# Sobolev spaces. Elliptic equations

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## 0 Introduction

The purpose of these notes is to introduce some basic functional and harmonic analysis tools (Sobolev spaces, singular integrals) and to explain how these tools are used in the study of elliptic partial differential equations. In a last part, we will introduce some basic variational methods, applied to existence of solutions of semi linear elliptic problems.

Many books and papers are at the origin of these notes:

*Sobolev spaces*

Robert A. Adams, John J.F. Fournier: Sobolev Spaces. 2nd ed, Elsevier 2003

Haïm Brezis, Analyse fonctionnelle. Théorie et applications, Masson 1983

Michel Willem, Analyse fonctionnelle élémentaire, Cassini 2003

Louis Nirenberg, On elliptic partial differential equations, Ann. Sc. Norm. Sup. Pisa 13 (1959), p. 116-162

Haïm Brezis, Augusto C. Ponce, Kato's inequality up to the boundary, Comm. Contemp. Math. 10 (2008), p. 1217–1241

Lars Hörmander, The Analysis of Linear Partial Differential Operators I, Springer, 1990

Richard Courant, David Hilbert, Methods of Modern Mathematical Physics, II, Interscience, 1962

Elliott H. Lieb, Michael Loss, Analysis, 2nd edition, American Mathematical Society, 2001

Moshe Marcus, Victor Mizel, Every superposition operator mapping one Sobolev space into another is continuous, J. Funct. Anal. 33 (1979), no. 2, 217–229

Xavier Lamy showed me a proof, much simpler than the one I initially found, of Lemma 1.98. I included his proof in the text

*Singular integrals*

Elias Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton University Press, 1993

Elias Stein, Guido Weiss, Introduction to Fourier analysis on Euclidean spaces, Princeton University Press, 1971

Lawrence Evans, Ronald Gariepy, Measure theory and fine properties of functions, CRC Press, 1992

*Linear elliptic equations*

David Gilbarg, Neil S. Trudinger, Elliptic partial differential equations of second order, 4th ed., Springer 2001

Quin Han, Fanghua Lin, Elliptic Partial Differential Equations, American Mathematical Society 2000

## 0.1 Notations

a) If $1 \leq p \leq \infty$, $p'$ stands for the conjugate of $p$ (thus $\dfrac{1}{p} + \dfrac{1}{p'} = 1$)

b) $N$ is the space dimension

c) $R^N_+ = \{x \in \mathbb{R}^N; x_N > 0\}$, $\mathbb{R}^N_- = \{x \in \mathbb{R}^N; x_N < 0\}$

d) $\Omega$ is an open set in $\mathbb{R}^N$. Unless stated otherwise, $\Omega$ is supposed connected (i. e., $\Omega$ is a domain)

e) $\alpha$, $\beta$ stand for multi-indices in $\mathbb{N}^N$. We let $|\alpha| = \displaystyle\sum_{j=1}^{N} |\alpha_j|$

f) $|A|$ is the (usually Lebesgue, sometimes Hausdorff) measure of $A$

g) $\fint$ denotes the average integral: $\displaystyle\fint_A f = \frac{1}{|A|} \int_A f$

h) $|\ |$ stands for the standard Euclidean norm. E. g., $|\nabla u| = \sqrt{\displaystyle\sum \left(\frac{\partial u}{\partial x_j}\right)^2}$

i) $\rho$ is a standard mollifier, i. e. a map s. t. $\rho \in C_c^\infty(\mathbb{R}^N; \mathbb{R})$, $\rho \geq 0$, $\displaystyle\int \rho = 1$, supp $\rho \subset B(0,1)$

j) $\nu$ stands for the unit outward normal at the boundary of a smooth domain $\Omega$

k) $A \Subset \Omega$ means that $\overline{A}$ is a compact subset of the open set $\Omega$

l) In principle, $K$ is always a compact set, but I may forget this here and there and let $K$ also denote a constant. Also in principle, $\omega$ is an open subset of $\Omega$

m) The subscript $_{loc}$ stands for the local version of the spaces we consider. This will not be defined each time, so that we content ourselves to give, once for all, an example:

$$W^{1,1}_{loc}(\Omega) = \{u \in L^1(\Omega); u_{|K} \in L^1, \nabla u_{|K} \in L^1, \forall\ K \Subset \Omega\}$$

n) We let $\omega_N$ denote the volume of the (Euclidean) unit ball in $\mathbb{R}^N$, and $\sigma_N$ denote the ($\mathscr{H}^{N-1}$ dimensional) measure of the (Euclidean) unit sphere. Recall that these two quantities are related by $\sigma_N = N\omega_N$

o) $\mathfrak{M}$ is the set of measurable functions

p) If $C$ is a ball in $\mathbb{R}^N$ (with respect to some norm), then we let $C^*$ denote the cube having the same center as $C$ and twice the same radius. Similarly, $C^{**}$ has four times the radius of $C$

# 1 Sobolev spaces

## 1.1 Motivation

Let $u$ solve the problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{1.1}$$

aka as the Dirichlet problem for the Poisson equation. Assume everything smooth. Multiply the first equation by $v \in C^2(\overline{\Omega})$ s. t. $v = 0$ on $\partial\Omega$ and integrate once by parts (i. e., use the first Green formula). We find that

$$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v = \int_\Omega fv,$$

i. e.,

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Omega fv = 0, \quad \forall\ v \in X,$$

where $X := \{v \in C^2(\overline{\Omega}); v = 0 \text{ on } \partial\Omega\}$. Formally, this is the same as $DJ(u) = 0$, where $J : X \to \mathbb{R}$ is given by

$$J(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega fu.$$

This suggests the following strategy intended to solve (1.1): minimize $J$ in $X$. Then the minimizer should solve (1.1). Good news: if $u$ is a minimizer, then it is indeed the solution of (1.1). Problem with that: the minimum need not exist. A first reason is that if we take a *minimizing sequence* (i. e., a sequence such that $J(u_n) \to \min$), then there is no reason to obtain some $u \in X$ s. t. $u_n \to u$ (whatever the sense we give to this convergence). Actually, Weierstrass showed that, if $f$ is continuous, then there may not be a minimum point of $J$ in $X$. A way to overcome this difficulty is to replace $X$ by a larger space, where the minimum is attained (but not necessarily by some $u \in X$). It turns out that the good candidate is the closure $Y$ of $X$ for the norm $u \mapsto \left(\int_\Omega |\nabla u|^2\right)^{1/2}$. As we will see later, this is a *Sobolev space.* Sobolev spaces are useful even when (1.1) *does have* a solution in $X$. Indeed, existence can be proved as follows: start by minimizing $J$ in $Y$ (this is always possible), then prove that $u \in X$ (this requires extra assumptions on $f$, and the proof of such type of results is the purpose of *regularity theory*). This is Hilbert's strategy.

## 1.2 Distributions

Distributions were formalized by Schwartz, but predecessors of this theory appear already in the works of Leray and Sobolev.

**1.1 Definition.** Let $\Omega \subset \mathbb{R}^N$ be an open set. A *distribution* on $\Omega$ is a linear functional $u$ on $C_c^\infty(\Omega)$ which is *continuous* in the following sense: for each compact $K \Subset \Omega$, there is a constant $C$ and an integer $k$ s. t.

$$|u(\varphi)| \leq C \sum_{|\alpha|\leq k} \sup |\partial^\alpha \varphi|, \quad \forall\, \varphi \in C_c^\infty(\Omega) \text{ s. t. } \operatorname{supp} \varphi \subset K.$$

The vector space of the distributions is denoted $\mathscr{D}'(\Omega)$.

**1.2 Example.** A continuous function $u$ defines a distribution, still denoted $u$, through the formula $u(\varphi) = \int_\Omega \varphi$. More generally, one can replace continuous by measurable and integrable on compacts. In this case, we may take $k = 0$ and $C = \int_K |u|$.

There is a good reason to keep the notation $u$ for the above distribution

**1.3 Proposition** (Localisation principle)**.** *Let $u, v \in L^1_{loc}(\Omega)$. Then $u = v$ a. e. iff the distributions defined by $u$ and $v$ are equal.*
*In other words, one can identify $u$ with the associated distribution.*

*Proof.* The only if part is clear. For the if part, we rely on the following fact that we will prove later

**1.4 Proposition.** *Let $\rho$ be a standard mollifier. Let $f \in L^1_{loc}(\Omega)$. Then $f * \rho_\varepsilon \to f$ a. e. as $\varepsilon \to 0$.*

**1.5 Remark.** Note that $f * \rho_\varepsilon$ is defined in $\Omega_\varepsilon = \{x \in \Omega;\ \mathrm{dist}\,(x, \partial\Omega) > \varepsilon\}$. Thus, for each $x \in \Omega$, $f*\rho_\varepsilon(x)$ is well defined for small $\varepsilon$ (smallness depending on $\varepsilon$).

Back to the proof of the localisation principle. With $f = u - v$, we have $\displaystyle\int f\varphi = 0$ for each $\varphi \in C^\infty_c(\Omega)$. In particular, $f * \rho_\varepsilon = 0$ for each $\varepsilon$. We conclude by letting $\varepsilon \to 0$. $\square$

**1.6 Example.** If $a \in \Omega$, then $\delta_a(\varphi) = \varphi(a)$ is a distribution (the *Dirac mass* at $a$). Indeed, we may take $k = 0$ and $C = 1$. When $a = 0$, we write $\delta$ rather than $\delta_0$.

**1.7 Example.** If $\Sigma$ is a $k$-dimensional submanifold of $\Omega$, then $\delta_\Sigma(\varphi) = \displaystyle\int_\Sigma \varphi\, d\mathscr{H}^k$ is a distribution (the *Dirac mass* on $\Sigma$). Indeed, we may take $k = 0$ and $C$ the Hausdorff measure of $\Sigma \cap K$.
More generally, we may consider the distribution $f\mathscr{H}^k\llcorner\Sigma$, where $f$ is locally integrable (with respect to $\mathcal{H}^k$) on $\Sigma$. This distribution acts through the formula $\varphi \mapsto \displaystyle\int_\Sigma f\varphi\, d\mathscr{H}^k$.

**1.8 Example.** All the above examples are special cases of measures: if $\mu$ is a *locally finite Borel measure* , then $\mu$ defines a distribution through the formula $\mu(\varphi) = \displaystyle\int_\Omega \varphi$. (Take $k = 0$ and $C = |\mu|(K)$.)

**1.9 Exercise.** It is not always possible to take $k = 0$. For example, let $u(\varphi) = \varphi'(0)$ (assuming that $0 \in \Omega$ and $N = 1$). Then $u \in \mathscr{D}'(\Omega)$, but it is not possible to take $k = 0$ when, say, $K$ is a closed interval centered at the origin.
Hint: consider $\varphi(nx)$ for a fixed $\varphi$.

The next result identifies (positive) measures.

**1.10 Proposition.** *Let $u \in \mathscr{D}'(\Omega)$. Then the following are equivalent:*

*(i) $u$ is "positive", in the sense that $u(\varphi) \geq 0$ if $\varphi \in C_c^\infty(\Omega)$ and $\varphi \geq 0$*

*(ii) There is some positive Radon measure $\mu$ s. t. $u(\varphi) = \int \varphi \, d\mu$, $\forall\ \varphi \in C_c^\infty(\Omega)$.*

*If these conditions are satisfied, we write $u \geq 0$.*

*Proof.* The implication *(ii)*$\Longrightarrow$*(i)* is clear. Conversely, let $K \Subset \Omega$ and let $\psi \in C_c^\infty(\Omega)$ be s. t. $0 \leq \psi \leq 1$ and $\psi \equiv 1$ in $K$. If $\varphi \in C_c^\infty(K)$, then

$$-\|\varphi\|_{L^\infty} u(\psi) \leq u(\varphi) \leq \|\varphi\|_{L^\infty} u(\psi),$$

so that $|u(\varphi)| \leq C(K)\|\varphi\|_{L^\infty}$, $\forall\ \varphi \in C_c^\infty(K)$. We find that $u$ has a linear positive continuous extension to $C_c(\Omega)$. We conclude via the Riesz representation theorem. $\square$

When $u \in C^1(\Omega)$, we have $\partial_j u(\varphi) = -u(\partial_j \varphi)$ (this is obtained by integration by parts). This suggests the following

**1.11 Definition.** If $u \in \mathscr{D}'(\Omega)$, we define $\partial_j u$ through the formula $\partial_j u(\varphi) = -u(\partial_j \varphi)$. More generally, $\partial^\alpha u(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi)$. This is still a distribution.
If $P(\partial) = \sum a_\alpha \partial^\alpha$, then we define $(P(\partial)u)(\varphi) = \sum (-1)^{|\alpha|} a_\alpha u(\partial^\alpha \varphi)$.
In the same vein, we define, for $u \in \mathscr{D}'(\Omega)$ and $a \in C^\infty(\Omega)$, the distribution $au$ by $(au)(\varphi) = u(a\varphi)$.

Two points are of interest: the result is still a distribution, and these definitions coincide with the usual ones when $u$ is smooth enough. For example, when $u \in C^1$, we have "old $\partial_j u$=new $\partial_j u$". When we want to emphasize such an equality, we write $\partial_{j,p} u = \partial_{j,d} u$ ($p$ stands for point derivative, $d$ for distributional derivative).

**1.12 Remark.** When there is a possible doubt, we let the subscript $p$ stand for point quantities, and $d$ for distributional ones.

**1.13 Exercise.** If $a \in C^\infty(\Omega)$ and $u \in \mathscr{D}'(\Omega)$, we have the following Leibniz rule

$$\partial^\alpha(au) = \sum_{\beta \leq \alpha} \frac{\alpha!}{\beta!(\alpha - \beta)!} \partial^\beta a \partial^{\alpha - \beta} u.$$

**1.14 Exercise.** If $f(x) = |x|$ (in $\mathbb{R}$), then $f' = \mathrm{sgn}$.

**1.15 Exercise.** If $f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$ (in $\mathbb{R}$), then $f' = \delta$.

**1.16 Exercise.** Let $\Omega_1$, $\Omega_2$ be smooth open sets s. t. $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \Sigma$, with $\Sigma$ smooth hypersurface. Let $u_i \in C^1(\Omega_i \cup \Sigma)$, $i = 1, 2$, and set $u = \begin{cases} u_1, & \text{in } \Omega_1 \\ u_2, & \text{in } \Omega_2 \end{cases}$. Then $\partial_j u = \partial_j u_1 \chi_{\Omega_1} + \partial_j u_2 \chi_{\Omega_2} + (u_2 - u_1)\nu_j \mathscr{H}^k \llcorner \Sigma$. Here, $\nu$ is the normal to $\Sigma$ directed from $\Omega_1$ to $\Omega_2$.

A slightly more involved result is the following

**1.17 Proposition.** *Assume that $N \geq 2$, $0 \in \Omega$. Let $f \in C^1(\Omega \setminus \{0\})$ be such that $\nabla_p u \in L^1_{loc}(\Omega)$. Then $\partial_{j,p} u = \partial_{j,d} u$.*

*Proof.* We have to prove that $\int_\Omega u \partial_j \varphi = -\int_\Omega \partial_{j,p} u \varphi$, $\varphi \in C_c^\infty(\Omega)$. We drop the subscript $p$. The key fact is that $\int_{S(0,\varepsilon)} |u| \, d\mathscr{H}^{N-1} = o(1)$ as $\varepsilon \to 0$. (This will imply $u \in L^1_{loc}(\Omega)$.) Assuming this for the moment, we argue as follows

$$\int_\Omega u \partial_j u \varphi = \lim_{\varepsilon \to 0} \int_{\Omega \setminus B(0,\varepsilon)} \partial_j u \varphi = \lim_{\varepsilon \to 0} \left( \int_{S(0,\varepsilon)} \nu_j u \varphi - \int_{\Omega \setminus B(0,\varepsilon)} u \partial_j \varphi \right)$$
$$= -\lim_{\varepsilon \to 0} \int_{\Omega \setminus B(0,\varepsilon)} u \partial_j \varphi = -\int_\Omega u \partial_j \varphi.$$

We now prove the key fact (and its consequence). Assume e. g., in what follows, that $B(0,1) \Subset \Omega$. We start by noting that (by dominated convergence)

$$\lim_{\varepsilon \to 0} \int_{B(0,1) \setminus B(0,\varepsilon)} \frac{\varepsilon^{N-1}}{|x|^{N-1}} |\nabla u| = 0.$$

We claim that

$$\int_{S(0,\varepsilon)} |u| \leq C\varepsilon^{N-1} + \int_{B(0,1) \setminus B(0,\varepsilon)} \frac{\varepsilon^{N-1}}{|x|^{N-1}} |\nabla u|.$$

This follows from

$$\int_{S(0,\varepsilon)} |u| = \varepsilon^{N-1} \int_{\mathbb{S}^{N-1}} |u(\varepsilon\omega)| \, d\mathscr{H}^{N-1}$$
$$= \varepsilon^{N-1} \int_{\mathbb{S}^{N-1}} \left| u(\omega) - \int_\varepsilon^1 \frac{d}{dr}[u(r\omega)] \right| \, d\mathscr{H}^{N-1}$$
$$\leq \varepsilon^{N-1} \int_{\mathbb{S}^{N-1}} |u| \, d\mathscr{H}^{N-1} + \int_{\Omega \setminus B(0,\varepsilon)} \frac{\varepsilon^{N-1}}{|x|^{N-1}} |\nabla u|.$$

Consequence: if $K \Supset \Omega$, then

$$\int_K |u| \leq \int_{B(0,1)} |u| + \sup_{K\setminus B(0,1)} |u| \leq \sup_{0\leq\varepsilon\leq 1} \int_{S(0,\varepsilon)} |u| + \sup_{K\setminus B(0,1)} |u| < \infty,$$

i. e., $u \in L^1_{loc}(\Omega)$. $\square$

In order to establish our next example, we need the following simple fact

**1.18 Proposition** (Delocalisation principle). *If $u = v$ on $\Omega_i$, $i \in I$, then $u = v$ on $\cup\Omega_i$.*

*Proof.* Let $\Omega = \cup\Omega_i$, $\varphi \in C^\infty_c(\Omega)$ and $\varphi_i$ be a finite partition of unity on supp $\varphi$ subordinated to the covering $(\Omega_i)$. Then $u(\varphi) = \sum u(\varphi_i\varphi) = \sum v(\varphi_i\varphi) = v(\varphi)$. $\square$

**1.19 Proposition.** *Let $\Sigma$ be a $k$-dimensional submanifold of $\Omega$ (here, $k \leq N-2$). Let $u \in C^1(\Omega \setminus \Sigma)$ be such that $\nabla_p u \in L^1_{loc}(\Omega)$. Then $\partial_{j,p}u = \partial_{j,d}u$. Conversely, if $\partial_{j,d}u \in L^1_{loc}(\Omega)$, then $\partial_{j,p}u \in L^1_{loc}(\Omega)$.*

Note that Proposition 1.17 is a special case (when $k = 0$) of the above result.

*Proof.* We start with a special case, later referred as the *standard case*. We let $\Omega = B_{\mathbb{R}^{N-k}}(0,1)\times(-1,1)^k$, $\Sigma = \{0\}\times(-1,1)^k$ and assume that $u \in C^1(\overline{\Omega}\setminus\Sigma)$. Write a point in $\mathbb{R}^N$ as $x = (x', x'')$, with $x' \in \mathbb{R}^{N-k}$, $x'' \in \mathbb{R}^k$. As in the proof of Proposition 1.17, we have

$$\int_{\{x\in\Omega;|x'|=\varepsilon\}} |u| \leq C\varepsilon^{N-k-1} + \int_\Omega \frac{\varepsilon^{N-k-1}}{|x'|^{N-k-1}}|\nabla u| \to 0 \text{ as } \varepsilon \to 0.$$

Consequently, we have $u \in L^1(\Omega)$ and

$$\int_{\{x\in\Omega;|x'|\leq\varepsilon\}} |u| = o(\varepsilon) \quad \text{ as } \varepsilon \to 0.$$

Let now $\psi \in C^\infty(\mathbb{R};\mathbb{R})$ be s. t. $\psi(t) = \begin{cases} 1, & \text{if } t \geq 1 \\ 0, & \text{if } t \leq 1/2 \end{cases}$. For $\varphi \in C^\infty_c(\Omega)$ we have

$$\begin{aligned}
\int_\Omega u\partial_j\varphi &= \lim_{\varepsilon\to 0}\int_\Omega u\psi(|x'|/\varepsilon)\partial_j\varphi = -\lim_{\varepsilon\to 0}\int_\Omega \partial_j(u\psi(|x'|/\varepsilon))\varphi \\
&= -\int_\Omega \partial_j u\varphi - \lim_{\varepsilon\to 0}\int_\Omega u\partial_j[\psi(|x'|/\varepsilon))]\varphi \\
&= -\int_\Omega \partial_j u\varphi - \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}O\left(\int_{\{x\in\Omega;|x'|\leq\varepsilon\}} |u|\right) = -\int_\Omega \partial_j u\varphi.
\end{aligned}$$

We now turn to the general case. We cover $\Omega$ with a family $(\Omega_i)$ of open sets s. t., for each $i$: either $\Omega_i \cap \Sigma = \emptyset$, or there is a $C^1$ diffeomorphism $\Phi_i : \overline{\Omega}_i \to \overline{\Omega}_0$ (here, $\Omega_0$ stands for the open set from the standard example) s. t. $\Phi(\Sigma \cap \Omega_i) = \{0\} \times \mathbb{R}^k$. In view of the delocalisation principle, it suffices to prove that $\partial_{j,p} u = \partial_{j,d} u$ in each $\Omega_i$. This equality is clear if $\Sigma$ does not meet $\Omega_i$. Otherwise, let $\varphi \in C_c^\infty(\Omega)$ be s. t. $\operatorname{supp} \varphi \subset \Omega_i$. Then

$$\int_{\Omega_i} u \partial_j \varphi = \lim_{\varepsilon \to 0} u\psi(|(\Phi_i(x))'|/\varepsilon)\partial_j \varphi = -\lim_{\varepsilon \to 0} \int_{\Omega_i} \partial_j (u\psi(|(\Phi_i(x))'|/\varepsilon))\varphi$$
$$= -\int_{\Omega_i} u\varphi - \lim_{\varepsilon \to 0} \int_{\Omega_i} u \partial_j(\psi(|(\Phi_i(x))'|/\varepsilon))\varphi = -\int_{\Omega_i} u\varphi.$$

Here, we use the fact that $|\partial_j(\psi(|(\Phi_i(x))'|/\varepsilon))| \le C/\varepsilon$ combined with

$$\int_{\{x \in \Omega_i; |(\Phi_i(x))'| \le \varepsilon\}} |u| \le C \int_{\{x \in \Omega_0; |x'| \le \varepsilon\}} |u \circ \Phi_i^{-1}| = o(\varepsilon) \text{ as } \varepsilon \to 0.$$

The last property is a consequence of the fact that $u \circ \Phi_i^{-1}$ falls into the standard case.
We end with the converse: since $u \in C^1(\Omega \setminus \Sigma)$, we have $\partial_{j,p} u = \partial_{j,d} u$ a. e. in $\Omega \setminus \Sigma$. Since $\Sigma$ is a null set, we find that $\partial_{j,p} u = \partial_{j,d} u$ a. e. in $\Omega$, so that $\partial_{j,p} u \in L^1_{loc}$. □

Another useful operation (in addition to differentiation and multiplication be smooth functions) is convolution.

**1.20 Definition.** If $u \in \mathscr{D}'(\mathbb{R}^N)$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$, we set $u * \varphi(x) = u(\varphi(x - \cdot))$.

