{\left(\frac{\exp(\varepsilon x)-1}{\varepsilon}\right)^{\ell}}_{:=f_{\ell,\varepsilon}(x)} = x^{\ell}, \forall\, x\in[0,1], \forall\, \ell\in\mathbb{N}; \tag{12}$$

here, the limit is pointwise. Noting that $f_{\ell,\varepsilon}$ belongs to span(A), $\forall\,\varepsilon\neq 0$, $\forall\,\ell\in\mathbb{N}$, in order to conclude it suffices to prove that the limit in (12) can be upgraded to a uniform limit. To prove this, we rely again on (8) and obtain, for $\varepsilon\le \ln 2$,

$$x^{\ell} \le \left(\frac{\exp(\varepsilon x)-1}{\varepsilon}\right)^{\ell} \le x^{\ell}(1+\varepsilon x)^{\ell} \le (1+\varepsilon)^{\ell} x^{\ell},$$

so that

$$\left\|\left(\frac{\exp(\varepsilon x)-1}{\varepsilon}\right)^{\ell} - x^{\ell}\right\|_{\infty} \le (1+\varepsilon)^{\ell}-1 \to 0 \text{ as } \varepsilon\to 0+. \qquad \square$$