One may prove the following

**1.21 Proposition.** *We have $u * \varphi \in C^\infty$ and $\partial^{\beta+\gamma}(u * \varphi) = (\partial^\beta u) * (\partial^\gamma \varphi)$.*

*Proof.* The second property is trivial. We sketch the argument leading to the first one. The key fact is continuity, which is obtained as follows: if $x_n \to x$, then there is a fixed compact $K$ s. t. $\operatorname{supp} \varphi(x_n - \cdot) \subset K$ for each $n$. Since $\partial^\alpha(\varphi(x_n - \cdot) - \varphi(x - \cdot)) \to 0$ uniformly in $K$, we find (using the definition of a distribution) that $u * \varphi(x_n) \to u * \varphi(x)$. Similarly, we have

$$\lim_{t \to 0} \frac{u * \varphi(x + te_j) - u * \varphi(x)}{t} = u * \partial_j \varphi(x).$$

By the key fact, the latter quantity is continuous in $x$. Thus $u * \varphi \in C^1$ and $\partial_j u * \varphi = u * \partial_j \varphi$. We continue by induction on the number of derivatives. □

**1.22 Example.** $\delta * \varphi = \delta$.

Convolution can be considered in other settings (not only $\mathscr{D}'(\mathbb{R}^N)$ *vs* $C_c^\infty(\mathbb{R}^N)$), e. g., $L^1$ *vs* $L^p$. We will come back to this later.

**1.23 Exercise.** A final point we discuss here is *extension*. If $u \in \mathscr{D}'(\Omega)$ and $\omega$ is an open subset of $\Omega$, then we may restrict $u$ to $\omega$ in the obvious way: we let $u$ act on $C_c^\infty(\omega)$. The converse is not always possible: given a distribution $u$ on $\omega$, it may not be the restriction of a distribution on $\Omega$. Check the following: if $\omega = (0,2)$ and $\Omega = \mathbb{R}$, there is no distribution on $\Omega$ whose restriction to $\omega$ is $u = \sum \delta_{1/n}$.

## 1.3 First properties of Sobolev spaces

### 1.3.1 Definition

**1.24 Definition.** Let $1 \le p \le \infty$. Then

$$W^{1,p} = W^{1,p}(\Omega) = \{u \in L^p(\Omega); \partial_j u \in L^p(\Omega)\}.$$

We endow this vector space with the norm $\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_j \|\partial_j u\|_{L^p}$.

When $p < \infty$, another possible (equivalent) norm is $\left(\|u\|_{L^p}^p + \sum_j \|\partial_j u\|_{L^p}^p\right)^{1/p}$.

When $p = 2$, $W^{k,2}$ is aka $H^1$.

Note that $L^p \subset L^1_{loc}$, so that $L^p$ functions are in $\mathscr{D}'(\Omega)$. Thus $\partial_j u$ makes sense (as a distribution).

**1.25 Example.** Assume that $\Omega = B_{\mathbb{R}^N}(0,1)$. Let $u(x) = |x|^{-\alpha}$, where $\alpha \in \mathbb{R}$. We claim that $u \in W^{1,p}$ iff $p(\alpha+1) < N$.
Indeed, to start with, we have $u \in L^p$ iff $p\alpha < N$. Next, we have $|\nabla_p u(x)| \sim |x|^{-\alpha-1}$, so that $\nabla_p u \in L^1_{loc}$ if and only if $\alpha < N-1$. If $\alpha \ge N-1$, then we cannot have $\nabla_d u \in L^1_{loc}$ (in view of Proposition 1.19). Thus we cannot have $u \in W^{1,p}$. If $\alpha < N-1$, then $\nabla_p u = \nabla_d u$, so that $\nabla u \in L^p$ iff $p(\alpha+1) < N$. We find that $u \in W^{1,p}$ iff $p(\alpha+1) < N$.

### 1.3.2 $W^{1,\infty}$ and *Lip*

**1.26 Theorem.** *We have $u \in W^{1,\infty}$ iff (possibly after redefining $u$ on a null set) $u$ is bounded and there is some $C > 0$ s. t. (P) $|u(x) - u(y)| \le C|x-y|$*

*whenever* $[x, y] \subset \Omega$.
*In the special case where* $\Omega$ *is convex, this is the same as* $u$ *bounded and Lipschitz. In this case,* $\|u\|_{W^{1,\infty}} \sim \|u\|_{L^\infty} + |u|_{Lip}$.

**1.27 Remark.** In particular, the theorem asserts that $u$ equals a. e. a continuous function. We will write, here and there, "$u$ is continuous" as a shorthand for "$u$ is equal a. e. to a continuous function".

*Proof.* Assume first that $u \in W^{1,\infty}$. Then

$$|\nabla(u * \rho_\varepsilon)| = |(\nabla u) * \rho_\varepsilon)| \leq \|\nabla u\|_{L^\infty} \|\rho_\varepsilon\|_{L^1} = \|\nabla u\|_{L^\infty}.$$

We find that the family $(u * \rho_\varepsilon)$ satisfies $|u * \rho_\varepsilon| \leq \|u\|_{L^\infty}$ and (by the mean value theorem) condition $(P)$. By Arzelà-Ascoli, we have (possibly after extraction) $u * \rho_\varepsilon \to v$ uniformly on compacts. Clearly, the limit is continuous, bounded (since $u$ is), and satisfies $(P)$. Since, on the other hand, we have $u * \rho_\varepsilon \to u$ a. e., we proved the only if part.
Conversely, by delocalisation we may assume $\Omega$ bounded and convex. We let as an exercise the fact that if $(P)$ holds, then $u * \rho_\varepsilon$ is $C$-Lipschitz. Thus $|\nabla(u * \rho_\varepsilon)| \leq C$. Thus the family $(\nabla(u * \rho_\varepsilon))$ is bounded in $L^\infty$ (and thus in $L^2$). It is a standard fact in functional analysis that, under such assumptions, possibly after passing to a subsequence, we have $\nabla(u * \rho_\varepsilon) \rightharpoonup f$ in $L^2$ for some $f$ s. t. $|f| \leq C$. Using the definition of weak convergence, we find that $\nabla_d u = f$.
On the way, we also proved norm equivalence when $\Omega$ is convex. $\square$

**1.28 Exercise.** Prove the "standard fact" mentioned above, which amounts to: if $f_n, f \in L^2(\Omega)$ are s. t. $|f_n| \leq C$ and $f_n \rightharpoonup f$ in $L^2$, then $|f| \leq C$ a. e. (here, $f_n, f$ may be vector-valued).
Hint: compute $\displaystyle\int_{|f|>C} f \cdot \frac{f}{|f|}$.

**1.29 Exercise.** Let $\Omega = \{x \in \mathbb{R}^2 \setminus \mathbb{R}_-; 1 < |x| < 2\}$ and $u(re^{\imath\theta}) = \theta$, $1 < r < 2$, $\theta \in (-\pi, \pi)$. Prove that $u \in W^{1,\infty}$, but $u \notin Lip$.

We will now see for the first time the role of the regularity of $\Omega$. In view of the above example, in general we do not have $W^{1,\infty} = Lip$. However, this holds under additional assumptions on $\Omega$. A deep question that will be systematically overlooked in what follows is the one of minimal assumptions that make "useful theorems" (embeddings, compactness, etc.) work. Most of time, we will consider "lazy assumptions" that make some rather simple proofs work. For sharper results in this direction, the books of Adams and Fournier, respectively Maz'ja, on Sobolev spaces, are good references.

**1.30 Theorem.** *Assume that $\Omega$ is a bounded Lipschitz domain. Then $W^{1,\infty} = Lip$, and $\|u\|_{W^{1,\infty}} \sim \|u\|_{L^\infty} + |u|_{Lip}$.*

*Proof.* When $\Omega$ is Lipschitz, the geodesic distance $d_\Omega$ in $\Omega$ is equivalent to the Euclidean one. (Recall that $d_\Omega(x, y)$ is the infimum of the length of all polygonal lines connecting $x$ to $y$. Such lines do exist, since $\Omega$ is a domain.) Clearly, the proof of Theorem 1.26 implies that $\|u\|_{W^{1,\infty}} \le \|u\|_{L^\infty} + N|u|_{Lip}$. Conversely, by the same theorem we have $|u(x) - u(y)| \le \|\nabla u\|_{L^\infty} d_\Omega(x, y)$. We conclude using the fact that the geodesic distance is of the same order as the Euclidean one. □

**1.31 Exercise.** Prove the equivalence between geodesic and Euclidean distance in Lipschitz bounded domains.
Hint: cover $\overline{\Omega}$ with balls which are: either contained in $\Omega$, or chart domains on which $\partial\Omega$ can be straightened. Extract a finite covering, then consider the Lebesgue number $r$ associated to this covering. Estimate $d_\Omega(x, y)$ by considering the cases $|x - y| \le r$ and $|x - y| > r$.

### 1.3.3 1D

We assume, e. g., that $0 \in \Omega$ and that $\Omega$ is an interval. The description of Sobolev spaces is basically a consequence of the following

**1.32 Theorem.** *Let $u, v \in L^1_{loc}(-1, 1)$. Then $u' = v$ iff there is some $C$ s. t. (possibly after redefining $u$ on a null set) $u(x) = C + \int_0^x v(t)\, dt$.*

*Proof.* It is straightforward (by definition+Fubini) that $u_0(x) = \int_0^x v(t)\, dt$ satisfies $u_0' = v$. The conclusion follows from the next lemma. □

**1.33 Lemma.** *Let $u \in \mathscr{D}'(\Omega)$ (with $\Omega \subset \mathbb{R}$ an interval). Then $u' = 0$ iff $u \in L^1_{loc}$ and $u = C$ a. e.*

*Proof.* The if part is clear. Conversely, fix $\varphi_0 \in C_c^\infty(\Omega)$ s. t. $\int \varphi_0 = 1$. Let $\varphi \in C_c^\infty(\Omega)$ ans set $\psi = \varphi - \left(\int \varphi\right) \varphi_0$. Then $\psi$ has zero average, which implies that $\psi = \zeta'$ for some $\zeta \in C_c^\infty(\Omega)$. We find that, with $C = u(\varphi_0)$, we have

$$u(\varphi) = \int C\varphi + u(\psi) = \int C\varphi - u'(\zeta) = \int C\varphi,$$

i. e., $u = C$ in $\mathscr{D}'(\Omega)$. We conclude via the localisation principle. □

**1.34 Remark.** From now on we identify maps with derivatives in $L^1_{loc}$ with their (existing and unique) continuous representative.

**1.35 Corollary.** *If $\Omega$ is bounded, then $W^{1,p} \hookrightarrow C(\overline{\Omega})$, with continuous embedding.*

*Proof.* Inclusion is clear. Continuity follows from the closed graph theorem (since, as we will se later, $W^{1,p}$ is a Banach space). $\square$

**1.36 Corollary.** *Assume that $u' \in L^1_{loc}$. Then $u$ has a (usual) derivative $u'_p$ a. e., and $u'_p = u'$ a. e.*

*Proof.* Let $u' = v$. Then, for a. e. $x \in \Omega$, we have (by the Lebesgue differentiation theorem) $u'_p(x) = \lim_{h\to 0} \dfrac{\int_x^{x+h} v(t)\,dt}{h} = v(x)$. $\square$

**1.37 Corollary.** *Lipschitz functions of one variable are differentiable a. e.*

The next result is due to Lebesgue.

**1.38 Theorem.** *Assume that $u$ is bounded. Then $u \in W^{1,1}$ iff $u$ is absolutely continuous.*

*Proof.* Recall that absolutely continuity means: for each $\varepsilon > 0$, there is some $-\Delta > 0$ s. t. if $(a_i, b_i)$ are disjoint intervals in $\Omega$ s. t. $\sum (b_i - a_i) < -\Delta$, then $\sum |u(b_i) - u(a_i)| < \varepsilon$.

The only if condition is a special case of Lebesgue's lemma applied to $v = u'$. This lemma asserts that if $|A| < -\Delta$, then $\int_A |v| < \varepsilon$ (provided $-\Delta$ is sufficiently small). If we take $A = \cup(a_i, b_i)$, then we recover the absolute continuity condition.

Conversely, if $u$ is absolutely continuous (AC), then the following facts are easy to check and left as an exercise:

a) If $u$ is AC, then $u$ is continuous and has bounded variation

b) If we let $u = u_1 - u_2$ be the Jordan decomposition of $u$ (i. e., $u_1(x) = \bigvee_0^x u$, $u_2 = u_1 - u$; these functions are non decreasing), then $u_1, u_2$ are AC

c) Write each $u_i$, $i = 1, 2$, as $u_i(x) = C_i + \mu_i((-\infty, x))$ for appropriate measures $\mu_i$. (This is possible since each $u_i$ is continuous and non decreasing.) Then $\mu_i$ has the property that if $\omega$ is an open set and $|\omega| < -\Delta$, then $\mu_i(\omega) < \varepsilon$

d) Consequently, if $\omega$ is a Borel null set, then $\mu_i(\omega) = 0$

e) By the Radon-Nikodym theorem, there is some $v_i \in L^1$ s. t. $u_i(x) = D_i + \int_0^x v_i(t)\, dt$

f) This implies at once that $u' = v_1 - v_2 \in L^1$.

$\square$

**1.39 Theorem.** *The following are equivalent, when $1 \le p < \infty$:*

*a) $u = 0$ on $\partial\Omega$*

*b) $u$ belongs to the closure, in $W^{1,p}$, of $C_c^\infty(\Omega)$*

*c) the extension of $u$ with the value 0 outside $\Omega$ is in $W^{1,p}(\mathbb{R})$.*

*Proof.* We assume that $\Omega$ is bounded, e. g., $\Omega = (-1, 1)$. The proofs have to be adapted to the other cases ($\Omega$ is a half line or $\mathbb{R}$).

*(a) implies (c).* Let $\tilde{}$ denote the extension with the value zero outside $\Omega$. Let $v = u'$. Then $\int_{-1}^1 v = 0$. We find that $\tilde{u}(x) = \int_{-1}^x \tilde{v}(t)\, dt$. This implies (c).

*(c) implies (b).* Let $w = \tilde{u}'$. We clearly have $w = 0$ outside $\Omega$ and $\int w = 0$. Let $(w_n) \subset C_c^\infty(\Omega)$ be s. t. $w_n \to w$ in $L^p$ and $\int w_n = 0$. Set $u_n(x) = \int_{-1}^x w_n(t)\, dt$. Then $u_n \in C_c^\infty(\Omega)$. We leave as an exercise that $u_n \to u$ in $W^{1,p}$.

*(b) implies (a).* Note that, if $u_1, u_2 \in W^{1,p}(\Omega)$ have derivatives $v_1, v_2$ and if $u_i(-1) = 0$, $i = 1, 2$, then $|u_1(x) - u_2(x)| \le \|v_1 - v_2\|_{L^1} \le C\|v_1 - v_2\|_{L^p}$. With this in mind, a Cauchy sequence (in $W^{1,p}$) $(u_n) \subset C_c^\infty(\Omega)$ converges uniformly. In particular, we must have $u(-1) = 0$ (and, similarly, $u(1) = 0$.) $\square$

### 1.3.4 $n$ dimensional case: basic properties

**1.40 Proposition.** *$W^{1,p}$ is a Banach space.*

*Proof.* Clearly, if $(u_n)$ is a Cauchy sequence, then $u_n \to u$ in $L^p$ and $\partial_j u_n \to v_j$ in $L^p$ for appropriate $u$ and $v_j$. It is obvious that $v_j = \partial_j u$. Finally, we clearly have $u_n \to u$ in $W^{1,p}$. $\square$

In the same vein

**1.41 Proposition.** *$W^{1,2}$ is a Hilbert space (with the second norm).*

**1.42 Proposition** (Approximation by regularization)**.** *Assume that either $\Omega = \mathbb{R}^N$ or $u$ vanishes outside some compact subset of $\Omega$. Let $1 \le p < \infty$ and $u \in W^{1,p}$. Then $u * \rho_\varepsilon \to u$ in $W^{1,p}$.*

*Proof.* This is straightforward using the fact that $\partial_j(u * \rho_\varepsilon) = (\partial_j u) * \rho_\varepsilon$. $\square$

**1.43 Exercise.** Let $u_0(x) = |x|$, $x \in (-1,1)$. Let $\psi \in C_c^\infty(-1,1)$ be s. t. $\psi = 1$ near the origin. Let $u = u_0\psi$. Then $u \in W^{1,\infty}$, $u$ is compactly supported, but there is no sequence of smooth maps converging to $u$ in $W^{1,\infty}$. In fact, prove that the closure of smooth maps consists precisely in $C^1$ maps.

**1.44 Proposition** (Approximation by cutoff and regularization)**.** *Let $1 \le p < \infty$. Then $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}^N)$.*

*Proof.* It suffices to prove that $C_c^\infty(\mathbb{R}^N)$ is dense in $C^\infty \cap W^{1,p}(\mathbb{R}^N)$ (next apply the previous proposition). Let $u \in C^\infty \cap W^{1,p}$. Let $\varphi \in C_c^\infty(\mathbb{R}^N)$ be s. t. $0 \le \varphi \le 1$, $\varphi = 1$ in $B(0,1)$, $\varphi = 0$ outside $B(0,2)$. If we set $u^\varepsilon = u\varphi(\varepsilon\cdot) \in C_c^\infty(\mathbb{R}^N)$, then

$$\|u-u^\varepsilon\|_{W^{1,p}} \le \|u\|_{L^p(\mathbb{R}^N\setminus B(0,1/\varepsilon))}+C\varepsilon\|u\|_{L^p}+\|\nabla u\|_{L^p(\mathbb{R}^N\setminus B(0,1/\varepsilon))} \to 0 \text{ as } \varepsilon \to 0.$$

$\square$

The next result is due to Meyers and Serrin.

**1.45 Theorem.** *Assume that $1 \le p < \infty$. Then $C^\infty(\Omega) \cap W^{1,p}$ is dense in $W^{1,p}$.*

*Proof.* Consider a sequence $(\Omega_i)$ of open sets s. t. $\Omega_{-1} = \emptyset$, $\Omega_i \Subset \Omega_{i+1}$ and $\cup\Omega_i = \Omega$. Let $(\zeta_i)$ be a partition of unity subordinated to the covering $(\Omega_{i+1} \setminus \overline{\Omega}_{i-1})$.

Let $u \in W^{1,p}$. Using approximation by regularization, for each $i$ there is some $w_i \in C^\infty$ s. t. supp $w_i \subset \Omega_{i+1} \setminus \overline{\Omega}_{i-1}$ and $\|w_i - \zeta_i u\|_{W^{1,p}} < 2^{-i-1}\varepsilon$. We now let $w = \sum w_i$. Then $w \in C^\infty$, since in the neighbourhood of a point at most four $w_i$'s do not vanish. If $\omega \Subset \Omega$, we find that $\|w-u\|_{W^{1,p}(\omega)} < \varepsilon$. We conclude by letting $\omega \to \Omega$. $\square$

### 1.3.5 Higher order spaces

**1.46 Definition.** Let $1 \leq p \leq \infty$ and $k = 1, 2, \ldots$ Then

$$W^{k,p} = W^{k,p}(\Omega) = \{u \in L^p(\Omega); \partial^\alpha u \in L^p(\Omega), |\alpha| \leq k\}.$$

We endow this vector space with the norm $\|u\|_{W^{k,p}} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p}$. When $p < \infty$, another possible (equivalent) norm is $\left( \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p}^p \right)^{1/p}$.

When $p = 2$, $W^{k,2}$ is aka $H^k$.

Straightforward generalizations of the previous results include

**1.47 Theorem.** *Assume that $\Omega$ is a Lipschitz bounded domain. Then $W^{k,\infty}$ consists precisely of $C^{k-1}$ maps s. t. $D^{k-1}$ is Lipschitz.*

**1.48 Theorem.** *If $1 \leq p < \infty$, then $C^\infty(\Omega) \cap W^{k,p}$ is dense in $W^{k,p}$.*

**1.49 Proposition** (Approximation by regularization)**.** *Assume that either $\Omega = \mathbb{R}^N$ or $u$ vanishes outside some compact subset of $\Omega$. Let $1 \leq p < \infty$ and $u \in W^{k,p}$. Then $u * \rho_\varepsilon \to u$ in $W^{k,p}$.*

**1.50 Proposition** (Approximation by cutoff and regularization)**.** *Let $1 \leq p < \infty$. Then $C_c^\infty(\mathbb{R}^N)$ is dense in $W^{k,p}(\mathbb{R}^N)$.*

Another obvious result that holds for each $k$ is

**1.51 Proposition.** *Assume that $a \in C^k(\Omega)$ has bounded derivatives up to the order $k$. If $u \in W^{k,p}(\Omega)$, then $au \in W^{k,p}(\Omega)$ and the usual Leibniz rule applies to the derivatives of $au$ up to the order $k$.*

*Proof.* This is clear when $u \in C^\infty \cap W^{k,p}$. The general case follows by approximation. □

**1.52 Exercise.** Let $\Omega_1$, $\Omega_2$ be smooth open bounded sets s. t. $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \Sigma$, with $\Sigma$ smooth hypersurface. Let $u_i \in C^k(\overline{\Omega}_i)$, $i = 1, 2$, be s. t. $u_1 = u_2$ on $\Sigma$. Set $u = \begin{cases} u_1, & \text{in } \Omega_1 \cup \Sigma \\ u_2, & \text{in } \Omega_2 \end{cases}$. Assume that $u \in C^{k-1}$. Then $u \in W^{k,p}$, $1 \leq p \leq \infty$, and, for $|\alpha| \leq k$, we have $\partial^\alpha u = \begin{cases} \partial^\alpha u_1, & \text{in } \Omega_1 \cup \Sigma \\ \partial^\alpha u_2, & \text{in } \Omega_2 \end{cases}$.

More generally, one may replace the condition $u_i \in C^k(\overline{\Omega}_i)$ by $u_i \in C^k(\Omega_i \cup \Sigma)$ and $\partial^\alpha u_i \in L^p(\Omega_i)$, $|\alpha| \leq k$.

Hint: consider first the case $k = 1$, then reduce the general case to this one.

**1.53 Exercise.** Let $\Omega$, $U$ be smooth bounded domains. Assume that $\Phi : \overline{\Omega} \to \overline{U}$ is a $C^k$ diffeomorphism. Then $u \in W^{k,p}(\Omega)$ iff $u \circ \Phi^{-1} \in W^{k,p}(U)$.
Hint: prove that the chain rule holds for $\partial^\alpha (u \circ \Phi^{-1})$, $|\alpha| \leq k$. (Second hint: start with a smooth $u$.)

### 1.3.6 Extensions, straightening

It happens to $W^{1,p}$ maps what happens to distributions: in general, it is impossible to extend a $W^{1,p}$ maps in $\Omega$ to a $W^{1,p}$ map in $\mathbb{R}^N$. Indeed, let $\Omega = (-1,1) \setminus \{0\}$, $u(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$. Then $u' = 0$, so that $u \in W^{1,1}$. However, $u$ does not have a $W^{1,1}$ extension to $\mathbb{R}$, and not even to $(-1,1)$. Indeed, otherwise $u$ would equal, a. e. in $(-1,1)$, a continuous function, which is impossible. As in the case where we discussed equality $W^{1,\infty} = Lip$, extension property holds if we impose some regularity on $\Omega$. Before examining that point, let us examine a simple (but not optimal) procedure that reduces the case of an arbitrary domain to the *standard case* where $\Omega$ is a half space, $\Omega = \mathbb{R}^N_+$. This procedure is the following: in order to establish a property, say $(P)$, for $\Omega$, do the following:

a) establish $(P)$ for $\Omega = \mathbb{R}^N_+$

b) establish, via diffeomorphism (=straightening of the boundary), $(P)$ in a neighbourhood of a given point of $\partial\Omega$

c) conclude via a partition of unity.

We will explain this in detail when $(P)$ is the extension problem. In other situations, we will concentrate ourselves on the heart of the matter, which concerns the standard case. The other steps are rather straightforward and will be left to the reader.

**1.54 Theorem.** *Assume that $1 \leq p < \infty$. Let $\Omega$ be a bounded $C^k$ domain. Then there is a linear continuous extension operator $P : W^{k,p}(\Omega) \to W^{k,p}(\mathbb{R}^N)$.*
*Here, "extension operator" means that $(Pu)_{|\Omega} = u$.*

**1.55 Remark.** Assumption on $\Omega$ is not optimal; Lipschitz, and even less, would be sufficient. We send to Adams and Fournier for sharper statements.

**1.56 Remark.** The theorem is also true when $p = \infty$ (Whitney's extension theorem), but this requires a separate proof and will be omitted here.

*Proof.* **Step 1.** The standard case $\Omega = \mathbb{R}^N_+$

Write a point $x \in \mathbb{R}^N$ as $x = (x', x_N)$; write also $\alpha \in \mathbb{N}^N$ as $\alpha = (\alpha', \alpha_N)$. Let $Pu(x) = \begin{cases} u(x), & \text{if } x_N > 0 \\ \displaystyle\sum_{j=1}^{k} a_j u(x, -jx_N), & \text{if } x_N < 0 \end{cases}$. Here, the real numbers $a_j$ will be fixed later. We claim that, for appropriate $a_j' s$ and $\alpha$ s. t. $|\alpha| \leq k$, we have

$$\partial^\alpha Pu(x) = \begin{cases} \partial^\alpha u(x), & \text{if } x_N \leq 0 \\ \displaystyle\sum_{j=1}^{k} a_j (-j)^{\alpha_N} \partial^\alpha u(x, -jx_N), & \text{if } x_N < 0 \end{cases}. \tag{1.2}$$

Assuming this, we leave to the reader the fact that $P$ has all the required properties. We start with the special case where, in addition to being in $W^{k,p}$, we assume that $u \in C^\infty(\overline{\mathbb{R}^N_+})$. In this case, we have $Pu \in C^{k-1}$ for appropriate $a_j$. Indeed, the derivatives up to the order $k-1$ from above and below $\mathbb{R}^{N-1} \times \{0\}$ will coincide if the $a_j$'s satisfy the system $\displaystyle\sum_{j=1}^{k} a_j (-j)^l = 1$, $l = 0, \ldots, k-1$. This (Vandermonde) system has a unique solution. E. g., when $k = 1$, $a_1 = 1$, and $P$ is the extension by reflection across $\mathbb{R}^{N-1} \times \{0\}$. By Exercise 1.52, (1.2) holds in this case.

We now turn to the case of a general $u$. The conclusion is a straightforward consequence of the following

**1.57 Proposition.** *Let $1 \leq p < \infty$. Then $C^\infty(\overline{\mathbb{R}^N_+}) \cap W^{k,p}(\mathbb{R}^N_+)$ is dense in $W^{k,p}(\mathbb{R}^N_+)$.*

*Proof.* It suffices to prove that the closure of $C^\infty(\overline{\mathbb{R}^N_+}) \cap W^{k,p}(\mathbb{R}^N_+)$ contains $C^\infty(\mathbb{R}^N_+) \cap W^{k,p}(\mathbb{R}^N_+)$ (and use the Meyers-Serrin theorem). This is a consequence of the fact that, if $u \in W^{k,p}(\mathbb{R}^N_+)$, then $u_\varepsilon \to u$ in $W^{k,p}$, where $u_\varepsilon(x) = u(x', x_N + \varepsilon)$. $\square$

**1.58 Exercise.** Check the above fact.
Hint: translations are continuous in $L^p$, $1 \leq p < \infty$.

We will need later the following fact: assume that supp $u \subset [-1, 1]^{N-1} \times [0, 1]$. Then supp $Pu \subset [-1, 1]^N$.

**Step 2.** The case where the support of $u$ lies near $\partial\Omega$

We may cover $\overline{\Omega}$ with a finite collection $\Omega_0, \ldots, \Omega_m$ of open sets s. t. : $\Omega_0 = \Omega$, and for each $i \geq 1$ there is a $C^k$ diffeomorphism $\Phi_i$ of $\overline{\Omega}_i$ onto $[-1, 1]^N$ s. t. $\Phi_i(\Omega_i \cap \Omega) = (-1, 1)^{N-1} \times (0, 1)$, $\Phi_i(\Omega_i \cap (\mathbb{R}^N \setminus \overline{\Omega})) = (-1, 1)^{N-1} \times (-1, 0)$,

$\Phi_i(\Omega_i \cap \partial\Omega) = (-1,1)^{N-1} \times \{0\}$. Let, for some $i \geq 1$, $u_i \in W^{k,p}(\Omega_i \cap \Omega)$ be compactly supported in $\overline{\Omega} \cap \Omega_i$. Let $v_i = u_i \circ \Phi_i^{-1}$ and consider $Pv_i$. In view of Exercise 1.53, we have $w_i = (Pv_i) \circ \Phi_i \in W^{k,p}(\mathbb{R}^N)$, and $w_i$ is an extension of $u_i$.

**Step 3.** Construction of the global extension

Consider a partition of unity $(\zeta_i)_{i=0,\ldots,m}$ subordinated to the covering $\Omega_0, \ldots, \Omega_m$. Let $u_i = \zeta_i u$. Then (with the notations of Step 2) $Pu = u_0 + \sum_{j=1}^{m} w_i$ has all the required properties. □

On the way, we proved the following

**1.59 Corollary.** *Assume $\Omega$ bounded and of class $C^k$. Let $1 \leq p < \infty$. Then $C^k(\overline{\Omega})$ is dense in $W^{k,p}(\Omega)$.*

**1.60 Corollary.** *With $\Omega$, $k$, $p$ as above, let $U$ be an open neighbourhood of $\overline{\Omega}$. Then we may choose $P$ s. t.* supp $Pu \subset U$.

*Proof.* Choose, in the proof of Theorem 1.54, $\Omega_i \Subset U$. □

**1.61 Remark.** In what follows, "smooth" will be a loose notion adapted to the statements, or rather to their proofs. E. g., consider the following statement: "Let $\Omega$ be smooth and bounded. Then $W^{1,1}(\Omega) \hookrightarrow L^{N/(N-1)}(\Omega)$." The proof goes as follows: we prove that $Pu \in L^{N/(N-1)}(\mathbb{R}^N)$, and next we restrict $Pu$ to $\Omega$. In order to work, the proof needs the extension theorem to apply. In this special case, smooth $= C^1$. We will not insist on the smoothness requirements; a dumb's rule is that $C^\infty$ will always suffice.

## 1.4 Inequalities, embeddings

### 1.4.1 Continuous embeddings

**1.62 Theorem** (Morrey)**.** *Assume that $N < p < \infty$. Let $\Omega$ be smooth and bounded (or $\mathbb{R}^N$, or a half space). Then $W^{1,p}(\Omega) \hookrightarrow C^\alpha(\overline{\Omega})$. Here, $\alpha = 1 - N/p \in (0,1)$.*

**1.63 Definition.** We will use the shorthand *standard domain* for $\Omega$ which is either smooth and bounded, or $\mathbb{R}^N$, or a half space.

*Proof.* It suffices to consider the case where $\Omega = \mathbb{R}^N$ and $u \in C_c^\infty(\mathbb{R}^N)$. Let $B$ a ball of radius $r$ containing the origin. Since, for $x \in \mathbb{R}^N$, we have $|u(x) - u(0)| \leq |x| \int_0^1 |\nabla u(tx)|\,dt$, we find that

$$\left| \fint_B u - u(0) \right| \leq Cr^{1-N} \int_0^1 \int_B |\nabla u(tx)|\,dxdt = Cr^{1-N} \int_0^1 \int_{tB} |\nabla u(y)| t^{-N}\,dydt$$
$$\leq Cr^{1-N+N/p'} \int_0^1 t^{N/p'-N} \|\nabla u\|_{L^p(tB)}\,dt \leq Cr^\alpha \|\nabla u\|_{L^p}.$$

Of course, the same holds if 0 is replaced by any other point. Let now $x, y \in \mathbb{R}^N$. Pick a a ball of radius $|x - y|$ containing both $x$ and $y$. We find that $|u(x) - u(y)| \leq C|x - y|^\alpha \|\nabla u\|_{L^p}$.

On the other hand, by taking $r = \dfrac{\|u\|_{L^p}}{\|\nabla u\|_{L^p}}$, we find that

$$|u(x)| \leq \left| \fint_B u \right| + Cr^\alpha \|\nabla u\|_{L^p} \leq C\|u\|_{L^p}^{1-N/p} \|\nabla u\|_{L^p}^{N/p}.$$

□

**1.64 Remark.** This embedding is optimal in the sense that the exponent $\alpha$ cannot be improved. Optimality, here and in the next theorems, is obtained via a *scaling argument.* Here it is how it works: assume that $W^{1,p} \subset C^\alpha$. Then this inclusion is continuous, by the closed graph theorem. Thus $|u(x) - u(y)| \leq C|x - y|^\alpha (\|\nabla u\|_{L^p} + \|u\|_{L^p})$, $\forall\ u \in W^{1,p}(\mathbb{R}^N)$. Fix now some $u$ and apply this estimate to $u(\lambda\cdot)$, $\lambda > 0$. It follows that

$$|u(x) - u(y)| \leq C|x - y|^\alpha (\lambda^{1-N/p-\alpha} \|\nabla u\|_{L^p} + \lambda^{-N/p-\alpha} \|u\|_{L^p}).$$

If $u$ is non constant and we take $x, y$ s. t. $u(x) \neq u(y)$, we find, by letting $\lambda \to \infty$, that $\alpha \leq 1 - N/p$.

**1.65 Remark.** Let us take a closer look to the estimates we obtained in $\mathbb{R}^N$. The higher order term in the $C^\alpha$ norm, namely $\sup\limits_{x \neq y} \dfrac{|u(x) - u(y)|}{|x - y|^\alpha}$, is estimated only by the higher order term in the $W^{1,p}$ norm, namely $\|\nabla u\|_{L^p}$. The lower term, $\|u\|_{L^\infty}$, is estimated by a portion of each term in the $W^{1,p}$ norm. This is typical for all the embeddings of this kind, and can be guessed by the scaling argument.

**1.66 Theorem.** *[Gagliardo-Nirenberg] Let $\Omega$ be a standard domain. Then $W^{1,1}(\Omega) \hookrightarrow L^{N/(N-1)}$ (with the convention $1/0 = \infty$).*

*Proof.* Assume $\Omega = \mathbb{R}^N$ and $u \in C_c^\infty(\mathbb{R}^N)$. For $j = 1, \ldots, N$, we have

$$\begin{aligned}|u(x)| &= \left| \int_{-\infty}^{x_j} \frac{\partial u}{\partial x_j}(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_N)\, dt \right| \\ &\le F_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \\ &:= \int_{\mathbb{R}} \left| \frac{\partial u}{\partial x_j}(x_1, \ldots, x_{j-1}, t, x_{j+1}, \ldots, x_N) \right| dt.\end{aligned}$$

We find that

$$|u(x)| \le G(x) := \left( \prod_j F_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N) \right)^{1/N}. \tag{1.3}$$

Noting that $\|F_j\|_{L^1} \le \|\nabla u\|_{L^1}$, we conclude via the next result. □

**1.67 Lemma.** *Let $F_1, \ldots, F_N \in L^1(\mathbb{R}^{N-1})$, and define $G$ as in* (1.3)*. Then $G \in L^{N/(N-1)}$ and $\|G\|_{L^{N/(N-1)}} \le \prod_j \|F_j\|_{L^1}^{1/N}$.*

*Proof.* The cases $N = 1, 2$ are trivial. Assuming that the lemma holds for $N$, we proceed as follows: let $\hat{x}_j := (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{N+1})$. By the induction hypothesis, we have, for fixed $x_{N+1}$

$$\int F_1(\hat{x}_1)^{1/(N-1)} \ldots F_N(\hat{x}_N)^{1/(N-1)}\, dx_1 \ldots dx_N \le \prod_{j=1}^{N} \|F_j(\hat{x}_j)\|_{L^1}^{1/(N-1)}.$$

On the other hand, Hölder's inequality implies that, for fixed $x_{N+1}$, we have

$$\begin{aligned}\int G(x)^{(N+1)/N}\, dx_1 \ldots dx_N \le &\|F_{N+1}\|_{L^1}^{1/N} \times \\ &\times \left( \int \prod_{j=1}^{N} F_j(\hat{x}_j)^{1/(N-1)}\, dx_1 \ldots dx_N \right)^{(N-1)/N}.\end{aligned}$$

Using again Hölder's inequality, we find that

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} G^{N+1/N} &\le \|F_{N+1}\|_{L^1}^{1/N} \int_{\mathbb{R}} \|F_1(\hat{x}_1)\|_{L^1}^{1/N} \ldots \|F_N(\hat{x}_N)\|_{L^1}^{1/N}\, dx_{N+1} \\ &\le \|F_{N+1}\|_{L^1}^{1/N} \|F_1\|_{L^1}^{1/N} \ldots \|F_N\|_{L^1}^{1/N};\end{aligned}$$

this is equivalent to the statement of the lemma. □

**1.68 Theorem.** *[Sobolev] Assume that* $1 < p < N$*. Let* $\Omega$ *be a standard domain. Then* $W^{1,p}(\Omega) \hookrightarrow L^{Np/(N-p)}(\Omega)$*.*

*Proof.* We start by noting that the conclusion of Theorem 1.66 holds for $u \in C_c^1$. Let $-\Delta > 1$ to be fixed later and let $u \in C_c^\infty(\mathbb{R}^N)$. Then $|u|^{-}\Delta \in C_c^1$, so that

$$\|u\|^{-}_{L^{-\Delta N/(N-1)}}\Delta = \||u|^{-}\Delta\|_{L^{N/(N-1)}} \le C\|\nabla |u|^{-}\Delta\|_{L^1} \le C\|\nabla u\|_{L^p}\|u\|^{-\Delta-1}_{L^{(-\Delta-1)p'}}. \tag{1.4}$$

If we take $-\Delta$ s. t. $-\Delta N/(N-1) = (-\Delta - 1)p'$, then we obtain that $\|u\|_{L^{Np/(N-p)}} \le C\|\nabla u\|_{L^p}$, $\forall\ u \in C_c^\infty(\mathbb{R}^N)$. We conclude as usual. □

**1.69 Definition.** For $1 \le p < N$, one usually denotes by $p^*$ the exponent $p^* = \dfrac{Np}{N-p}$.

**1.70 Theorem.** *Assume that* $N \ge 2$ *and let* $\Omega$ *be a standard domain. Then* $W^{1,N}(\Omega) \not\hookrightarrow L^\infty(\Omega)$*. However, we have* $W^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$*,* $p \le q < \infty$*.*

*Proof.* Assume, e. g., that $\Omega = B(0,1)$. Let $u(x) = |\ln|x||^\alpha$. Here, $0 < \alpha < 1 - 1/N$. Then $|\nabla_d u| \sim |x|^{-1}|\ln|x||^{\alpha-1}$, by Proposition 1.17. Our choice of $\alpha$ implies that $u \notin L^\infty$ and $u \in W^{1,N}$.

Using (1.4) with $p = N$ and $-\Delta = N$ (this choice of $-\Delta$ is made in order to have $(-\Delta - 1)p' = N$), we find that

$$\|u\|_{L^{N^2/(N-1)}} \le C_0\|\nabla u\|^{1/N}_{L^N}\|u\|^{(N-1)/N}_{L^N}. \tag{1.5}$$

We next use again (1.4), but this time we take $-\Delta$ s. t. $(-\Delta - 1)p' = N^2/(N-1)$, i. e., $-\Delta = N + 1$. With the help of (1.5), we find that $\|u\|_{L^{(N+1)N/(N-1)}} \le C_1\|\nabla u\|^{\alpha_1}_{L^N}\|u\|^{1-\alpha_1}_{L^N}$ for some $\alpha_1 \in (0,1)$. We continue with $-\Delta = N + k$, $k = 2, 3, \ldots$, and find by induction that $\|u\|_{L^{(N+k)N/(N-1)}} \le C_k\|\nabla u\|^{\alpha_k}_{L^N}\|u\|^{1-\alpha_k}_{L^N}$ for some $\alpha_k \in (0,1)$. Therefore, $\|u\|_{L^{(N+k)N/(N-1)}} \le C_k\|u\|_{W^{1,N}}$. Let now $N \le q < \infty$. Then there is some large $k$ s. t. $N \le q < (N+k)N/(N-1)$. If $\theta \in [0,1)$ is s. t. $\dfrac{1}{q} = \dfrac{\theta}{N} + \dfrac{1-\theta}{(N+k)N/(N-1)}$, then, by Hölder's inequality, we have

$$\|u\|_{L^q} \le \|u\|^\theta_{L^N}\|u\|^{1-\theta}_{L^{(N+k)N/(N-1)}} \le C_q\|u\|_{W^{1,N}}.$$

□

This basic embeddings give birth to many others, which are obtained by combining the above theorems. Rather then giving a long and uninformative list, let us rather give some examples.

**1.71 Example.** Let $\Omega$ be a standard domain in $\mathbb{R}^3$. Then $W^{2,4}(\Omega) \hookrightarrow C^{1,1/4}(\overline{\Omega})$.

Indeed, if $u \in W^{2,4}$, then $\nabla u \in W^{1,4}$. It follows that $\nabla u \in C^{1/4}(\overline{\Omega})$, so that $u \in C^{1,1/4}(\overline{\Omega})$.

**1.72 Proposition.** *If $kp = N$ and $\Omega$ is a standard domain, then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for $p \le q < \infty$.*

*In the exceptional case $k = N$, $p = 1$, we also have $W^{N,1}(\Omega) \hookrightarrow C(\overline{\Omega})$.*

*However, the embedding $W^{k,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ does not hold when $k = 1, 2, \ldots, N-1$ and $kp = N$.*

*Proof.* Counterexamples to the embedding $W^{k,p}(\Omega) \hookrightarrow L^\infty(\Omega)$ are obtained by considering, as in the proof of Theorem 1.70, maps of the form $|x|^\alpha|\ln|x||^\beta$ (with appropriate $\alpha$ and $\beta$).

The validity of the embeddings $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ is established by induction on $k$ (one has to apply Theorem 1.66 or 1.68 to $D^{k-1}u$).

Finally, the embedding $W^{N,1} \hookrightarrow C(\overline{\Omega})$ is obtained, in the model case $\Omega = \mathbb{R}^N$, from the identity

$$u(x) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_N} \frac{\partial^N}{\partial_1 \ldots \partial_N} u(t_1, \ldots, t_N)\, dt_N \ldots dt_1,$$

valid for $u \in C_c^\infty(\mathbb{R}^N)$. $\square$

### 1.4.2 Compact embeddings

We start by noting that, once the embeddings in the preceding section established, they imply yet another family of embeddings, obtained via Hölder's inequality. For example: if $N \ge 2$, then $W^{1,1} \hookrightarrow L^{N/(N-1)} \cap L^1$, so that $W^{1,1} \hookrightarrow L^p$, $1 \le p < N/(N-1)$. We will call such an embedding *suboptimal.* Another example: in 2D, $W^{1,4} \hookrightarrow C^{1/3}$. By contrast, the estimates established in the previous section will be referred as *optimal.*

**1.73 Exercise.** An optimal embedding cannot be compact.
Hint: consider $\varphi(n\cdot)$ for a fixed $\varphi$.

**1.74 Exercise.** Assume that $\Omega = \mathbb{R}^N$ (or $\mathbb{R}^N_+$). Then an embedding (optimal or not) cannot be compact.
Hint: consider $\varphi(ne_1 + \cdot)$ for a fixed $\varphi$.

**1.75 Theorem** (Rellich-Kondratchov)**.** *Assume that $\Omega$ is smooth and bounded. Then suboptimal embeddings are compact.*

*Proof.* We will not prove this theorem in all cases (the list is too long). However, we consider the two cases which lead to the general one. These cases are:

**Case 1.** Assume that $p > N$. Let $0 < \beta < \alpha = 1 - N/p$. Then $W^{1,p} \hookrightarrow C^\beta$ is compact

**Case 2.** Assume that $1 \le p < N$. Let $1 \le q < p^* = Np/(N-p)$. Then $W^{1,p} \hookrightarrow L^q$ is compact

Before starting, let us note that we may consider only functions supported in a fixed compact (this follows by the properties of the extension operator associated to a bounded domain).

**Case 1.** In this case, we simply rely on $W^{1,p} \hookrightarrow C^\alpha$ combined with

**1.76 Lemma.** *Let $U$ be bounded. Let $0 < \beta < \alpha < 1$. Then $C^\alpha(\overline{\Omega}) \hookrightarrow C^\beta(\overline{\Omega})$ is compact.*

*Proof.* Let $(u_n)$ be a bounded sequence in $C^\alpha(\overline{\Omega})$. By Arzelà-Ascoli, up to a subsequence we have $u_n \to u$ uniformly, for some $u \in C^\alpha(\overline{\Omega})$. We prove that $u_n \to u$ in $C^\beta(\overline{\Omega})$. We may assume that $u = 0$. Then $\|u_n\|_{L^\infty} \to 0$. On the other hand

$$|u_n(x) - u_n(y)| \le \max\left\{\frac{2\|u_n\|_{L^\infty}}{|x-y|^\beta}|x-y|^\beta, \|u_n\|_{C^\alpha}|x-y|^{\alpha-\beta}|x-y|^\beta\right\}.$$

Thus

$$\|u_n\|_{C^\beta} \le o(1) + C\max_{x\neq y}\left\{\frac{\|u_n\|_{L^\infty}}{|x-y|^\beta}, \|u_n\|_{C^\alpha}|x-y|^{\alpha-\beta}\right\} \to 0 \text{ as } n \to \infty.$$

□

**Case 2.** This case relies on the following

**1.77 Exercise.** Let $X$ be a complete metric space. Let $(x_n) \subset X$. Assume that: for each $\varepsilon > 0$, there is a sequence $(y_n) \subset X$ s. t. $d(x_n, y_n) \le \varepsilon$ and $(y_n)$ is relatively compact.

Then the sequence $(x_n)$ is relatively compact.

**1.78 Exercise.** We have $\|u * \rho_\varepsilon\|_{W^{k,p}} \le \|u\|_{W^{k,p}}$, $\forall\, \varepsilon > 0$ and $u \in C_c^\infty(\mathbb{R}^N)$.

We consider a bounded sequence $(u_n) \subset W^{1,p}(U)$ s. t. supp $u_n \subset K$, where $K \Subset U$. By the two preceding exercises, we may further assume that $u_n \in C_c^\infty(\mathbb{R}^N)$. Let, for fixed $\varepsilon > 0$, $v_n = u_n * \rho_\varepsilon$. Then $(v_n)$ is bounded in $C^\infty(\mathbb{R}^N)$, and thus relatively compact in $W^{1,p}(U)$, by Arzelà-Ascoli. It remains to establish a uniform bound $\|u_n - v_n\|_{L^q} \le f(\varepsilon)$, where $f(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We drop the subscript $n$. We start with

$$
\begin{aligned}
|v(x) - u(x)| &= \left| \int (u(x - \varepsilon y) - u(x))\rho(y)\, dy \right| \\
&\le \int |u(x - \varepsilon y) - u(x)|\rho(y)\, dy \\
&\le C \int_{B(0,1)} |u(x - \varepsilon y) - u(x)|\, dy \\
&\le C \int_{B(0,1)} \int_0^1 \varepsilon |\nabla u(x - t\varepsilon y)|\, dt dy.
\end{aligned} \tag{1.6}
$$

By integration, we find that

$$
\|v - u\|_{L^1} \le C\varepsilon \|\nabla u\|_{L^1}. \tag{1.7}
$$

Let now $1 \le q < p^*$. Let $\theta \in (0, 1]$ be s. t. $\dfrac{1}{q} = \dfrac{\theta}{p}1 + \dfrac{1-\theta}{p^*}$. Then

$$
\begin{aligned}
\|u - v\|_{L^q} &\le \|u - v\|_{L^1}^{\theta} \|u - v\|_{L^{p^*}}^{1-\theta} \le C\|u - v\|_{L^1}^{\theta} \|\nabla(u - v)\|_{L^p}^{1-\theta} \\
&\le C\varepsilon^{\theta} \|\nabla u\|_{L^1}^{\theta} \|\nabla u\|_{L^p}^{1-\theta} \le C\varepsilon^{\theta}.
\end{aligned}
$$

□

### 1.4.3 Equivalent norms

As a starter, we recall the following

**1.79 Exercise.** Let $u$ be a measurable function which is locally constant in the domain $\Omega$. Then $u$ is constant a. e.
Hint: consider the set $\{x \in \Omega; u = C \text{ a. e. in a neighbourhood of } x\}$.

**1.80 Proposition.** *Let $\Omega$ be a domain. Let $u \in \mathscr{D}'(\Omega)$ be s. t. $\nabla u = 0$. Then $u$ is constant.*

*Proof.* It suffices to prove that $u$ is locally constant. In particular, we may assume that $\Omega$ is a cube, say $(-2, 2)^N$, and prove that $u$ is constant in $(-1, 1)^N$. Fix $\psi \in C_c^\infty(-2, 2)$ s. t. $\psi = 1$ in a neighbourhood of $[-1, 1]$. We argue by induction (the case $N = 1$ is settled, *cf* Lemma 1.33). Define $v$ through the formula

$$
v(\eta) = u(\eta(x_1, \dots, x_{N-1})\psi(x_N)) = u(\eta \otimes \psi), \quad \forall \eta \in C_c^\infty((-1, 1)^{N-1}).
$$

Clearly, $\nabla v = 0$, so that $v = C$. Let now $\varphi \in C_c^\infty((-1,1)^N)$ and set $\eta(x_1, \dots, x_{N-1}) := \displaystyle\int \varphi(x)\, dx_1$. Then $\zeta := \varphi - \eta \otimes \psi \in C_c^\infty((-2,2)^N)$ and $\displaystyle\int \zeta\, dx_1 = 0$. It follows that $\zeta = \partial_1 - \Delta$ for some $-\Delta \in C_c^\infty((-2,2)^N$. Consequently,

$$u(\varphi) = u(\partial_1 - \Delta + \eta \otimes \psi) = v(\eta) = C(\eta) = C \int \varphi,$$

i. e., $u = C$. □

**1.81 Proposition** (Poincaré)**.** *Let $\Omega$ be smooth bounded connected. Then $u \mapsto \left| \displaystyle\int u \right| + \|\nabla u\|_{L^p}$ is an equivalent norm on $W^{1,p}$.*

*Proof.* By the preceding proposition, the above quantity is a norm. Denote it by [ ]. It is easy to see that $[u] \le C\|u\|_{W^{1,p}}$. Conversely, argue by contradiction: assume that there is a sequence $(u_n) \subset W^{1,p}$ s. t. $\left| \displaystyle\int u_n \right| + \|\nabla u_n\|_{L^p} \le 1/n$ and $\|u_n\|_{W^{1,p}} = 1$. Then, up to a subsequence, $u_n \to u$ in $L^p$, while $\nabla u_n \to 0$ in $L^p$. We find that $\nabla u = 0$, so that $u$ is a constant. On the other hand, we have $\displaystyle\int u_n \to 0$, so that $u = 0$. Therefore, $u_n \to 0$ in $W^{1,p}$. This contradicts the fact that $\|u_n\|_{W^{1,p}} = 1$. □

If we apply the above proposition to $u - \displaystyle\fint u$, we find the usual form of Poincaré's inequality

**1.82 Corollary** (Poincaré)**.** *Assume that $\Omega$ is smooth and bounded and let $1 \le p < \infty$. Then*

$$\|u - \fint u\|_{L^p(\Omega)} \le C\|\nabla u\|_{L^p(\Omega)}. \tag{1.8}$$

**1.83 Lemma.** *Let $\Omega$ be a domain. Let $u \in \mathscr{D}'(\Omega)$ be s. t. $D^k u = 0$. Then $u$ is a polynomial of degree $\le k-1$.*

*Proof.* We know this for $k = 0$. We argue by induction. Since $D^{k-1}\partial_j u = 0$, there is some polynomial $P_j$ of degree $\le k-2$ s. t. $\partial_j u = P_j$. Since $\partial_k \partial_j u = \partial_j \partial_k u$, we find that $\partial_k P_j = \partial_j P_k$. By the Poincaré lemma, in each ball $B \subset \Omega$ there is some polynomial $P$ (possibly depending on $B$) of degree $\le k-1$ s. t. $\partial_j P = P_j$ in $B$, $j = 1, \dots, N$. By the uniqueness principle for analytic maps, this $P$ does not depend on $B$.

Finally, we have $\nabla(u - P) = 0$, so that $u = P + C$. □

**1.84 Proposition.** *Let $\Omega$ be smooth and bounded. Let $L : L^p \to \mathbb{R}^M$ be a linear continuous functional s. t. Ker $L$ does not contain any non zero polynomial of degree $\leq k-1$. Then $u \mapsto |Lu| + \|D^k u\|_{L^p}$ is an equivalent norm in $W^{k,p}$.*

*Proof.* The proof is similar to the one of the Proposition 1.81. The more difficult part: we argue by contradiction and obtain, via the compact embedding $W^{k,p} \hookrightarrow W^{k-1,p}$, the existence of a $u \in W^{k,p}$ s. t. $\|u\|_{W^{k-1,p}} = 1$, $Lu = 0$ and $D^k u = 0$. Thus $u = 0$ (since $u$ is a polynomial of degree $\leq k-1$ and $u$ is in the kernel of $L$). This contradicts the fact that $\|u\|_{W^{k-1,p}} = 1$. $\square$

Similarly, we have the following result whose proof will be omitted.

**1.85 Proposition.** *Let $1 \leq p, q, r \leq \infty$, $0 < m < k$. Then $\|u^{(m)}\|_{L^q(0,1)} \leq C(\|u\|_{L^p(0,1)} + \|u^{(k)}\|_{L^r(0,1)})$.*

The next result is different in nature (since we are not in a compact embedding situation).

**1.86 Proposition.** *The norm $u \mapsto \|u\|_{L^p} + \|D^k u\|_{L^p}$ is an equivalent norm in $W^{k,p}(\mathbb{R}^N)$ (or in $\mathbb{R}^N_+$).*

*Proof.* This is a consequence of the following theorem (applied to $p = q = r$ and $0 = l < m < k$). $\square$

**1.87 Theorem** (Gagliardo-Nirenberg)**.** *Let $1 \leq p, q, r \leq \infty$, $0 \leq l \leq m \leq k$ be s. t. the following compatibility conditions*

$$m = \theta l + (1-\theta)k, \frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r} \quad (\textit{for some } \theta \in [0,1])$$

*are satisfied. Then, for $u : \mathbb{R}^N \to \mathbb{R}$, we have*

$$\|D^m u\|_{L^q} \leq C\|D^l u\|_{L^p}^{\theta} \|D^k u\|_{L^r}^{1-\theta} \leq C(\|D^l u\|_{L^p} + \|D^k u\|_{L^r}).$$

*In particular, if $u \in \mathscr{D}'(\mathbb{R}^N)$ is s. t. $D^l u \in L^p$ and $D^k u \in L^r$, then $D^m u \in L^q$.*

**1.88 Remark.** Note that $\theta = \dfrac{k-m}{k-l}$, so that $q$ is determined by the other parameters. More specifically, the compatibility condition is equivalent to requiring that $q$ is given by $\dfrac{1}{q} = \dfrac{k-m}{k-l}\dfrac{1}{p} + \dfrac{m-l}{k-l}\dfrac{1}{r}$.

*Proof.* The statement will be reduced to a special case.

First reduction: assume that the theorem holds when $u$ is smooth. Then it holds for every $u$. To see this, it suffices to note that, when $u \in L^s$ (even for $s = \infty$!) we have $\|u * \rho_\varepsilon\|_{L^s} \to \|u\|_{L^s}$ as $\varepsilon \to 0$.

Second reduction: it suffices to know that the inequalities hold when $l = 0$. (Then apply the inequality not to $u$, but to $D^l u$.)

Third reduction: it suffices to prove the result when $l = 0$, $k = 2$. The general case is obtained by induction on $k - l$. (See Exercise 1.91.)

Thus we take $l = 0$, $m = 1$, $k = 2$.

Fourth reduction, the most important one: it suffices to consider the case $N = 1$. Indeed, assuming the case $N = 1$ settled, we estimate $\partial_j u$, e. g., when $j = N$. Write $\mathbb{R}^N \ni x = (x', x_N)$. Then $\|\partial_N u(x', \cdot)\|_{L^q} \leq C\|u(x', \cdot)\|_{L^p}^{1/2} \|\partial_N^2 u(x', \cdot)\|_{L^r}^{1/2}$. If we integrate this inequality w. r. t. $x'$ and use Hölder's inequality with exponents $\dfrac{2p}{q}$ and $\dfrac{2p}{r}$, then we find the desired estimate. The key fact is that the above exponents are conjugate to each other (check!).

We have thus reduced the theorem to the following special case: if $1 \leq p, q, r \leq \infty$ and $\dfrac{1}{q} = \dfrac{1}{2p} + \dfrac{1}{2r}$, and if $u \in C^\infty(\mathbb{R})$, then $\|u'\|_{L^q} \leq C\|u\|_{L^p}^{1/2} \|u''\|_{L^r}^{1/2}$ (this is the one dimensional case). Yet another reduction: it suffices to consider the case where $\mathbb{R}$ is replaced by $\mathbb{R}_+$. We may also assume that $\|u''\|_{L^r} = 1$. Finally, we present the argument when $p, r$ (and thus $q$) are finite. The adaptation to the remaining cases is straightforward.

By the Proposition 1.85 (with $k = 2$, $m = 1$), we have $\|v^{'}\|_{L^q(0,1)} \leq C(\|v\|_{L^p(0,1)} + \|v^{''}\|_{L^r(0,1)})$ for every $v \in C^\infty([0, 1])$. By scaling (i. e., applying this to $u(x + \ell \cdot)$), we find that, for each interval $I$ of length $\ell$, we have

$$\|u^{'}\|_{L^q(I)} \leq C(\ell^{-\alpha}\|u\|_{L^p(I)} + \ell^\alpha \|u^{''}\|_{L^r(I)}) := C(A(\ell) + B(\ell)). \tag{1.9}$$

Here, $\alpha := 1 - \dfrac{1}{2r} + \dfrac{1}{2p} > 0$.

Fix some $\varepsilon > 0$. If we take a look at the quantities involved in (1.9), we see that, when $\ell \to \infty$, we have $A(\ell) \to 0$ and $B(\ell) \to \infty$. Thus $B(\ell) > A(\ell)$ for large $\ell$. We define a first $\ell$, say $\ell_1$, as follows: if $A(\varepsilon) < B(\varepsilon)$, then we take $\ell_1 = \varepsilon$ and $I_1 = (0, \varepsilon)$; we say that this interval is of type I. Otherwise, pick the first $\ell_1 > \varepsilon$ s. t. $A(\ell_1) = B(\ell_1)$. In this case, we take $I_1 = (0, \ell_1)$; this is an interval of type II. Then start again, but at $\ell_1$, not at the origin; call the new intervals $I_2, \ldots$.

Fix next another number, say $L > 0$. We stop the construction of intervals when these intervals cover $(0, L)$. This is achieved after a finite number of steps (since each interval is of length $\geq \varepsilon$). Let $(0, L) \subset I_1 \cup \ldots \cup I_m$. Note the following: if $I_i$ is type I, then $\|u'\|_{L^q(I_i)} \le C\varepsilon^\alpha \|u''\|_{L^r(I_i)}$. On the other hand, if $I_i$ is of type II, then $\|u'\|_{L^q(I_i)} \le C\|u\|^{1/2}_{L^p(I_i)}\|u''\|^{1/2}_{L^r(I_i)}$. We find that

$$\|u'\|^q_{L^q(0,L)} \le C \sum_{\text{type I}} \varepsilon^{\alpha q}\|u''\|^q_{L^r(I_i)} + C \sum_{\text{type II}} \|u\|^{q/2}_{L^p(I_i)}\|u''\|^{q/2}_{L^r(I_i)}.$$

By Hölder's inequality with conjugate exponents $\dfrac{2p}{q}$ and $\dfrac{2p}{r}$, we find that the second sum is at most $C\|u\|^{q/2}_{L^p}\|u''\|^{q/2}_{L^r}$.

In order to estimate the first sum, we consider two cases:

a) if $r > 1$, then $\alpha q > 1$ (check!) and we estimate the sum with $C(L/\varepsilon+1)\varepsilon^{\alpha q}$. (Here, we use the fact that we have at most $L/\varepsilon + 1$ intervals)

b) if $r = 1$, then we estimate the sum with $C\varepsilon^{\alpha q}\sum \|u''\|_{L^1} \le C\varepsilon^{\alpha q}$.

If, in these estimates, we let first $\varepsilon \to 0$, next $L \to \infty$, then we obtain the desired inequality. $\square$

**1.89 Exercise.** There is a flaw in the above proof: it may happen that $B(\ell) \equiv 0$. Prove that, in this case, $u' = 0$ in $(0, \infty)$, and conclude.

**1.90 Exercise.** Prove the theorem in the remaining cases (where one or both of $p, r$ are infinite).

**1.91 Exercise.** The purpose of this exercise is to explain how to prove the Gagliardo-Nirenberg for arbitrary $k, l, m$. As explained in the proof of Theorem 1.87, we may assume that $N = 1$, $l = 0$, and $k \ge 3$.

a) Prove that the inequalities hold under the additional assumption that $u \in C^\infty_c$.
Hint: let, for $0 \le j \le k$, $q_j$ be defined by the formula $\dfrac{1}{q_j} = \dfrac{1 - j/m}{p} + \dfrac{j/m}{r}$.
Start from $\|u^{(j)}\|_{L^{q_j}} \le C\|u^{(j-1)}\|^{1/2}_{L^{q_{j-1}}}\|u^{(j+1)}\|^{1/2}_{L^{q_{j+1}}}$, $j = 1, \dots, k-1$, and proceed by induction on $k$

b) Let now $u$ be arbitrary. Set $u_\varepsilon = u * \rho_\varepsilon$. If $u \in L^p$ and $u^{(k)} \in L^r$, prove that $u_\varepsilon^{(j)} \in L^p \cap L^\infty$, $\forall\, j \in \mathbb{N}$. Prove also that, in the special case where $r = 1$, we have in addition that $\lim\limits_{|x|\to\infty} u_\varepsilon^{(k-1)}(x) = 0$

c) Let $\varphi \in C^\infty_c(\mathbb{R})$ be s. t. $0 \le \varphi \le 1$ and $\varphi(0) = 1$. By applying the Gagliardo-Nirenberg inequalities (compactly supported case) to $x \mapsto u_\varepsilon(x)\varphi(\delta x)$, prove that these inequalities hold for $u_\varepsilon$.
Hint: consider separately the exceptional cases where $r = 1$ or $q = 1$

d) Conclude.

**1.92 Definition.** We let, for $1 \le p < \infty$, $W_0^{1,p}(\Omega) := \overline{C_c^\infty(\Omega)}^{W^{1,p}}$. When $p = 2$, we write $H_0^1(\Omega)$ rather than $W_0^{1,p}(\Omega)$.

**1.93 Theorem** (Poincaré)**.** *Let $\Omega$ be bounded in one direction. Then $u \mapsto \|\nabla u\|_{L^p}$ is an equivalent norm on $W_0^{1,p}(\Omega)$.*

*Proof.* The hypothesis is that there is some unit vector $v$ and some finite number $\ell > 0$ s. t. for each $w \in v^\perp$, the set $\{t \in \mathbb{R}; w + tv \in \Omega\}$ is contained in an interval of length $\ell$. Since the statement we want to prove is invariant by isometries (check!), we may assume that $v = e_N$. Fix $x' \in \mathbb{R}^{N-1}$ and $u \in C_c^\infty(\Omega)$. The support of $u(x', \cdot)$ is contained in some $(a, b)$, with $b - a \le \ell$. Since $u(x', x_N) = \displaystyle\int_a^{x_N} \partial_N u(x', t)\, dt$, we find that $\|u(x', \cdot)\|_{L^p} \le C\|\partial_N u(x', \cdot)\|_{L^p}$, with $C$ depending only on $\ell$ and $p$. By integration, we find that $\|u\|_{L^p} \le C\|\nabla u\|_{L^p}$. By density, we find that the preceding estimate holds in $W_0^{1,p}(\Omega)$. $\square$

**1.94 Exercise.** Use the above argument combined with the Rellich-Kondratchov theorem in order to prove the following: Assume that $\Omega$ is smooth and bounded. Let $k \in \mathbb{N}$, $k \ge 2$. Then $u \mapsto \|\nabla u\|_{L^p} + \|D^k u\|_{L^p}$ is an equivalent norm in $W^{k,p} \cap W_0^{1,p}(\Omega)$.

## 1.5 Traces

We discuss here the properties of the "restrictions" of Sobolev maps to hyper surfaces, e. g., to the boundary of a smooth domain. Note that giving a meaning to the value of a Sobolev map on a hypersurface requires some thought, since *a priori* such maps are only defined a. e.

In what follows, we will state the results for general domains, but content ourselves to prove the results we state in the model case where $\Omega = \mathbb{R}_+^N$; we already explained how to obtain the case of a standard domain from the model case. We identify $\partial\mathbb{R}_+^N$ with $H = R^{N-1}$.

We start with the following

**1.95 Proposition.** *Let $\Omega$ be a standard domain and let $\Sigma := \partial\Omega$. Then the map $u \mapsto u_{|\Sigma}$, initially defined from $C^\infty(\overline{\Omega})$ into $C^\infty(\Sigma)$, extends uniquely by density to a linear map (called* trace map*) $u \mapsto \mathrm{tr}\ u$ from $W^{1,p}(\Omega)$ into $L^p(\Sigma)$, for $1 \le p < \infty$.*

**1.96 Remark.** It is easy to see that, when $\Omega$ is standard, we have tr $W^{1,\infty}(\Omega) = Lip(\Sigma)$.

*Proof.* It suffices to consider the case where $\Omega = \mathbb{R}^N_+$ and $u \in C^\infty_c(\overline{\mathbb{R}^N_+})$. Fix a function $\varphi \in C^\infty_c(\mathbb{R})$ s. t. $\varphi(0) = 1$ and supp $\varphi \subset (-1, 1)$. If $u \in C^\infty_c(\overline{\mathbb{R}^N_+})$, then $v = u\varphi(x_N) \in C^\infty_c(\overline{\mathbb{R}^N_+})$ and $u_{|H} = v_{|H}$. In addition, it is clear that $\|v\|_{W^{1,p}} \le C\|u\|_{W^{1,p}}$. It therefore suffices to prove that $\|v_{|H}\|_{L^p} \le C\|v\|_{W^{1,p}}$. This follows from

$$\int_H |v(x',0)|^p dx' = \int_H \left| \int_0^1 \partial_N v(x',t)dt \right|^p dx' \le \int_{H\times(0,1)} |Dv|^p \le \|Dv\|^p_{L^p}.$$

$\square$

When $1 < p < \infty$, the above proposition is not sharp, in the following sense: if $f$ is an arbitrary map in $L^p(\mathbb{R}^{N-1})$, we cannot always find a map $u \in W^{1,p}$ s. t. tr $u = f$. In other words, the trace map is not onto between the spaces we consider.
Our next task is to determine the image of the trace map.

**1.97 Definition.** For $0 < s < 1$ and $1 \le p < \infty$, we define

$$W^{s,p} = W^{s,p}(\mathbb{R}^N) = \{f \in L^p(\mathbb{R}^N)\ ;\ \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}} dx\, dy < \infty\},$$

equipped with the norm

$$\|f\|_{W^{s,p}} = \|f\|_{L^p} + \left( \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}} dx\, dy \right)^{1/p}.$$

We let the reader check that $W^{s,p}$ is a Banach space.

The main result of this section states that tr $W^{1,p}(\mathbb{R}^N_+) = W^{1-1/p,p}(\mathbb{R}^{N-1})$ (and a similar result holds when $\Omega$ is standard). We start with some preliminary results.

**1.98 Lemma.** *$C^\infty(\mathbb{R}^N) \cap W^{s,p}(\mathbb{R}^N)$ is dense into $W^{s,p}(\mathbb{R}^N)$ for $0 < s < 1$ and $1 \le p < \infty$.*

*Proof.* Let $\rho$ be a standard mollifier. We will prove that, if $u \in W^{s,p}$, then $u_\varepsilon = u * \rho_\varepsilon \to u$ in $W^{s,p}$ as $\varepsilon \to 0$. Clearly, $u_\varepsilon \to u$ in $L^p$. It remains to prove that, with $v_\varepsilon = u_\varepsilon - u$, we have

$$I_\varepsilon = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_\varepsilon(x)-v_\varepsilon(y)|^p}{|x-y|^{N+sp}}\, dxdy = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_\varepsilon(x+h)-v_\varepsilon(x)|^p}{|h|^{N+sp}}\, dxdh \to 0.$$

Let $w(x,h) := \dfrac{u(x+h)-u(x)}{|h|^{N/p+s}}$, which belongs to $L^p(\mathbb{R}^{2N})$. Let $\tau_{\varepsilon,z}w(x,h) := w(x-\varepsilon z, h)$, $\varepsilon > 0$, $|z| \le 1$. Then $\|\tau_{\varepsilon,z}w - w\|_{L^p} \to 0$ as $\varepsilon > 0$, uniformly in $z$. We have

$$\frac{|v_\varepsilon(x+h)-v_\varepsilon(x)|^p}{|h|^{N+sp}} = \frac{\left|\displaystyle\int \rho(z)[u(x-\varepsilon z+h)-u(x-\varepsilon z)-u(x+h)-u(x)]\,dz\right|^p}{|h|^{N+sp}}$$
$$\le C\int_{|z|\le 1} |\tau_{\varepsilon,z}w(x,h)-w(x,h)|^p\,dz.$$

We find that

$$I \le C \sup_{|z|\le 1} \|\tau_{\varepsilon,z}w - w\|_{L^p}^p \to 0 \quad \text{as } \varepsilon \to 0.$$

□

**1.99 Lemma.** *If* $u \in C(\overline{\mathbb{R}^N_+}) \cap W^{1,p}$, *then* $\operatorname{tr} u = u_{|H}$.

*Proof.* Extend $u$ to $\mathbb{R}^N$ by symmetry across $\mathbb{R}^{N-1}$. This extension, still denoted $u$, is continuous and in $W^{1,p}$ (check!).

Let $\rho$ be a standard mollifier and set $u_\varepsilon = \rho(\varepsilon\cdot)(u * \rho_\varepsilon)$. Clearly, $u_\varepsilon \in C_c^\infty$ and $u_\varepsilon \to u$ in $W^{1,p}$. Thus $u_{\varepsilon|H} = \operatorname{tr} u_\varepsilon \to \operatorname{tr} u$ in $L^p$ (and thus in $\mathcal{D}'$). On the other hand, $u_{\varepsilon|H}$ converges to $u_{|H}$ uniformly on compacts (and thus in $\mathcal{D}'$), whence the conclusion. □

The same argument leads to the following variant

**1.100 Lemma.** *Let $\Omega$ be a standard domain. Assume that $u \in W^{1,p}(\Omega)$ is continuous in an open neighbourhood $U$ of $x_0 \in \Sigma$. Then* $\operatorname{tr} u = u_{|\Sigma}$ *in* $U \cap \Sigma$.

In the statement of the above lemma, we use the following

**1.101 Definition.** Assume that $\Omega$ is smooth and bounded. Then

$$W^{s,p}(\Sigma) = \left\{ u \in L^p(\Sigma); \iint_{\Sigma\times\Sigma} \frac{|u(x)-u(y)|^p}{|x-y|^{N-1+sp}}\,dxdy < \infty \right\},$$

equipped with the natural norm.

### 1.5.1 Trace of $W^{1,p}$, $1 < p < \infty$

**1.102 Theorem** (Gagliardo). *Let $p \in (1, \infty)$ and let $\Omega$ be a standard domain of boundary $\Sigma$.*
*a) If $u \in W^{1,p}(\Omega)$, then $\mathrm{tr}\ u \in W^{1-1/p,p}(\Sigma)$ and $\|\mathrm{tr}\ u\|_{W^{1-1/p,p}} \leq C\|u\|_{W^{1,p}}$.*
*b) Conversely, let $f \in W^{1-1/p,p}(\Sigma)$. Then there is some $u \in W^{1,p}(\Omega)$ s. t. $\mathrm{tr}\ u = f$. In addition, we may pick $u$ s. t. $\|u\|_{W^{1,p}} \leq C\|\mathrm{tr}\ u\|_{W^{1-1/p,p}}$.*

**1.103 Remark.** Let $T : W^{1,p}(\Omega) \to W^{1-1/p,p}(\Sigma)$, $Tu = \mathrm{tr}\ u$. $T$ is linear, and the above theorem implies that $T$ is continuous and onto. Then the last conclusion in *b)* follows from the open mapping principle (each linear continuous map which is onto between two Banach spaces has a bounded right inverse). However, we will see during the proof a stronger conclusion: we will construct in *b)* a *linear* right inverse, i. e. , the map $f \mapsto u$ in *b)* will be linear.

*Proof.* We will consider the model case $\Omega = \mathbb{R}^N_+$.
*a)* By density, it suffices to prove that

$$\|u_{|H}\|_{W^{1-1/p,p}} \leq C\|u\|_{W^{1,p}} \quad \forall\ u \in C^\infty_c(\overline{\mathbb{R}^N_+}).$$

We start by noting that we already know that $\|u_{|H}\|_{L^p} \leq C\|u\|_{W^{1,p}}$; thus it suffices to establish, with $f(x') = u(x', 0)$, the inequality

$$I = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x'+h') - f(x')|^p}{|h'|^{N+p-2}}\, dh'dx' \leq C \int_{\mathbb{R}^N_+} |\nabla u(x)|^p\, dx. \quad (1.10)$$

The starting point is the inequality

$$|f(x'+h')-f(x')| \leq |f(x'+h')-u(x'+h'/2, |h'|/2)|+|f(x')-u(x'+h', |h'|/2)|,$$

which implies that $I \leq C(I_1 + I_2)$, where

$$I_1 = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x'+h') - u(x'+h'/2, |h'|/2)|^p}{|h'|^{N+p-2}}\, dx'dh',$$

$$I_2 = \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x') - u(x'+h'/2, |h'|/2)|^p}{|h'|^{N+p-2}}\, dx'dh'.$$

If we perform, in $I_1$, the change of variables $x' + h' = y'$, next we change $h'$ into $-h'$, we see that $I_1 = I_2$, and thus

$$I \leq C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x') - u(x'+h'/2, |h'|/2)|^p}{|h'|^{N+p-2}}\, dh'dx'.$$

Changing $h'$ into $2k'$ and applying the Leibniz-Newton formula, we find that

$$I \le C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \left( \int_0^{|k'|} |\nabla u(x' + t(k'/|k'|), t)| \right)^p |k'|^{-(N+p-2)} \, dk' dx'.$$

Expressing $k'$ in polar coordinates, we find that

$$I \le C \int_{\mathbb{R}^{N-1}} \int_{S^{N-2}} \int_0^\infty \left( \int_0^s |\nabla u(x' + t\omega, t)| dt \right)^p s^{-p} \, ds ds_\omega dx'.$$

Applying, for fixed $x'$ and $\omega$, Hardy's inequality in to the double integral in $s$ and $t$, we find that

$$I \le C \int_{\mathbb{R}^{N-1}} \int_{S^{N-2}} \int_0^\infty |\nabla u(x' + t\omega, t)|^p \, dt ds_\omega dx'.$$

Integrating, in the above inequality, first in $x'$, next in $\omega$, we find that

$$I \le C \int_{\mathbb{R}^{N-1}} \int_0^\infty |\nabla u(x', t)|^p dt \, dx' = \int_{\mathbb{R}^{N+}} |\nabla u(x)|^p \, dx \le C \|\nabla u\|_{L^p}^p.$$

*b)* It suffices to construct a linear map $f \mapsto u$, $f \in C^\infty(\mathbb{R}^{N-1}) \cap W^{1-1/p,p}$, $u \in W^{1,p}(\mathbb{R}^N_+)$, s. t. tr $u = f$ and $\|u\|_{W^{1,p}} \le C\|f\|_{W^{1-1/p,p}}$. We fix a standard mollifier $\rho$ in $\mathbb{R}^{N-1}$ and a function $\varphi \in C^\infty(\mathbb{R})$ s. t. $\varphi(0) = 1$, $0 \le \varphi \le 1$ and supp $\varphi \subset (-1, 1)$. We define, for $t > 0$, $v(x', t) = f * \rho_t(x')$ and $u(x', t) = v(x', t)\varphi(t)$. We extend $u$ to $\overline{\mathbb{R}^N_+}$ by setting $u(x', 0) = f(x')$. Clearly, the map $f \mapsto u$ is linear and $u \in C^\infty(\mathbb{R}^N \setminus H)$. In addition, $u \in C(\overline{\mathbb{R}^N_+})$ when $f$ is continuous. We also note that, for a fixed $t > 0$, Young's inequality implies that $\|f * \rho_t\|_{L^p} \le \|f\|_{L^p}$, and thus $\|u\|_{L^p} \le \|f\|_{L^p}$. It remains to prove that $\nabla u$ satisfies

$$\int_{\mathbb{R}^{N-1} \times \mathbb{R}_+} |\nabla u(x', t)|^p \, dx' dt \le C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x' + y') - f(x')|^p}{|y|^{N+p-2}} \, dy' dx' + C\|f\|_{L^p}^p.$$

For $1 \le j \le N - 1$, we have $|\partial_j u| \le |\partial_j v|$. On the other hand, $|\partial_N u| \le C|v|\chi_{\mathbb{R}^{N-1} \times [0,1)} + |\partial_N v|$. Since $\||v|\chi_{\mathbb{R}^{N-1} \times [0,1)}\|_{L^p} \le \|u\|_{L^p}$, it suffices to prove the estimate

$$\int_{\mathbb{R}^{N-1}} \int_0^\infty |\nabla v(x', t)|^p \, dt dx' \le C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x' + y') - f(x')|^p}{|y|^{N+p-2}} \, dy' dx'.$$

Let $1 \le j \le N - 1$. Since $\displaystyle\int \partial_j \rho = 0$, we have

$$\begin{aligned} \partial_j v(x', t) &= t^{-N} \int f(y')(\partial_j \rho)((x' - y')/t) \, dy' \\ &= t^{-N} \int [f(y') - f(x')](\partial_j \rho)((x' - y')/t) \, dy', \end{aligned}$$

so that

$$|\partial_j v(x',t)| \leq \frac{C}{t^N} \int_{B(0,t)} |f(x'+y') - f(x')|\, dy'.$$

We next claim that $\displaystyle\int \frac{d}{dt}[\rho_t(x')]dx' = 0$. This follows from the fact that $\displaystyle\int \rho_t \equiv 1$. Thus

$$|\partial_N v(x',t)| = |\int [f(y') - f(x')] \frac{d}{dt}[\rho_t(x'-y')]\, dy'| \leq \frac{C}{t^N} \int_{B(0,t)} |f(x'+y') - f(x')|\, dy',$$

since $\displaystyle\left|\frac{d}{dt}\rho_t\right| \leq Ct^{-N}$. We find that

$$|\nabla v(x',t)| \leq \frac{C}{t^N} \int_{B(0,t)} |f(x'+y') - f(x')|\, dy',$$

and therefore it suffices to establish the estimate

$$\begin{aligned} I &= \int_{\mathbb{R}^{N-1}} \int_0^\infty \left( \int_{B(0,t)} |f(x'+y') - f(x')|\, dy' \right)^p t^{-Np}\, dt dx' \\ &\leq C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x'+y') - f(x')|^p}{|y|^{N+p-2}}\, dy' dx'. \end{aligned}$$

This is done as in the proof of Lemma 1.98: Hölder's inequality applied to the integral over $B(0,t)$ implies that

$$I \leq C \int_{\mathbb{R}^{N-1}} \int_0^\infty \int_{B(0,t)} |f(x'+y') - f(x')|^p\, dy'\, t^{-N-p+1}\, dt dx'.$$

Fubini's theorem yields

$$\begin{aligned} I &\leq C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \int_{|y'|}^\infty t^{-N-p+1}\, dt dx' dy' \\ &= C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x'+y') - f(x')|^p}{|y|^{N+p-2}}\, dy' dx'. \end{aligned}$$

□

On the way, we established the following

**1.104 Corollary.** *Let $1 < p < \infty$. Let $f \in W^{1-1/p,p}(\mathbb{R}^{N-1})$ and set , for $t > 0$, $u(x',t) = f * \rho_t(x')\varphi(t)$. Then $u \in W^{1,p}(\mathbb{R}^N_+)$ and* tr $u = f$.

### 1.5.2 Trace of $W^{1,1}$

We start with some auxiliary results needed in the proof of the fact that the trace of $W^{1,1} = L^1$.

**1.105 Lemma.** *Let $u \in W^{1,p} \cap W^{1,q}(\mathbb{R}^N_+)$. Then the two traces of $u$ on $H$ (one in $W^{1,p}$, the other one in $W^{1,q}$), coincide.*

*Proof.* We start by extending, as in the proof of Lemma 1.99, $u$ to $\mathbb{R}^N$. If $\rho$ is a standard mollifier, then $u_\varepsilon = \rho(\varepsilon \cdot) u * \rho_\varepsilon$ converges (as $\varepsilon \to 0$) to $u$ both in $W^{1,p}$ and in $W^{1,q}$. Since, for $u_\varepsilon \in C_c^\infty$, both traces coincide, we obtain the result by passing to the limits. $\square$

The same argument leads to the following result.

**1.106 Lemma.** *Let $u \in W^{1,p}(\mathbb{R}^N_+)$. For $-\Delta \neq 0$ and $x' \in \mathbb{R}^{N-1}$, we have* tr $u(-\Delta \cdot -x') = (\text{tr } u)(-\Delta \cdot -x')$.

**1.107 Lemma.** *Let $f$ be the characteristic function of a cube in $\mathbb{R}^{N-1}$. Then $f \in W^{1-1/p,p}(\mathbb{R}^{N-1})$ for $1 < p < 2$.*

*Proof.* We may assume that $C = (-l, l)^{N-1}$. Since a change of norms in $\mathbb{R}^{N-1}$ does not affect the fact that a given map belongs to some Sobolev space, we compute the $W^{1-1/p,p}$ norm w. r. t. the $l^\infty$ norm in $\mathbb{R}^{N-1}$. For this norm, we have

$$\|f\|^p_{W^{1-1/p,p}} \sim \int_{|x'|<l} \int_{|y'|>l} \frac{dx'\, dy'}{|x'-y'|^{N+p-2}} + l^{(N-1)p}.$$

If $|x'| < l$ and $|y'| > l$, then $y' \in \mathbb{R}^{N-1} \setminus B(x', l - |x'|)$, and therefore

$$\int_{|y'|>l} \frac{dy'}{|x'-y'|^{N+p-2}} \leq \int_{|z'|>l-|x'|} \frac{dz'}{|z'|^{N+p-2}} = C \int_{l-|x'|}^{\infty} r^{-p} = C(l - |x'|)^{1-p}.$$

Since $p < 2$, we find that

$$\begin{aligned} \|f\|^p_{W^{1-1/p,p}} &\leq C \int_{|x'|<l} (l - |x'|)^{1-p} + C l^{(N-1)p} \\ &\leq C \int_{\{|x_j| \leq x_1 < l,\ j=1,\ldots,N-2\}} \frac{dx'}{(l-x_1)^{p-1}} + C l^{(N-1)p} < \infty. \end{aligned}$$

$\square$

**1.108 Lemma.** *Let $C$ be a cube of size $l$ in $\mathbb{R}^{N-1}$ and set $a = \dfrac{1}{|C|}\chi_C$. Then there is a map $u \in W^{1,1}$ s. t.* tr $u = a$ *and*

$$\|u\|_{L^1} \le c\,l \quad \text{ and } \|\nabla u\|_{L^1} \le c. \tag{1.11}$$

*Proof.* We start with the case where $C$ is the unit cube (or any other cube of size 1). We fix a $p \in (1,2)$. Since $a \in W^{1-1/p,p}$, we have $a = \text{tr } u_0$ for some $u \in W^{1,p}$. In addition, Corollary 1.104 implies that we may assume $u_0$ compactly supported. Thus $u \in W^{1,1}$ and tr $u_0 = a$ (computed in $W^{1,1}$). Let now $C$ be an arbitrary cube, which we may assume with sides parallel to the unit cube $Q$. Let $C = x' + (0,l)^{N-1}$. Set $u = l^{-(N-1)}u_0(l^{-1}(\cdot - x'))$. Then $u \in W^{1,1}$ and tr $u = a$. Inequality (1.11) follows from the identities $\|u\|_{L^1} = l\|u_0\|_{L^1}$ and $\|\nabla u\|_{L^1} = \|\nabla u_0\|_{L^1}$. □

**1.109 Theorem** (Gagliardo)**.** *Let $\Omega$ be a standard domain, of boundary $\Sigma$. Let $f \in L^1(\Sigma)$. Then there is some $u \in W^{1,1}(\Omega)$ s. t.* tr $u = f$ *and* $\|u\|_{W^{1,1}} \le C\|f\|_{L^1}$.

**1.110 Remark.** This time, the map $f \mapsto u$ we construct *is not linear.*

*Proof.* We consider the case $\Omega = \mathbb{R}^N_+$.

The main ingredient is the following: if $f \in L^1(\mathbb{R}^{N-1})$, then we may write, in $L^1$, $f = \sum -\Delta_n a_n$, where:

(i) each $a_n$ is of the form $a_n = \dfrac{1}{|C_n|}\chi_{C_n}$;

(ii) each cube $C_n$ is of size at most 1;

(iii) $\sum |-\Delta_n| \le C\|f\|_{L^1}$.

Assuming that this can be achieved, here is the end of the proof: the preceding lemma implies that each $a_n$ is the trace of some $u_n \in W^{1,1}$ s. t. $\|u_n\|_{W^{1,1}} \le C$. The linearity of the trace and property (iii) imply that the map $u = \sum -\Delta_n u_n \in W^{1,1}$ satisfies tr $u = f$ and $\|u\|_{W^{1,1}} \le C\|f\|_{L^1}$.

It remains to perform the decomposition $f = \sum -\Delta_n a_n$. For each $j \in \mathbb{N}$, let $\mathcal{F}_j$ be the grid of cubes of size $2^{-j}$, with sides parallel to the coordinate axes and having the origin among the edges. We define the linear map $T_j : L^1 \to L^1$, $T_j f(x) = \fint_C f$ if $x \in C \in \mathcal{F}_j$. Clearly, $T_j$ is of norm 1. We claim that, for each $f \in L^1$, we have $T_j f \to f$ in $L^1$ as $j \to \infty$. This is clear when $f \in C_c^\infty$; the case of a general $f$ follows by approximation using the fact that $\|T_j\| = 1$. We may thus find an increasing sequence of indices, $(j_k)$, s. t. $\|f_{j_0}\|_{L^1} + \sum \|f_{j_{k+1}} - f_{j_k}\|_{L^1} \le C\|f\|_{L^1}$. We claim that $f_{j_{k+1}} - f_{j_k} = \sum -\Delta_n^k a_n^k$, where each $a_n^k$ is of the form $\dfrac{1}{|C|}\chi_C$ for some cube of size at most 1 and $\sum |-\Delta_n^k| = \|f_{j_{k+1}} - f_{j_k}\|_{L^1}$. Indeed, $f_{j_{k+1}} - f_{j_k}$ is constant on each cube $C \in \mathcal{F}_{j_{k+1}}$, and thus $f_{j_{k+1}} - f_{j_k} = \sum\limits_{C\in\mathcal{F}_{j_{k+1}}} (f_{j_{k+1}} - f_{j_k})_{|C}\chi_C$, so that $f_{j_{k+1}} - f_{j_k} = \sum\limits_{C\in\mathcal{F}_{j_{k+1}}} -\Delta_C \dfrac{1}{|C|}\chi_C$, with $-\Delta_C = (f_{j_{k+1}} - f_{j_k})_{|C}|C|$. We find that

$$\begin{aligned}\|f_{j_{k+1}} - f_{j_k}\|_{L^1} &= \sum_{C\in\mathcal{F}_{j_{k+1}}} \int_C |f_{j_{k+1}} - f_{j_k}| = \sum_{C\in\mathcal{F}_{j_{k+1}}} |C||(f_{j_{k+1}} - f_{j_k})_{|C}| \\ &= \sum_{C\in\mathcal{F}_{j_{k+1}}} |-\Delta_C|.\end{aligned}$$

Similarly, we may write $f_{j_0} = \sum -\Delta_n^0 a_n^0$, where each $a_n^0$ is of the form $\dfrac{1}{|C|}\chi_C$ for some cube of size at most 1 and $\sum |-\Delta_n^0| = \|f_{j_0}\|_{L^1}$.
Finally, we write $f = \sum\limits_k \sum\limits_n -\Delta_n^k a_n^k$, and this decomposition has the properties (i)-(iii). $\square$

### 1.5.3 Traces for higher order spaces

**1.111 Definition.** Fort $k \in \mathbb{N}$, $0 < \sigma < 1$ and $1 \le p < \infty$, we let

$$W^{k+\sigma,p}(\mathbb{R}^N) = \{u \in W^{k,p}(\mathbb{R}^N); D^k u \in W^{\sigma,p}\},$$

equipped with the natural norm.

**1.112 Definition.** Assuming that $\Omega$ is smooth and bounded, we define, for $k$ and $\sigma$ as above, $W^{k+\sigma,p}(\Sigma)$ as follows: we cover $\partial\Omega$ with a finite number of chart domains $\Sigma_i$, which can be straightened via $C^k$ diffeomorphisms $\Phi_i$. We consider a partition of unity $1 = \sum \zeta_i$ subordinated to the covering $\Sigma_i$ and define

$$W^{k+\sigma,p}(\Sigma) = \{u : \Sigma \to \mathbb{R}; (\zeta_i u)\circ\Phi_i^{-1} \in W^{k+\sigma,p}(\mathbb{R}^{N-1}) \text{ for each } i\},$$

endowed with the natural norm.

One can prove that the definition is *intrinsic*, i. e., does not depend on the choice of $\Sigma_i$ and $\Phi_i$. This will be omitted here.

**1.113 Theorem.** *Let $\Omega$ be a standard domain. Let $1 < p < \infty$ and let $k \in \mathbb{N}^*$. Then* tr $W^{k,p}(\Omega) = W^{k-1/p,p}(\Sigma)$, *and the trace map has a continuous linear right inverse.*

*Proof.* We may assume that $k \geq 2$. It suffices to consider the model case $\Omega = \mathbb{R}^N_+$.

We start by noting that the trace operator commutes with horizontal derivatives: if $u \in W^{k,p}(\mathbb{R}^N_+)$ and if $\alpha \in \mathbb{N}^N$ is s. t. $\alpha_N = 0$ and $|\alpha| \leq k-1$, then tr $\partial^\alpha u = \partial^\alpha$ tr $u$. (This is proved by density, starting with the case where $u \in C^\infty_c(\overline{\mathbb{R}^N_+})$.) This implies at once that tr $u \in W^{k-1,p}$ and $D^{k-1}$ tr $u \in W^{1-1/p,p}(\mathbb{R}^{N-1})$.

Conversely, let $f \in W^{k-1/p,p}(\mathbb{R}^{N-1})$. Define again $v(x',t) = f * \rho_t(x')$ and $u(x',t) = v(x',t)\varphi(t)$. We will prove that $u \in W^{k,p}(\mathbb{R}^N_+)$. We go to the most difficult part, which consists in proving the fact that $D^{k-1}u \in W^{1-1/p,p}$. (The reader may try to prove by his own means that $D^j u \in L^p$, $0 \leq j \leq k-1$.) This will be a consequence of the fact that $D^{k-1}v$ is expressed as in terms of $D^{k-1}f$. More specifically, if $|\alpha| = k-1$, then

$$\partial^\alpha v(x',t) = \partial^\alpha \int f(x'-ty')\rho(y')\,dy',$$

and this quantity may be rewritten as $\displaystyle\sum_{\beta' \in \mathbb{N}^{N-1}, |\beta'| \leq k-1} (\partial^{\beta'} f) * \rho_t^{\beta'}(x)$, for appropriate compactly supported $\rho^{\beta'}$. This implies that

$$\partial^\alpha u(x',t) = \sum_{|\beta'|+\gamma \leq k-1} c_{\beta',\gamma}(\partial^{\beta'} f) * \rho_t^{\beta'}(x)\varphi^\gamma(t).$$

Using this form, we conclude, as in the proof of Theorem 1.102, that $D^{k-1}u \in W^{1-1/p,p}$. □

**1.114 Remark.** It is not true that tr $W^{k,1}(\Omega) = W^{k-1,1}(\Sigma)$, $k = 2, 3, \ldots$. In fact, we have the strict inclusion tr $W^{k,1}(\Omega) \subsetneq W^{k-1,1}(\Sigma)$, $k = 2, 3, \ldots$. This has been proved by Brezis and Ponce when $N = 2$ and $k = 2$, but their argument adapts to every $k$ and $N$.

An exciting open problem: find tr $W^{2,1}(\mathbb{R}^N_+)$.

### 1.5.4 Integration by parts. Functions with zero trace

We start with the following

**1.115 Theorem.** *[Integration by parts] Let $1 \le p \le \infty$ and let $\Omega$ be a standard domain. Then we have*

$$\int_\Omega u\partial_j\varphi = \int_{\partial\Omega} \nu_j \mathrm{tr}\, u\varphi - \int_\Omega \partial_j u\varphi, \quad \forall\ u \in W^{1,p}(\Omega), \varphi \in C_c^\infty(\Omega). \tag{1.12}$$

*Proof.* The conclusion being local, we may assume that $p < \infty$. Identity (1.12) is clear when $u \in C^\infty(\overline{\Omega}) \cap W^{1,p}$. The general case follows by density. □

The remaining part of this section is devoted to the proof of the following generalization of Theorem 1.39.

**1.116 Theorem.** *Let $\Omega$ be a standard domain and let $1 \le p < \infty$. For $u \in W^{1,p}(\Omega)$, the following assertions are equivalent:*

*i)* $u \in W_0^{1,p}(\Omega)$

*ii)* $\mathrm{tr}\, u = 0$

*iii) the map* $\tilde{u} = \begin{cases} u, & in\ \Omega \\ 0, & in\ \mathbb{R}^N \setminus \Omega \end{cases}$ *belongs to* $W^{1,p}(\mathbb{R}^N)$.

*Proof.* $i) \Longrightarrow ii)$ follows immediately from the continuity of the trace map.
$ii) \Longrightarrow iii)$ It suffices to prove that $\nabla_d\tilde{u} = \widetilde{\nabla_d u}$. This equality holds in $\mathbb{R}^N \setminus \partial\Omega$, so that it suffices to establish it in a neighbourhood of $\partial\Omega$. Let $U$ be a standard domain which does not intersect $\overline{\Omega}$ and s. t. $V := \overline{\Omega} \cup U$ is at the same time a neighbourhood of $\partial\Omega$ and a standard domain. [When $\Omega = \mathbb{R}^N_+$, take $U = \mathbb{R}^N_-$; when $\Omega$ is smooth and bounded, take $U = \{x \in R^N \setminus \overline{\Omega}; \mathrm{dist}\,(x, \Omega) < \varepsilon\}$, for sufficiently small $\varepsilon > 0$.] For $\varphi \in C_c^\infty(V)$, we have (using integration by parts) $\displaystyle\int_V u\partial_j\varphi = -\int_\Omega \partial_j u\varphi$, i. e., we have $\nabla_d\tilde{u} = \widetilde{\nabla_d u}$.
$iii) \Longrightarrow i)$ We treat the case where $\Omega$ is smooth and bounded; the case where $\Omega = \mathbb{R}^N_+$ is simpler. We consider (for small $\varepsilon > 0$) a family maps $\Phi_\varepsilon : \mathbb{R}^N \to \mathbb{R}^N$ with the following properties:

a) $\Phi_\varepsilon$ is a diffeomorhism, and $\Phi_\varepsilon$ maps a neighbourhood $U_\varepsilon$ of $\overline{\Omega}$ into a relatively compact subset $V_\varepsilon$ of $\Omega$

b) $\Phi_\varepsilon \to I_N$ and $D\Phi_\varepsilon \to I_N$ uniformly as $\varepsilon \to 0$.

[The construction of $\Phi_\varepsilon$ will be sketched in Exercise 1.133.] Let $u_\varepsilon := \tilde{u} \circ \Phi_\varepsilon^{-1}$. Then $u_\varepsilon \in W^{1,p}(\Omega)$ and supp $u_\varepsilon \Subset \Omega$ (check!). It follows that $u_\varepsilon \in W_0^{1,p}(\Omega)$. (Hint: approximate $u_\varepsilon$ by regularization.) Finally, we have $u_\varepsilon \to u$ in $W^{1,p}(\Omega)$; this follows with the help of b). $\square$

On the way, we established the following

**1.117 Corollary.** *Let $\Omega$ be a standard domain, $1 \le p < \infty$ and let $u \in W_0^{1,p}(\Omega)$. Let $\tilde{u}$ denote the extension of $u$ with the value $0$ outside $\Omega$. Then* $\partial_j \tilde{u} = \begin{cases} \partial_j u, & \text{in } \Omega \\ 0, & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$.

## 1.6 Sobolev spaces from a functional analysis viewpoint

**1.118 Proposition.** *Let $k \in \mathbb{N}^*$ and $1 \le p \le \infty$. Then the space $W^{k,p}(\Omega)$ is*

*a) uniformly convex (and thus reflexive) when $1 < p < \infty$*

*b) separable when $1 \le p < \infty$.*

*c) The same assertions hold for $W_0^{1,p}(\Omega)$ and $W^{k,p} \cap W_0^{1,p}(\Omega)$.*

*Proof.* The map $W^{k,p}(\Omega) \mapsto (\partial^\alpha u)_{|\alpha| \le k}$ is an isometry onto a subspace of $[L^p(\Omega)]^M$, for an appropriate integer $M$. This implies a) and b) (and thus the separability part in c)). Reflexivity in c) follows from a) combined with the fact that $W_0^{1,p}(\Omega)$, respectively $W^{k,p} \cap W_0^{1,p}(\Omega)$, are closed in $W^{1,p}(\Omega)$, respectively in $W^{k,p}(\Omega)$. $\square$

**1.119 Theorem.** *Assume that $1 < p \le \infty$ and $k \in \mathbb{N}^*$. Let $(u_n) \subset W^{k,p}(\Omega)$ be a bounded sequence. Then there is some $u \in W^{k,p}(\Omega)$ s. t., possibly after passing to a subsequence, we have*

*a) $u_n \to u$ in $W^{k-1,p}_{loc}(\Omega)$*

*b) $\|u\|_{W^{k,p}} \le \liminf \|u_n\|_{W^{k,p}}$.*

*c) If, in addition to the initial hypotheses, $\Omega$ is smooth and bounded, then we may enhance the conclusion* a) *to $u_n \to u$ in $W^{k-1,p}(\Omega)$*

d) *If, in addition to the initial hypotheses,* $1 < p < \infty$ *and* $(u_n) \subset W_0^{1,p}(\Omega)$, *then* $u \in W_0^{1,p}(\Omega)$.

*Proof.* Assume first that $1 < p < \infty$. Possibly after passing to a further subsequence, $u_n$ converges to some $v \in W^{k,p}$ in the weak topology (since $W^{k,p}$ is reflexive). Since $C_c^\infty(\Omega)$ is in the dual of $W^{k,p}$, we find that $\displaystyle\int u_n\varphi \to \int v\varphi$, $\forall\ \varphi \in C_c^\infty(\Omega)$ (in more official terms, $u_n \to v$ in $\mathscr{D}'(\Omega)$). On the other hand, we have $\|v\|_{W^{k,p}} \leq \liminf \|u_n\|_{W^{k,p}}$. If in addition, $(u_n) \subset W_0^{1,p}(\Omega)$, then $v \in W_0^{1,p}(\Omega)$ (since $W^{k,p} \cap W_0^{1,p}$ is convex and closed, and thus weakly closed, in $W^{k,p}$).
Assume next that $\Omega$ is smooth and bounded. Then, up to a subsequence, $u_n$ converges strongly in $W^{k-1,p}$ to some $u \in W^{k-1,p}$ (by the Rellich-Kondratchov theorem). This implies that $\displaystyle\int u_n\varphi \to \int u\varphi$, $\forall\ \varphi \in C_c^\infty(\Omega)$.
Assume now that $1 < p < \infty$ and that $\Omega$ is smooth and bounded. Then $u = v$, and the theorem is proved in this special case.
Assume next that $1 < p < \infty$; we do not assume $\Omega$ bounded. Cover $\Omega$ with a countable family of balls, $(B_i)_{i\in\mathbb{N}}$. Possibly after passing to a subsequence, $(u_n)$ converges, in $W^{k-1,p}(B_1)$, to $v$. The same holds in each $B_j$; by the diagonal procedure, we may find a subsequence, still denoted $(u_n)$, s. t. $u_n \to v$ in $W_{loc}^{k-1,p}$. This proves the theorem when $p < \infty$.
Assume next that $p = \infty$. We cover $\Omega$ with an increasing sequence of smooth bounded relatively compact sets, $\omega_j$. By a diagonal argument, we may assume that $u_n \to u$ in $W^{k-1,\infty}(\omega_j)$, for each $j$, and that $\|u\|_{W^{k,l}(\omega_j)} \leq \liminf \|u_n\|_{W^{k,l}(\omega_j)}$, for each $j$ and $l$. By letting first $l \to \infty$, next $j \to \infty$, we find that $\|u\|_{W^{k,\infty}} \leq \liminf \|u_n\|_{W^{k,\infty}}$. □

The situation is dramatically different when $p = 1$; see Exercise 1.134.

We end this section with the characterization of the dual of certain Sobolev spaces.

**1.120 Theorem.** *Let* $1 \leq p < \infty$. *Then the dual of* $W^{1,p}(\mathbb{R}^N)$ *is* $L^{p'}(\mathbb{R}^N) + \operatorname{div} L^{p'}(\mathbb{R}^N;\mathbb{R}^N)$, *in the following sense:* $T \in (W^{1,p})^*$ *iff there are* $f \in L^{p'}$ *and* $F \in L^{p'}(\mathbb{R}^N)$ *s. t.* $T = f + \operatorname{div} F$, *i. e.,* $T(u) = \displaystyle\int fu - \int F \cdot \nabla u$, $\forall\ u \in W^{1,p}$

a) *Let* $1 \leq p < \infty$. *Then the dual of* $W_0^{1,p}(\Omega)$ *is* $L^{p'}(\Omega) + \operatorname{div} L^{p'}(\Omega;\mathbb{R}^N)$

b) *Let* $1 \leq p < \infty$. *Assume that* $\Omega$ *is bounded. Then the dual of* $W_0^{1,p}(\Omega)$ *is* $\operatorname{div} L^{p'}(\Omega;\mathbb{R}^N)$.

*In all these assertions, we have*

$$\|T\| = \inf\{\|f\|_{L^{p'}} + \|F\|_{L^{p'}}; T = f + \text{div } F\}. \tag{1.13}$$

*Proof.* Case a) is a special case of case b). In case b), inclusion $\supset$ and inequality $\leq$ in (1.13) are clear. For the opposite inclusion and inequality, we consider the isometry $\Phi : W_0^{1,p}(\Omega) \to [L^p(\Omega)]^{N+1}$, $\Phi(u) = (u, \nabla u)$. Let $X$ denote its range. If $T \in (W_0^{1,p})^*$, then $T \circ \Phi^{-1} \in X^*$. By Hahn-Banach, $T$ has a norm preserving extension $V$ to $[L^p(\Omega)]^{N+1}$. Therefore, there are $f, F \in L^{p'}$ s. t.

$$V(v, \tilde{v}) = \int fv + \int F \cdot \tilde{v}, \quad \forall\ (v, \tilde{v}) \in L^p \times [L^p]^N,$$

and $\|f\|_{L^{p'}} + \|F\|_{L^{p'}} = \|T\|$. We easily find that $T = f + \text{div } F$.

In case c), we repeat the same proof, but starting this time with the map $u \mapsto \nabla u$. □

## 1.7 Exercises

These may be more than simple exercises...

**1.121 Exercise.** We discuss here the notion of support of a distribution. We define the complement of supp $u$, where $u \in \mathscr{D}'(\Omega)$, as the union of the open sets on which $u$ vanishes, i. e.: $\Omega \setminus \text{supp } u$ is the largest $\omega$, open subset of $\Omega$, s. t. $u(\varphi) = 0$ for each $\varphi \in C_c^\infty(\omega)$.

a) If $u \in L^1_{loc}$, prove that supp $u$ is the smallest closed set $F$ s. t. $u = 0$ a. e. in $F$

b) We let $\mathscr{E}'$ denote the set of compactly supported distributions in $\mathbb{R}^N$. Prove that there is a natural notion of convolution $u*v$ when $u \in \mathscr{D}'(\mathbb{R}^N)$ and $v \in \mathscr{E}'$. Prove that this convolution has the expected properties. In particular, there is a natural notion of $E * u$ when $E \in L^1_{loc}$ and $u \in \mathscr{E}'$

c) Prove other good sense properties: if $u \in C^\infty$ and $v \in \mathscr{E}'$, then $u*v \in C^\infty$. Convolution commutes with derivatives

d) Prove that supp $u * v \subset$ supp $u+$ supp $v$.

If you do not want to think, but need the results alluded here, take a look at Hörmander, Section 4.2. Same applies to the next result.

**1.122 Exercise** (Schwartz)**.** Let $u \in \mathscr{D}'(\mathbb{R}^N)$ be s. t. supp $u = \{0\}$. Prove that $u = \sum_{finite} c_\alpha \partial^\alpha \delta$.

**1.123 Exercise.** We discuss here "convergence" of distributions. If $(u_n) \subset \mathscr{D}'(\Omega)$, we say that $u = \lim u_n$ if $u(\varphi) = \lim_n u_n(\varphi)$, $\forall\ \varphi \in C_c^\infty(\Omega)$. (Thus we assume that the limit exists and is finite for each $\varphi$.) We define similarly $u = \lim_{\varepsilon \to 0} u_\varepsilon$.

a) Prove that $u$ is *automatically* a distribution.
Hint: apply the Banach-Steinhaus theorem to the Fréchet space $C_c^\infty(K)$, where $K \Subset \Omega$.

b) Prove that, if $u \in \mathscr{D}'(\mathbb{R}^N)$, then $\lim_{\varepsilon \to 0} u * \rho_\varepsilon = u$

c) Prove that, if $u_n \to u$ in $\mathscr{D}'(\mathbb{R}^N)$ and $\varphi \in C_c^\infty(\mathbb{R}^N)$, then $u_n * \varphi \to u * \varphi$ in $C^\infty(\mathbb{R}^N)$

d) Symmetrically, prove the following: if $u_n \to u$ in $\mathscr{D}'(\mathbb{R}^N)$ and if there is some compact $K$ s. t. supp $u_n \subset K$, $\forall\ n$, and if $E \in C^\infty(\mathbb{R}^N)$, then $E * u_n \to E * u$ in $C^\infty(\mathbb{R}^N)$.

**1.124 Exercise** (Converse to dominated convergence)**.** Prove that, if $u_n \to u$ in $L^p$ then, possibly after passing to a subsequence, there is some $v \in L^p$ s. t. $|u_n| \le v$ a. e. and $u_n \to u$ a. e.

**1.125 Exercise.** Assume that $u_n \to u$ in $L^p(\mathbb{R}^N)$. Prove that, possibly after passing to a subsequence, $u_n(x', \cdot) \to u(x', \cdot)$ in $L^p(\mathbb{R})$ for a. e. $x' \in \mathbb{R}^{N-1}$.

**1.126 Exercise.** Use the preceding result in order to derive the following "Fubini type" characterization of Sobolev maps: for $1 \le p < \infty$, we have $u \in W^{1,p}(\mathbb{R}^N)$ if an only if:

a) $u$ is (say, Borel) measurable and belongs to $L^p(\mathbb{R}^N)$

b) for a. e. $x' \in R^{N-1}$, $u(x', \cdot) \in W^{1,p}(\mathbb{R})$ (+ similar conclusion when $x'$ is replaced by $(x_1, \dots x_{j-1}, x_{j+1}, \dots, x_N)$)

c) the map $(x', x_N) \mapsto \partial_N u(x', x_N)$ is in $L^p(\mathbb{R}^N)$ (+similar...)

Same if $W^{1,p}$ is replaced by $W^{1,p}_{loc}$.

**1.127 Exercise.** In this exercise, we discuss the importance of the presence of the $L^p$ norm of $u$ in the definition of the norm in $W^{1,p}(\Omega)$.

a) Prove that, in $\mathbb{R}^N$, if $\nabla u \in L^p$, then $u$ need not be in $L^p$.

b) Find an example of bounded open set $\Omega$ and of map $u$ in $\Omega$ s. t. $\nabla u \in L^p$, but $u \notin L^p$
Hint: let $\Omega$ have infinitely many connected components

c) However, there is a "local" implication: if $\nabla u \in L^p_{loc}$, then $u \in L^p_{loc}$
Hint: let $E$ denote "the" fundamental solution of the Laplace equation. Let $K \Subset \Omega$ and let $\zeta \in C_c^\infty(\Omega)$ equal 1 in a neighbourhood of $K$. Then we have the identity $\zeta u = \sum_j \partial_j E * (\zeta \partial_j u) + \sum_j \partial_j E * (u \partial_j \zeta)$. The second sum is $C^\infty(K)$ (why?). In order to estimate the first sum, apply a local form of Hölder's inequality (use the fact that $\nabla E$ is locally in $L^1$). For more details, see Hörmander, Theorem 4.5.8

d) There is also a global implication, which holds under the assumption that $\Omega$ is smooth and bounded: if $\nabla u \in L^p$, then $u \in L^p$. [Note, however, that $\|u\|_{L^p}$ cannot be controlled by $\nabla u$: consider, e. g., the case where $u$ is constant.] We give here the main lines of a proof in a special case. The general case, which we do not discuss here, reduces to the special case below after flattening of the boundary. For more details, see e. g. the first chapter in Maz'ja. We assume, for simplicity, that $N = 2$ and $\Omega = (0,1)^2$. We are going to prove that, if $\nabla u \in L^p(\Omega)$, then $u$ has an extension, still denoted by $u$, to $(-1,2)^2$ s. t. $\nabla u \in L^p$. If this can be achieved then, by the previous item, we have $u \in L^p(\Omega)$.

- Prove that $u$ has a trace on $(0,1) \times \{0\}$, in the following sense: for each $\varphi \in (0,1)$, $u$ has a trace, which belongs to $L^p_{loc}$, on $(0,1) \times \{\varepsilon\}$. If we denote this trace by $v_\varepsilon$, then $v_\varepsilon$ converges, as $\varepsilon \to 0$, to some $v \in L^p_{loc}$. We call this $v$ the trace of $u$ on $(0,1) \times \{0\}$.
- Prove that the following integration by parts formula holds: if $\varphi \in C_c^\infty((0,1) \times [0,1))$, then $\int_\Omega u\, \partial_j \varphi = -\int_0^1 v(x)\varphi(x,0)\,dx - \int_\Omega \partial_j u\, \varphi$.
- Extend $u$ to $(0,1) \times (-1,1)$ by reflection across $(0,1) \times \{0\}$: set $u(x,y) = \begin{cases} u(x,y), & \text{if } y > 0 \\ u(x,-y), & \text{if } y < 0 \end{cases}$. Prove that this extension has its gradient in $L^p$.
- Repeat this procedure and conclude.

**1.128 Exercise** (Leibniz rule for Lipschitz maps)**.** Prove that, if $u, v \in Lip_{loc}$, then $uv \in Lip_{loc}$ and $\nabla_d(uv) = u\nabla_p v + v\nabla_p u$. (Recall that locally Lipschitz functions have point first order partial derivatives a. e.)

**1.129 Exercise.** Let $\Phi : \Omega \to \omega$ be a homeomorphism s. t. $\Phi, \Phi^{-1} \in W^{k,\infty}_{loc}$. Prove that, for $1 \le p \le \infty$, we have $u \in W^{k,p}_{loc}(\omega)$ iff $u \circ \Phi \in W^{k,p}_{loc}(\Omega)$.
Which is the global version of this statement?
Special case: $W^{1,p}$ is invariant under composition with a bi-Lipschitz homeomorphism.

**1.130 Exercise** (Chain rule for Lipschitz maps)**.** Prove that, if $u, v \in Lip_{loc}$, then $\partial_j[u \circ v] = \sum_k [\partial_{k,p}u] \circ v \ \partial_{j,p}v$.
Which is the higher order analogue of this result?

**1.131 Exercise.** Starting from the estimate (1.6), prove the following generalization of (1.7): if $u \in W^{1,p}(\mathbb{R}^N)$, then $\|u * \rho_\varepsilon - u\|_{L^p} \le C\varepsilon\|\nabla u\|_{L^p}$, $1 \le p \le \infty$.

**1.132 Exercise.** This exercise is related to the previous one. Let $1 \le p \le \infty$. For $h \in \mathbb{R}^N$ and $u$ defined in $\mathbb{R}^N$, let $\tau_h u = u(\cdot - h)$.

a) If $u \in W^{1,p}$, then $\|\tau_h u - u\|_{L^p} \le |h|\|\nabla u\|_{L^p}$. (And, of course, $u \in L^p$.)

b) The "converse" is not true in general, i. e., if $u \in L^p$ and $\|\tau_h u - u\|_{L^p} \le C|h|$ for each $h$, then we need not have $u \in W^{1,p}$. (Take $p = 1$, $N = 1$ and $u = \chi_{(0,1)}$.)

c) However, the converse is true when $1 < p \le \infty$: if $u \in L^p$ and $\|\tau_h u - u\|_{L^p} \le C|h|$ for each $h$, then $u \in W^{1,p}$. In addition, $\|\partial_j u\|_{L^p} \le C$.

Hints: prove the assertion when $u$ is smooth. Apply the conclusion to $u * \rho_\varepsilon$, then let $\varepsilon \to 0$. Use, as an ingredient, that $L^p$ $(1 < p \le \infty)$ is the dual of a separable space, and therefore a bounded sequence contains a *-weakly convergent subsequence.

**1.133 Exercise.** We construct here the family of deformations of a smooth bounded domain $\Omega$ required in the proof of Theorem 1.116. This is a standard, but difficult construction. Assume that $\Omega \in C^k$ is bounded; here, $k \ge 2$. Let $d(x) := \operatorname{dist}(x, \partial\Omega)$. We also let $\tilde{d}$ be the signed distance, $\tilde{d}(x) = \begin{cases} d(x), & \text{if } x \notin \Omega \\ -d(x), & \text{if } x \in \Omega \end{cases}$.

a) Prove that there is some $\delta > 0$ s. t., if $d(x) < \delta$, then $x$ has a unique projection on $\partial\Omega$, i. e., the set $\{y \in \partial\Omega; d(x) = |y - x|\}$ is reduced to a single point.

   From now on, we consider only points s. t. $d(x) < \delta$. We let $\Pi(x)$ denote the unique $y \in \partial\Omega$ s. t. $|y - x| = d(x)$

b) Prove that $x = \Pi(x) + \tilde{d}(x)\nu(y)$ (with $\nu$ the outward normal to $\partial\Omega$).

c) Prove that

$$(-\delta, \delta) \times \partial\Omega \ni (t, y) \mapsto y + t\nu(y) \in \Omega_\delta := \{x \in \mathbb{R}^N; d(x) < \delta\}$$

is a $C^{k-1}$ diffeomorphism.

d) Let $0 < \varepsilon < \min\{\delta/2, 1\}$. Let $\psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ be s. t. $\psi_\varepsilon(t) = \begin{cases} t - \varepsilon^2, & \text{if } t \le \varepsilon^2 \\ t, & \text{if } t \ge \varepsilon \end{cases}$,

$\psi'_\varepsilon > 0$ and $|\psi'_\varepsilon - 1| \le C\varepsilon$. Let

$$\Phi_\varepsilon(x) = \begin{cases} x, & \text{if } x \in \mathbb{R}^N \setminus \Omega_\delta \\ \Pi(x) + \psi_\varepsilon(\tilde{d}(x))\nu(\Pi(x)), & \text{if } x \in \Omega_\delta \end{cases}.$$

Prove that

(a) $\Phi_\varepsilon$ is a $C^{k-1}$ diffeomorphism

(b) $\Phi_\varepsilon$ maps an appropriate neighborhood of $\overline{\Omega}$ into a compact subset of $\Omega$

(c) $\Phi_\varepsilon \to I_N$ and $D\Phi_\varepsilon \to I_N$ uniformly.

**1.134 Exercise.** This exercise is an introduction to the BV space.

a) Consider the sequence $(u_n) \subset W^{1,1}(-1, 1)$, $u_n(x) = \begin{cases} nx, & \text{if } 0 < x < 1/n \\ 0, & \text{if } x \le 0 \\ 1, & \text{if } x \ge 1/n \end{cases}$.

This sequence is bounded in $W^{1,1}$. However, it does not have a subsequence converging, strongly in $L^1$, to some $u \in W^{1,1}$. In other words, Theorem 1.119 does not hold for $p = 1$.

b) The space $\mathrm{BV}(\Omega)$ (for Bounded Variation) is defined as $\{u \in L^1(\Omega); \nabla u \in \mathscr{M}\}$, where $\mathscr{M}$ is the space of finite signed Borel measure on $\Omega$. This is a Banach space with the norm $u \mapsto \|u\|_{L^1} + |\nabla u|_{\mathscr{M}}$. (Here, $|\ |_{\mathscr{M}}$ stands for the total variation of a signed measure.) The space $W^{1,1}(\Omega)$ is a strict closed subspace of $\mathrm{BV}(\Omega)$.

c) Let $\Omega_\varepsilon = \{x \in \Omega; \text{dist } (x, \partial\Omega) > \varepsilon\}$. Let $u \in L^1(\Omega)$. Then $u \in \mathrm{BV}(\Omega)$ if and only if there is some $C > 0$ s. t. $\|\tau_h u - u\|_{L^1(\Omega_\varepsilon)} \le C\varepsilon$, for each $\varepsilon > 0$ and for each $h \in \mathbb{R}^N$ s. t. $|h| = \varepsilon$.
Hint: a bounded sequence in $L^1$ has a subsequence $*$-weakly converging (in the sense of measures) to some finite Borel measure $\mu$.

d) Use the preceding property in order to find a linear continuous extension operator $P$ :BV($R^N_+$) →BV($\mathbb{R}^N$).

e) Prove that, if $\Omega$ is smooth and bounded, then there is a linear continuous extension operator from BV($\Omega$) into BV($\mathbb{R}^N$).

f) Assuming $\Omega$ smooth and bounded, prove that the embedding BV($\Omega$) $\hookrightarrow$ $L^1(\Omega)$ is compact.

g) Same for the embedding BV($\Omega$) $\hookrightarrow L^q(\Omega)$, $1 < q < N/(N-1)$.
Hint: start by establishing the embedding BV($\Omega$) $\hookrightarrow L^{N/(N-1)}(\Omega)$.

h) Prove now the following substitute of Theorem 1.119: if $(u_n)$ is a bounded sequence in BV($\Omega$), then, up to a subsequence, we have $u_n \to u$ in $L^1_{loc}$, where $u \in$BV($\Omega$) and $\|u\|_{\text{BV}} \leq \liminf \|u_n\|_{\text{BV}}$. If, in addition, $\Omega$ is smooth and bounded, then we may replace convergence in $L^1_{loc}$ by convergence in $L^1$.

# 2 Linear elliptic equations

The problem we deal in this part is $\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$. Equation $-\Delta u = 0$ the Laplace equation; its solution are *harmonic distributions* (we will quickly replace "distributions" by "functions"). Equation $-\Delta u = f$ is the Poisson equation. In order to have uniqueness of a solution, these equations have to be supplemented by a boundary condition. There are three main types of boundary conditions associated to the Laplace operator $-\Delta$: Dirichlet ($u$ is given on $\partial\Omega$), Neuman (the normal derivative is given) and Robin (or mixed: a linear combination of $u$ and the normal derivative of $u$ is given). We shall deal here exclusively with the Dirichlet boundary condition; DP will be a shorthand for D̲irichlet P̲roblem. It is not that the three problems are the same: it would take too much time to treat all of them. For a brief treatment of the Neumann problem, as well as some references for the Robin problem, we send to Gilbarg and Trudinger. The main guide about these conditions is that they ensure existence and uniqueness of solutions, provided the domain and the data are "good enough".

A special case is the DP is the DP for the Laplace equation; DPL in short. The solution of the DPL (if it exists) is called the *harmonic extension* of $g$. The harmonic extension is defined by the requirements $u \in C(\overline{\Omega})$, $-\Delta u = 0$ in $\Omega$, $u = g$ on $\partial\Omega$.

## 2.1 Harmonic functions

We recall the following well-known results:

**2.1 Lemma** (Green's formulae)**.** *Let $\Omega$ be of class $C^1$ and let $u, v \in C^2(\overline{\Omega})$. We assume, in addition, that either $\Omega$ is bounded, or that one of the maps $u$ or $v$ is compactly supported.*

*a)* (First formula) $\displaystyle\int_\Omega u\Delta v = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - \int_\Omega \nabla u \cdot \nabla v$

b) (Second formula) $\displaystyle\int_\Omega u\Delta v = \int_{\partial\Omega} u\frac{\partial v}{\partial \nu} - \int_{\partial\Omega} v\frac{\partial u}{\partial \nu} + \int_\Omega v\Delta u.$

**2.2 Lemma.** *a) For $f \in C^2(0,\infty)$, we set $u(x) = f(|x|)$, $x \in \mathbb{R}^N \setminus \{0\}$. Then $\Delta u = f''(|x|) + \dfrac{N-1}{|x|} f'(|x|)$*

*b) In the special case* $f_N(r) = \begin{cases} \dfrac{-1}{2\pi}\ln r, & \text{if } N = 2 \\ \dfrac{1}{(N-2)\sigma_N r^{N-2}}, & \text{if } N \geq 3 \end{cases}$*, the corresponding $u$'s are harmonic in $\mathbb{R}^N \setminus \{0\}$*

*c) If we let* $E(x) = E_N(x) = \begin{cases} \dfrac{-1}{2\pi}\ln |x|, & \text{if } N = 2 \\ \dfrac{1}{(N-2)\sigma_N |x|^{N-2}}, & \text{if } N \geq 3 \end{cases}$*, then $E$ is a fundamental solution of $-\Delta$, i. e., $-\Delta E = \delta$.*
*We will call this $E$ "the" fundamental solution of $-\Delta$.*

*Proof.* Only the last statement requires a proof. We have $\dfrac{x}{|x|} \cdot \nabla E(x) = -\dfrac{1}{\sigma_N |x|^{N-1}}$, $\displaystyle\int_{S(0,\varepsilon)} E \to 0$ as $\varepsilon \to 0$ and $E \in L^1_{loc}$ (check!). Using Green's second formula, we have, for $\varphi \in C_c^\infty(\mathbb{R}^N)$:

$$
\begin{aligned}
(-\Delta E)(\varphi) = E(-\Delta\varphi) &= -\lim_{\varepsilon \to 0} \int_{R^N \setminus B(0,\varepsilon)} E\Delta\varphi \\
&= -\lim_{\varepsilon \to 0} \int_{\partial(R^N \setminus B(0,\varepsilon))} \left( E\frac{\partial \varphi}{\partial \nu} - \varphi\frac{\partial E}{\partial \nu} \right) \\
&= \lim_{\varepsilon \to 0} \int_{S(0,\varepsilon)} \left( E\,\frac{x}{|x|} \cdot \nabla\varphi - \varphi\,\frac{x}{|x|} \cdot \nabla E \right) \\
&= \lim_{\varepsilon \to 0} \int_{S(0,\varepsilon)} \varphi\,\frac{1}{\sigma_N \varepsilon^{N-1}} = \varphi(0) = \delta(\varphi).
\end{aligned}
$$

□

As a consequence, we obtain the following result (valid, more generally, for linear constant coefficients elliptic operators, see Hörmander, Theorems 7.12 to 7.1.22 and the remarks following these theorems).

**2.3 Theorem.** *a) Harmonic distributions are analytic.*

*b) Let $(u_n)$ be a sequence of harmonic functions converging in $\mathscr{D}'(\Omega)$ to some distribution $u$. Then $u$ is harmonic and the convergence takes place in $C^\infty$.*

In particular, harmonic distributions are $C^\infty$, fact known as *Weyl's lemma.*

*Proof.* We first prove that harmonic functions are $C^\infty$. Fix a compact $K \Subset \Omega$ and let $\varphi \in C_c^\infty(\Omega)$ equal 1 in $K_\varepsilon = \{x; \text{dist } (x, K) < 2\varepsilon\}$. (Note that $K_\varepsilon \Subset \Omega$ for small $\varepsilon$.) Set $f := -\Delta(\varphi u) \in C^\infty$. This $f$ is supported in $\mathbb{R}^N \setminus K_\varepsilon$. Let $\psi_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ be s. t. supp $\psi_\varepsilon \subset B(0, \varepsilon)$ and $\psi_\varepsilon - 0$ near the origin. Split $E = F_\varepsilon + G_\varepsilon$, where $F_\varepsilon = E\psi_\varepsilon$. Then

$$\varphi u = \delta * (\varphi u) = (-\Delta E) * (\varphi u) = F_\varepsilon * (-\Delta(\varphi u)) + G_\varepsilon * (-\Delta(\varphi u)) \equiv A_\varepsilon + B_\varepsilon.$$

We next note that $B_\varepsilon \in C^\infty$. On the other hand, $A_\varepsilon$ is supported in $B(0, \varepsilon)+$ supp $\Delta(\varphi u)$ (*cf* Exercise 1.127, last item). Since we have supp $\Delta(\varphi u) \subset \mathbb{R}^N \setminus K_\varepsilon$, we find that $A_\varepsilon \equiv 0$ in $K$. Thus $u \in C^\infty$.

We next prove that $u$ is analytic. Let $K$, $\varepsilon$, $\varphi$ and $f$ as above. Note that now we know that $f \in C^\infty$. If $x \in K$ and $f(y) \neq 0$, then we have $|x - y| > 2\varepsilon$. Consequently, in $K$, $u$ is given by

$$u(x) = \int E(x - y) f(y) \, dy. \tag{2.1}$$

Since $E$ is analytic in $\mathbb{R}^N \setminus \{0\}$, $u$ has a (several complex variables) holomorphic extension to a neighborhood $U$ of $\mathbb{R}^N \setminus \{0\}$ in $\mathbb{C}^N$. From this it follows easily that the r. h. s. of (2.1) is holomorphic in a neighborhood of $K$. In particular, $u$ itself is analytic.

We now turn to property b). Clearly, $u$ is harmonic, and thus $u \in C^\infty$. We may assume that $u = 0$ (otherwise, replace $u_n$ by $u_n - u$). Let, with $K$, $\varepsilon$ and $\varphi$ as above, $f_n := -\Delta(\varphi u_n)$; note that $f_n \to 0$ in $\mathscr{D}'(\mathbb{R}^N)$ and that supp $f_n$ is contained in the fixed compact set $L =$supp $\nabla\varphi$. Then $\varphi u_n = F_\varepsilon * f_n + G_\varepsilon * f_n \equiv A_{\varepsilon,n} + B_{\varepsilon,n}$. By the Exercise 1.123, we have $B_{\varepsilon,n} \to 0$ in $C^\infty(\mathbb{R}^N)$. On the other hand, we have $A_{\varepsilon,n} = 0$ in $K$. Thus $u_n \to 0$ in $C^\infty(K)$. □

**2.4 Corollary.** *Let $(u_n)$ be a sequence of harmonic functions. If $u_n \to u$ uniformly on compacts, or more generally, if $u_n \to u$ in $L^1_{loc}$, then $u_n \to u$ in $C^\infty$.*

**2.5 Definition.** A fonction $u \in C^2(\Omega)$ is superharmonic if $-\Delta u \geq 0$, respectively subharmonic if $-\Delta u \leq 0$.

**2.6 Theorem** ( Mean value theorem)**.** *Let $x \in \Omega$ and let $0 < R <$ dist $(x, \partial\Omega)$. Then*

*a) if $u$ is harmonic, then* $u(x) = \fint_{B(x,R)} u = \fint_{S(x,R)} u$

*b) if $u$ is superharmonic, then* $u(x) \geq \fint_{B(x,R)} u \geq \fint_{S(x,R)} u$

*c) if $u$ is subharmonic, then* $u(x) \leq \fint_{B(x,R)} u \leq \fint_{S(x,R)} u$.

*Proof.* We assume, e. g., that $x = 0$ and that $u$ is superharmonic (the case where $u$ is subharmonic is similar, and the case where $u$ is harmonic is obtained by combining the two other ones). Let $f : ]0, R] \to \mathbb{R}$, $f(r) = \fint_{S(0,r)} u$. Then $f \in C^2(]0, R])$ and $f$ can be extended by continuity with the value $u(0)$ at $r = 0$. We are going to prove that $f$ is non increasing. Indeed, with $\nu$ the outward normal at $B(0, r)$, the Gauss-Green formula yields

$$\begin{aligned} f'(r) &= \frac{d}{dr} \fint_{S(0,1)} u(ry)\, d\mathscr{H}^{N-1}(y) = \fint_{S(0,1)} (\nabla u)(ry) \cdot y\, d\mathscr{H}^{N-1}(y) \\ &= \fint_{S(0,r)} \frac{\partial u}{\partial \nu} = \frac{1}{|S(0,r)|} \int_{B(0,r)} \Delta u \leq 0. \end{aligned}$$

On the other hand, we have

$$\int_{B(0,R)} u = \int_0^R \sigma_N r^{N-1} f(r)\, dr = \omega_N R^N f(\xi) = |B(0,R)| f(\xi) \text{ with } \xi \in [0, R].$$

We find that b) holds. $\square$

**2.7 Proposition.** *A subharmonic function having a maximum point in a domain is constant.*
*Similar statements for a superharmonic function having a minimum point or for a harmonic function having an extreme point.*

*Proof.* Let $M = \max u$ et $A = \{x \in \Omega\ ;\ u(x) = M\}$. Then $A$ is closed and non empty. In order to conclude, it suffices to prove that $A$ is open. If $x \in A$ and if $0 < R < \text{dist }(x, \partial\Omega)$, then

$$M = u(x) \leq \fint_{B(x,R)} u \leq \fint_{B(x,R)} M = M, \tag{2.2}$$

so that $B(x, R) \subset A$. $\square$

**2.8 Proposition** (Maximum principle)**.** *Let $\Omega$ be bounded and let $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

*a) If $u$ is subharmonic, then $u \leq \sup\limits_{\partial\Omega} u$*

*b) If $u$ is superharmonic, then $u \geq \inf\limits_{\partial\Omega} u$.*

*Proof.* It suffices to consider the case where $\Omega$ is connected and bounded and where $u$ is subharmonic (see the next exercise). If $u$ has no maximum point in $\Omega$, then the conclusion is clear. Otherwise, $u$ is constant in $\Omega$ and the conclusion is once again clear. $\square$

**2.9 Exercise.** Let $\Omega$ be bounded. Let $\omega$ be a connected component of $\Omega$. Prove that $\partial\omega \subset \partial\Omega$.

**2.10 Corollary** (Uniqueness of the solution of the DP)**.** *If $\Omega$ is bounded, then the DP with datum $g \in C(\partial\Omega)$ has at most a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$.*

**2.11 Proposition.** *Let $\Omega$ is bounded and let $f \in C(\Omega)$, $g \in C(\partial\Omega)$. If $u \in C^2\Omega) \cap C(\overline{\Omega})$ solves DP, then $|u| \leq \sup\limits_{\partial\Omega} |g| + C \sup\limits_{\Omega} |f|$, where $C$ depends only on $\Omega$.*

*Proof.* Let $R > 0$ be s. t. $\Omega \subset B(0, R)$. Let $v(x) := \dfrac{R^2 - |x|^2}{2N} \sup\limits_{\Omega} |f| + \sup\limits_{\partial\Omega} |\varphi|$. Then $v$ satisfies $-\Delta v \geq -\Delta u$ in $\Omega$ and $v_{|\partial\Omega} \geq u_{|\partial\Omega}$. The maximum principle implies that $u \leq v$. Similarly, we have $u \geq -v$. Thus $|u| \leq v$, so that $|u| \leq \sup\limits_{\partial\Omega} |\varphi| + \dfrac{R^2}{2N} \sup\limits_{\Omega} |f|$. $\square$

**2.12 Theorem** (Poisson's formula)**.** *Let $B = B(x_0, R)$ be a (Euclidean) ball. Then each $g \in C(\partial\Omega)$ has (exactly) a harmonic extension, given by*

$$u(x) = \frac{R^2 - |x - x_0|^2}{\sigma_N R} \int_{S(x_0,R)} \frac{g(y)}{|x - y|^N}\, d\mathscr{H}^{N-1}(y). \tag{2.3}$$

*Proof.* We may assume that $x_0 = 0$. Let $v$ denote the r. h. s. of (2.3). It suffices to prove that $v$ is harmonic in $B$ and satisfies $\lim_{x \to z} v(x) = g(z)$ for each $z \in S(0, R)$. Set, for $x \in B$ and $y \in S(0, R)$, $P(x, y) = \dfrac{R^2 - |x|^2}{\sigma_N R |x - y|^N}$ (this is the *Poisson kernel*). Define $S(x) = \displaystyle\int_{S(0,R)} P(x, y)\, d\mathscr{H}^{N-1}(y)$. The following properties are easily proved:

a) $\Delta_x P = 0$

b) $\Delta S = 0$

c) $P, S > 0$

d) $P, S \in C^\infty$.

By combining a) and d), we find at once that $v \in C^\infty$ and that $\Delta v = 0$.
On the other hand, $S$ depends only on $|x|$. Indeed, if $A$ is an isometry of $\mathbb{R}^N$, then

$$\begin{aligned} S(Ax) &= \frac{R^2 - |Ax|^2}{\sigma_N R} \int_{S(0,R)} \frac{d\mathscr{H}^{N-1}(y)}{|Ax - y|^N} \\ &= \frac{R^2 - |x|^2}{\sigma_N R} \int_{S(0,R)} \frac{d\mathscr{H}^{N-1}(y)}{|x - A^{-1}y|^N} = S(x), \end{aligned}$$

since the Hausdorff measure is invariant by isometries. If we set $f(r) = S(r, 0, \ldots, 0)$, then $f \in C^\infty([0, R))$. With $r = |x| > 0$, we then have $0 = \Delta S(x) = f''(r) + \dfrac{N-1}{r} f'(r)$. We find that $S$ is constant (check!). Since $S(0) = 1$, we find
e) $S \equiv 1$.

Let $\delta > 0$. If $z \in S(0, R)$ and $x \in B(0, R)$ are s. t. $|x - z| < \delta/2$, we find that

$$\int_{\{y \in S(0,R)\ ;\ |y-z| > \delta\}} P(x, y)\, d\mathscr{H}^{N-1}(y) \le \frac{R^2 - |x|^2}{\sigma_N R} \int_{S(0,R)} (2/\delta)^N = C_\delta (R^2 - |x|^2).$$

We find that $P$ satisfies the additional property
f) $\displaystyle\lim_{x \to z} \int_{|y-z| > \delta} P(x, y)\, d\mathscr{H}^{N-1}(y) = 0$.
It follows that

$$\begin{aligned} |v(x) - g(z)| &= \left| \int_{S(0,R)} P(x, y)(g(y) - g(z)) \right| \\ &\le \int_{|y-z| \le \delta} P(x, y)|g(y) - g(z)| + 2 \int_{|y-z| > \delta} P(x, y) \|g\|_{L^\infty}, \end{aligned}$$

so that

$$|v(x) - g(z)| \leq \sup_{|y-z|\leq\delta} |g(y) - g(z)| + 2\|g\|_{L^\infty} \int_{|y-z|>\delta} P(x,y). \tag{2.4}$$

If, in (2.4), we first let $x \to z$, then we let $\delta \to 0$, we find that $\lim\limits_{x\to z} v(x) = g(z)$. $\square$

We continue with some straightforward consequences of Poisson's formula.

**2.13 Proposition** (Basic gradient estimate)**.** *Let $u \in C(\overline{B}(x_0,R))$ be harmonic in $B(x_0,R)$. Then*

$$|\nabla u(x_0)| \leq \frac{N}{R} \sup_{B(x_0,R)} |u|. \tag{2.5}$$

*Proof.* We may assume that $x_0 = 0$. Without loss of generality, we may assume that $\nabla u(0) = (\partial_1 u(0), 0, \ldots, 0)$. If we let $g = u_{|S(0,R)}$, then we have

$$\partial_1 u(0) = \frac{N}{\sigma_N R} \int_{S(0,R)} \frac{g(y)}{|y|^{N+1}} \frac{y_1}{|y|}\, d\mathscr{H}^{N-1}(y). \tag{2.6}$$

We obtain (2.5) by taking absolute values in (2.6). $\square$

**2.14 Proposition.** *A continuous function satisfying the mean value theorem is harmonic.*

*Proof.* Let $B$ be a ball s. t. $\overline{B} \subset \Omega$. Let $u_0$ be the harmonic extension of $u_{|\partial B}$. In $B$, $u - u_0$ satisfies the mean value theorem, and therefore it also satisfies the maximum principle (whose proof relies only on the mean value theorem). We find that $u \leq u_0$. Similarly, $u \geq u_0$. Therefore, $u$ is harmonic in $B$. $\square$

**2.15 Proposition.** *A uniformly bounded of harmonic functions contains a sequence converging in $C^\infty$.*

*Proof.* By the basic gradient estimate, the sequence $(\nabla u_n)$ is uniformly bounded on compacts. Using Ascoli's theorem (plus a diagonal procedure), we find that, up to a subsequence, $(u_n)$ converges uniformly on compacts (and thus in $C^\infty$). $\square$

**2.16 Theorem** (Liouville)**.** *A harmonic function in $\mathbb{R}^N$ which is bounded from either above or below is constant.*

*Proof.* We may assume that $u$ is bounded from below. By adding a constant to $u$ if necessary, we may assume that $u \geq 0$. Let $x, y \in \mathbb{R}^N$, $r > |x-y|$ and set $R = r + |x-y|$. Then the following inclusion holds: $B(x,r) \subset B(y,R)$. Therefore, we have

$$\begin{aligned} u(x) &= \frac{1}{|B(x,r)|} \int_{B(x,r)} u \leq \frac{1}{|B(x,r)|} \int_{B(y,R)} u \\ &= \left( \frac{r + |x-y|}{r} \right)^N \fint_{B(y,R)} u = \left( \frac{r + |x-y|}{r} \right)^N u(y). \end{aligned}$$

If we let $r \to \infty$, then we find that $u(x) \leq u(y)$, $\forall\ x, y \in \mathbb{R}^N$, so that $u$ is constant. $\square$

Our next task is to extend the notions of super- and subharmonicity to distributions.

**2.17 Definition.** A distribution $u \in \mathscr{D}'(\Omega)$ is

a) subharmonic if $-\Delta u \leq 0$

b) superharmonic if $-\Delta u \geq 0$.

**2.18 Exercise.** Prove that, in the special case where $u \in C^2$, the above definition coincides with Definition 2.5.

We will use these notions in the special case where $u$ is, in addition, continuous. However, the results we prove hold without this assumption. The continuity assumption is removed in the exercises section.

**2.19 Theorem.** *Let $u \in C(\Omega)$. Then the following properties are equivalent:*

*a) $u$ is subharmonic*

*b) if $B$ is a ball s. t. $\overline{B} \subset \Omega$, and if $v$ is the harmonic extension of $u_{|\partial B}$, then $u \leq v$ in $B$*

*c) $u$ satisfies the mean value theorem for subharmonic functions, i. e., if $x \in \Omega$ and if $0 < r < \operatorname{dist}(x, \partial\Omega)$, then $u(x) \leq \fint_{S(x,r)} u$.*

*Proof.* a)$\Longrightarrow$b) We smoothen $u$. The smooth functions $u_\varepsilon := u * \rho_\varepsilon$ satisfy $-\Delta u_\varepsilon = (-\Delta u) * \rho_\varepsilon \le 0$ in $\Omega_\varepsilon = \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$. If we fix $B$ and $x \in B$ then, for small $\varepsilon$, we have $u_\varepsilon(x) \le v_\varepsilon(x)$, where $v_\varepsilon$ is the harmonic extension of the restriction of $u_\varepsilon$ to $\partial B$. Since $u_\varepsilon \to u$ uniformly on $\partial B$ as $\varepsilon \to 0$, Poisson's formula implies that $v_\varepsilon(x) \to v(x)$. We find that $u(x) \le v(x)$.

b)$\Longrightarrow$c) If $v$ is the harmonic extension of $u_{|\partial B}$, then $u(x) \le v(x) = \fint_{S(x,r)} v = \fint_{S(x,r)} u$.

c)$\Longrightarrow$a) The idea is to smoothen $u$ and to prove that $-\Delta u_\varepsilon \le 0$ in $\Omega_\varepsilon$. Indeed, if this holds, then, by passing to the limits, we find that $-\Delta u \le 0$ (check!). To this purpose, we start by proving that $u_\varepsilon(x) \le \fint_{S(x,r)} u_\varepsilon$ if $0 < r < \operatorname{dist}(x, \partial\Omega) - \varepsilon$. The identity $\fint_{S(x,r)} u(y-z)\, d\mathscr{H}^{N-1}(y) = \fint_{S(x-z,r)} u$ and Fubini's theorem imply that

$$\begin{aligned}\fint_{S(x,r)} u_\varepsilon &= \fint_{S(x,r)} (\int (u(y-z)\rho_\varepsilon(z)\, dz)) = \int \fint_{S(x-z,r)} u\rho_\varepsilon(z)\, dz \\ &\ge \int u(x-z)\rho_\varepsilon(z)\, dz = u_\varepsilon(x).\end{aligned}$$

We next fix some $x \in \Omega$ and set, for small $r_0$, $f(r) = \fint_{S(x,r)} u_\varepsilon$, $0 < r < r_0$. Starting from the identity $f(r) = \dfrac{1}{\sigma_N} \displaystyle\int_{S(0,1)} u_\varepsilon(x + r\omega)\, d\mathscr{H}^{N-1}(\omega)$, it is easy to see that this $f$ satisfies:

(i) $f \in C^\infty([0, r_0))$ (if we extend $f$ at the origin with the value $f(0) = u_\varepsilon(x)$)

(ii) $f'(r) = \displaystyle\fint_{S(0,1)} \sum_j \partial_j u_\varepsilon(x + r\omega)\omega_j\, d\mathscr{H}^{N-1}(\omega)$

(iii) $f''(r) = \displaystyle\fint_{S(0,1)} \sum_{j,k} \partial_j\partial_k u_\varepsilon(x + r\omega)\omega_j\omega_k\, d\mathscr{H}^{N-1}(\omega)$.

We obtain that $f(r) \ge f(0)$, $f'(0) = \displaystyle\fint_{S(0,1)} \sum_j \partial_j u_\varepsilon(x)\omega_j\, d\mathscr{H}^{N-1}(\omega)$, $f''(0) = \displaystyle\fint_{S(0,1)} \sum_{j,k} \partial_j\partial_k u_\varepsilon(x)\omega_j\omega_k\, d\mathscr{H}^{N-1}(\omega)$.

Since $f'(0)$ is the integral of an odd function, we find that $f'(0) = 0$. Consequently, we must have $f''(0) \geq 0$. Since (again by parity considerations) we have $\fint_{S(0,1)} \partial_j \partial_k u_\varepsilon(x) \omega_j \omega_k \, d\mathscr{H}^{N-1}(\omega) = 0$ when $j \neq k$, we find that

$$0 \leq f''(0) = \sum_j c_j \, \partial_j \partial_j u_\varepsilon(x).$$

Here, $c_j = \fint_{S(0,1)} \omega_j^2 \, d\mathscr{H}^{N-1}(\omega)$. Since the Hausdorff measure on the sphere is invariant w. r. t. isometries, we find that $c_j = c_k > 0$, $\forall\ j, k$. Finally, we have $f''(0) = c \, \Delta u_\varepsilon(x) \geq 0$, where $c > 0$. It follows that $\Delta u_\varepsilon(x) \geq 0$. □

**2.20 Corollary.** *A continuous subharmonic distribution satisfies the maximum principle.*

*Proof.* If we integrate over the inequality c) in the above theorem, we find that $u(x) \leq \fint_{B(x,r)} u$ for $0 < r < \text{dist}\,(x, \partial\Omega)$. In turn, this (new) inequality is the only ingredient in the proof of the maximum principle. □

**2.21 Proposition.** *If $u$, $v$ are continuous and subharmonic, then so is their maximum.*

*Proof.* Let $w = \max(u, v)$. Then $w$ is continuous. On the other hand, let $x \in \Omega$. If, say, $w(x) = u(x)$ and if we let $0 < r < \text{dist}\,(x, \partial\Omega)$, then $w(x) \leq \fint_{S(x,r)} u \leq \fint_{S(x,r)} w$. □

A similar result is the following

**2.22 Lemma** (Harmonic lifting)**.** *Let $B$ be a ball s. t. $\overline{B} \subset \Omega$. Let $v$ be continuous and subharmonic. Let $v_B = \begin{cases} v, & \text{in } \Omega \setminus B \\ w, & \text{in } \overline{B} \end{cases}$, where $w$ is the harmonic extension of $v_{|\partial B}$. Then $v_B$ is subharmonic.*

The map $v_B$ is the *harmonic lifting* of $v$ in $B$.

*Proof.* Let $C$ be a ball s. t. $\overline{C} \subset \Omega$. We have to prove that, if $U$ is harmonic in $C$ and continuous in $\overline{C}$, and if $\overline{v} \leq U$ on $\partial C$, then $\overline{v} \leq U$ in $C$. If $C \subset B$ or $C \cap \overline{B} = \emptyset$, the conclusion is obvious. Otherwise, we have $w \geq v$ in $B$, so that $U \geq v$ on $\partial C \cap B$. It follows that $U \geq v$ on $\partial C$, property which in turn implies that $U \geq v$ in $C$. Consequently, we have $U \geq w$ on $\partial(C \cap B)$, and therefore $U \geq w$ in $C \cap B$. Finally, we have $U \geq v_B$ in $C$. □

## 2.2 Existence of the harmonic extension

In this part, we examine the problem DPL. Given $g \in C(\partial\Omega)$, we look for a solution $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ of $\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$. We will give an iff condition for the existence of the harmonic extension in a bounded domain $\Omega$ (Perron's theorem). It turns out that this condition is difficult to check. We will next give sufficient conditions for existence. Though these conditions are not sharp, they have the merit of exhibiting large classes of domains in which the harmonic extension exists. Warning: the harmonic extension does not exist in an arbitrary domain. Second warning: we assume throughout this section that $\Omega$ is a bounded domain.

We start with an example of DPL which cannot be solved

**2.23 Example** (Dirichlet problem without solution)**.** Let $N = 2$, $\Omega = B(0,1) \setminus \{0\}$, $g(x) = \begin{cases} 1, & \text{if } |x| = 1 \\ 0, & \text{if } x = 0 \end{cases}$. Then $g$ does not have a harmonic extension.

Indeed, argue by contradiction. Since $g$ is rotation invariant, so has to be $u$ (by uniqueness). Thus $u$ is of the form $u(x) = f(|x|)$, where $f \in C([0,1]) \cap C^2((0,1))$, $f(0) = 0$, $f(1) = 1$ and (since $u$ is harmonic) $f''(r) + \frac{1}{r}f'(r) = 0$ in $(0,1)$. We find that $f(r) = a + b\ln r$ for some constants $a, b \in \mathbb{R}$. This is not compatible with the requirements $f(0) = 0$ and $f(1) = 1$.

Set

$$S_g = \{v \in C(\overline{\Omega})\ ;\ v \text{ continuous, subharmonic and } v_{|\partial\Omega} \le g\}.$$

Note that $S_g$ is non empty (take $v = -C$, for large $C$). Note also that, by the maximum principle, we have $v \le \sup g$, $\forall\ v \in S_g$.

**2.24 Theorem** (Perron)**.** *Let* $u = \sup\limits_{v \in S_g} v$. *Then*

*a) u is harmonic*

*b) If g admits a harmonic extension, then this extension is u.*

We say that this $u$ (which, unlike the harmonic extension, always exists) is the *Perron solution* to DPL.

*Proof.* Part b) is a straightforward consequence of a): if $w$ is the harmonic extension of $g$, then $w \in S_g$ and thus $w \leq u$. On the other hand, each $v \in S_g$ satisfies $v \leq w$, so that $u \leq w$.
Let $x \in \Omega$. Consider a ball $B$ s. t. $x \in B \subset \overline{B} \subset \Omega$. Consider a sequence $(v_n) \subset S_g$ s. t. $v_n(x) \to u(x)$. We may assume that the sequence $(v_n)$ is uniformly bounded. Indeed, on the one hand the maximum principle implies that $v_n \leq \sup\limits_{\partial\Omega} g$. On the other hand, we may replace $v_n$ by $\max\{v_n, \inf\limits_{\partial\Omega} g\}$. Finally, we may also assume that $v_n$ is harmonic in $B$: for otherwise, replace $v_n$ by $(v_n)_B$. Since the $v_n$'s are harmonic and uniformly bounded in $B$, the sequence $(v_n)$ converges, up to a subsequence, uniformly on compacts of $B$ to some harmonic function $v$.

If we prove that $v = u$ in $B$, then we are done (recall that we want to prove that $u$ is harmonic). Argue by contradiction: otherwise, there is some $y \in B$ s. t. $v(y) \neq u(y)$, which implies that $v(y) < u(y)$. Consider a sequence $(w_n) \subset S_g$ s. t. $w_n(y) \to u(y)$. The sequence of subharmonic functions defines by $V_n = (\max\{v_n, w_n\})_B$ converges, up to a subsequence and uniformly on the compacts of $B$, to a harmonic function $V$. Clearly, we have $v(x) = V(x)$ and $v \leq V$. The maximum principle implies that $v = V$ in $B$. This contradicts the fact that $v(y) < V(y) = u(y)$. □

**2.25 Definition.** A barrier at $x_0 \in \partial\Omega$ is a function $w : \overline{\omega} \to \mathbb{R}$ s. t.:

a) $w(x_0) = 0$

b) $w > 0$ in $\overline{\Omega} \setminus \{x\}$

c) $w$ is continuous and superharmonic.

**2.26 Definition.** If each $g \in C(\partial\Omega)$ has a harmonic extension, we say that DPL is solvable in $\Omega$.

**2.27 Theorem** (Perron)**.** *DPL is solvable in $\Omega$ iff there is a barrier at each point $x_0 \in \partial\Omega$.*

*Proof.* $\Longrightarrow$ Let $x_0 \in \partial\Omega$. Let $g(x) = |x - x_0|$, $x \in \partial\Omega$. Then the harmonic extension $w$ of $g$ is a barrier at $x_0$ (the fact that $w > 0$ in $\overline{\Omega} \setminus \{x_0\}$ follows from the maximum principle).
$\Longleftarrow$ Let $x_0 \in \partial\omega$ and let $w$ be a barrier at $x_0$. Let $g \in C(\partial\Omega)$ and let $\varepsilon > 0$. It is easy to see that, for large $C$, we have $g(x_0) - \varepsilon - Cw \leq g \leq g(x_0) + \varepsilon + Cw$ on $\partial\Omega$. The maximum principle implies that, for each $v \in S_g$, we have $v \leq g(x_0) + \varepsilon + Cw$ in $\overline{\Omega}$. It follows that the Perron solution $u$ satisfies $u \leq g(x_0) + \varepsilon + Cw$ in $\overline{\Omega}$ . On the other hand, we have $g(x_0) - \varepsilon - Cw \in S_g$,

and therefore $u \geq g(x_0) - \varepsilon - Cw$. We find that $g(x_0) - \varepsilon - Cw(x) \leq u(x) \leq g(x_0) + \varepsilon + Cw(x)$ in $\overline{\Omega}$. If we let first $x \to x_0$, next $\varepsilon \to 0$, we find that $\lim\limits_{x \to x_0} u(x) = g(x_0)$. □

**2.28 Proposition.** *Assume that, for each $x_0 \in \partial\Omega$, there is some ball $B$ s. t. $\overline{B} \cap \overline{\Omega} = \{x_0\}$. Then the DPL is solvable in $\Omega$.*

If the above condition is fulfilled, we say that $\Omega$ satisfies the *exterior sphere condition.*

*Proof.* Assume that $B = B(y, r)$. Then $x \mapsto w(x) = E(x - y) - E(x_0 - y)$, where $E$ is the fundamental solution of $-\Delta$, is a barrier at $x_0$. □

**2.29 Corollary.** *DPL is solvable in a convex domain.*

*Proof.* Let $x_0 \in \partial\omega$, with $\Omega$ convex. By the geometric form of the Hahn-Banach theorem, there is some hyperplane $H$ separating $\{x_0\}$ from $\Omega$. Up to an isometry, we may assume that $H = \{x_N = 0\}$, $x_0 = 0$ et $\Omega \subset \{x_N \geq 0\}$. We find that $\overline{\Omega} \subset \{x_N \geq 0\}$, and thus $B(-e_N, 1)$ is an exterior sphere to $\Omega$ at $x_0$. □

**2.30 Corollary.** *DPL is solvable in $C^2$ domains.*

*Proof.* By the next result, a $C^2$ domain has the exterior sphere property. □

**2.31 Lemma.** *A $C^2$ open set satisfies the exterior sphere condition.*

No boundedness is required here.

*Proof.* Let $x_0 \in \partial\Omega$. Consider an open set $V$ containing $x_0$ and s. t., for some $\varphi \in C^2(V; \mathbb{R})$ we have: $\varphi = 0$ in $\partial\Omega \cap V$, $\varphi > 0$ in $\Omega \cap V$, $\varphi < 0$ in $(\mathbb{R}^N \setminus \overline{\Omega}) \cap V$ and $\nabla\varphi \neq 0$. With no loss of generality, we may assume that $x_0 = 0$, $\nabla\varphi(0) = e_N$. We are going to prove that $B(-\varepsilon e_N, \varepsilon)$ is exterior for sufficiently small $\varepsilon$. This is equivalent to proving that, for such a $\varepsilon$ and for $y \in \mathbb{S}^{N-1}$ , we have $\varphi(-\varepsilon e_N + \varepsilon y) < 0$ except when $y = e_N$. Taylor's formula gives

$$\begin{aligned}\varphi(-\varepsilon e_N + \varepsilon y) &= \varepsilon(y_N - 1) + O(\varepsilon^2 |y - e_N|^2)\\ &\leq \varepsilon(y_N - 1) + C\varepsilon^2((y_N - 1)^2 + (1 - y_N^2))\\ &\leq \varepsilon(1 - y_N)(-1 + 4C\varepsilon).\end{aligned}$$

For small $\varepsilon$, we find that $\varphi(-\varepsilon e_N + \varepsilon y) < 0$ unles $y_N = 1$. In turn $y_N = 1$ is the same as $y = e_N$. □

**2.32 Proposition.** *Let $B$ be a ball centered at $x_0 \in \partial\Omega$. Assume that there is some $w_0$ s. t. $w_0 > 0$ in $\overline{\Omega} \cap \overline{B} \setminus \{x_0\}$, $w \in C(\overline{\Omega} \cap \overline{B})$, $w(x_0) = 0$ and $w$ superharmonic in $\Omega \cap B$. Then there is a barrier at $x_0$.*

A function $w_0$ as above is a *local barrier*. Thus the above proposition can be rephrased as local barrier implies barrier.

*Proof.* Let $m = \min\{w_0(x); x \in \partial B \cap \overline{\Omega}\} > 0$. Then $w := \min\{w_0, m\}$ is a barrier at $x_0$. □

**2.33 Definition.** A cone at $x_0$ is a set of the form

$$C = C_{v,a} := \{x \in \mathbb{R}^N; (x - x_0) \cdot v \geq a|x - x_0|\}.$$

Here, $a \in (0, 1)$ and $v$ is a unit vector.
A local cone is a set of the form $K = K_{v,a,r} = \{x \in C_{v,a}; |x| \leq r\}$.
An open set satisfies the exterior cone condition if for each $x_0 \in \partial\Omega$ there is a local cone $K$ with vertex $x_0$ s. t. $K \cap \overline{\Omega} = \{x_0\}$.

The proof of the next result is only sketched. A full proof would require ingredients which we will not develop here.

**2.34 Theorem.** *If $\Omega$ satisfies the exterior cone condition, then DPL is solvable in $\Omega$.*

*Proof.* Let $-\Delta_{\mathbb{S}^{N-1}}$ denote the Laplace-Beltrami operator on $\mathbb{S}^{N-1}$. We will make use of two facts. The first one is the "separation of variables" formula

$$\begin{aligned}\Delta(f(r)g(\omega)) = &\left(f''(r) + \frac{N-1}{r} f'(r)\right) g(\omega) \\ &+ \frac{f(r)}{r^2} \Delta_{\mathbb{S}^{N-1}} g(\omega), \quad r > 0, \omega \in \mathbb{S}^{N-1},\end{aligned} \tag{2.7}$$

valid if $f \in C^2((0, +\infty))$ and $g \in C^2(\mathbb{S}^{N-1})$. The same holds for $\omega \in U$ provided $g \in C^2(U)$, where $U$ is an open subset of $\mathbb{S}^{N-1}$. This formula is the generalization to higher dimensions of the identity

$$\Delta(u(r, \theta)) = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta},$$

valid in two dimensions.

The second ingredient is the following: given $U \subset \mathbb{S}^{N-1}$ a smooth open domain (different from $\mathbb{S}^{N-1}$), there is some $\mu > 0$ and there is some $g \in C^\infty(\overline{U})$ satisfying $g > 0$ in $U$, $g = 0$ on $\partial U$ and $-\Delta_{\mathbb{S}^{N-1}} g = \mu g$. Though we do not present here the proof of this result, it is a straightforward adaptation to the "curved" case of the fact (proved in the regularity theory part) that the same holds in a smooth bounded domain $U \subset\ R^N$ for the standard Laplace operator $-\Delta$.

We the help of these two ingredients, we may conclude as follows:. Assume, e. g., that $x_0 = 0 \in \partial\Omega$. Let $U = \{x \in \mathbb{S}^{N-1}; x \cdot v < a\}$. Let $g$, $\mu$ associated as above to $U$. Let $\lambda > 0$ solve $\lambda(\lambda + N - 2) = \mu$. Let $u(x) = |x|^\lambda g(x/|x|)$. Then we claim that $u$ is a local barrier at the origin. The only fact which requires a proof is $-\Delta u \geq 0$ in $\Omega \cap B$, where $B$ is a small ball centered at the origin. This follows from the fact that the choice of $\mu$ combined with (2.7) implies that we actually have $\Delta u = 0$ in $\{r\omega; r > 0, \omega \in U\}$. □

**2.35 Remark.** It would be nice to find an elementary self contained proof of the above result. I do not know such a proof, but it could possibly be obtained along the following lines. We may assume that $x_0 = 0$ and $v = e_N$. Look for a local barrier of the form $|x|^\lambda g(x_N/|x|)$. Here, $\lambda > 0$ is to be fixed later and $g$ is smooth in $[-1, 1-\varepsilon]$, for sufficiently small $\varepsilon$. Plug this into the equation. It follows that it suffices to find $g$ satisfying $g(1-\varepsilon) = 0$, $g > 0$ in $[-1, 1-\varepsilon)$ and

$$(1 - t^2)g''(t) + (N - 2)tg'(t) \leq -\mu g(t) \tag{2.8}$$

for some $\mu > 0$. Finding a simple solution to the above result requires then finding an explicit solution to (2.8).

**2.36 Corollary.** *DPL is solvable in Lipschitz domains.*

*Proof.* Possibly after taking an isometry, we may assume that $x_0 = 0$. We may also assume that there is some $\delta > 0$ and, in a neighborhood $V$ of the origin in $\mathbb{R}^{N-1}$, there is some Lipschitz map $\varphi$ s. t. $\Omega \cap [V \times (-\delta, \delta)] = \{(x', x_N); x_N > \varphi(x')\}$. Let $v = -e_N$ and let $a \in (0, 1)$ to be fixed later. If $x \cdot v \geq a|x|$, then $x_N \leq -\dfrac{a}{(1-a^2)^{1/2}}|x'|$. If, in addition, we have $|x'|$ sufficiently small, then $y_N \geq -C|x'|$, whenever $(x', y_N) \in \overline{\Omega}$; here, $C$ is the Lipschitz constant of $\varphi$. With the choice $\dfrac{a}{(1-a^2)^{1/2}} > C$, we find that the local cone $K_{-e_N,a,r}$ intersects $\overline{\Omega}$ only at the origin (provided $r$ is sufficiently small). □

We discuss, in the exercises section, several existence/non existence results. For a more detailed discussion, see, e. g., Gilbarg and Trudinger, pp. 26-28 et pp. 206-209. (And the references there, especially those related to potential theory.)

## 2.3 Exercises

**2.37 Exercise.** We discuss here an alternative proof of the fact that harmonic functions are analytic. We take for granted the fact that they are continuous.

a) Prove, by induction on $m := |\alpha|$, that, if $u \in C(\overline{B}(x_0, r))$ is harmonic in $B(x_0, r)$, then

$$|\partial^\alpha u(x_0)| \le \frac{m^m N^m}{r^m} \sup_{\overline{B}(x_0,r)} |u|.$$

Hint: use Poisson's formula.

b) Prove that the Taylor series of $u$ at some point $x_0 \in \Omega$ converges to $u$ in $B(x_0, \delta)$, for sufficiently small $\delta > 0$.
Hint: use Stirling's formula in order to estimate $n$!

**2.38 Exercise** (Strong maximum principle)**.** In a domain, a harmonic function having a local maximum or minimum is constant.

**2.39 Exercise** (Schwarz's reflection principle)**.** Let $u \in C(\overline{\mathbb{R}^N_+})$ be harmonic in $\mathbb{R}^N_+$. Assume that $u$ vansishes on $\partial\mathbb{R}^N_+$. We extend $u$ to $\mathbb{R}^N$ by setting $v(x', x_N) = \begin{cases} u(x', x_N), & \text{if } x_N \ge 0 \\ -u(x', -x_N), & \text{if } x_N \le 0 \end{cases}$.

a) Prove that, if $K \Subset \mathbb{R}^{N-1}$, then $x_N \nabla u(x', x_N) \to 0$ as $x_N \searrow 0$ uniformly in $x' \in K$

b) Prove that $v$ is harmonic

c) Prove that, if $u$ is bounded, then $u = 0$

d) Consequence: DPL has at most one bounded solution in $\mathbb{R}^N_+$. What if we remove the boundedness condition?

**2.40 Exercise** (Poisson's formula in $\mathbb{R}^N_+$)**.** We identify $\partial\mathbb{R}^N_+$ with $\mathbb{R}^{N-1}$. Let $g \in C_b(\mathbb{R}^{N-1})$. Prove that, in $\mathbb{R}^N_+$, the DPL with datum $g$ has exactly one bounded solution, given by

$$u(x) = c_N x_N \int_{\mathbb{R}^{N-1}} \frac{g(y)}{|x - y|^N}\, dy.$$

Here, the constant $c_N$ is given by $c_N = 1/\left(\sigma_{N-2} \int_0^\infty \frac{r^{N-2}}{(1 + r^2)^{N/2}}\, dr\right)$.

**2.41 Exercise.** a) If $u$ is harmonic and non negative in $B(0,R)$, prove that $u(x) \leq 3^N u(y)$ for each $x, y \in B(0, R/4)$

b) (Harnack's inequality) Let $u$ be harmonic and non negative in the domain $\Omega$. If $K \Subset \Omega$, prove that there is some $C > 0$ depending only on $K$ (thus not on $u$) s. t. $u(x) \leq Cu(y)$, $\forall\ x, y \in K$.

**2.42 Exercise.** We give here explicit examples of operators having a fundamental solution which is smooth outside the origin. The general theory of such (hypoelliptic) operators can be found in Hörmander, vol. II, Chapter 11.
Let $u \in \mathscr{D}'$ satisfying either the homogeneous heat equation $\partial_t u - \Delta_x u = 0$ in $\mathbb{R}^N \times \mathbb{R}$ or the Cauchy Riemann equation $\partial_x u + \imath \partial_y u = 0$ in $\mathbb{R}^2$. Prove that $u \in C^\infty$.

**2.43 Exercise** (Characterization of subharmonic distributions)**.** Recall that a function $u : \Omega \to [-\infty, \infty[$ is *upper semi continuous* (u. s. c.) if $u^{-1}(]-\infty, a[)$ is open for every $a \in \mathbb{R}$.

a) If $f$ is u. s. c., then $u$ is Borel and achieves its maximum on each compact $K \Subset \Omega$

b) If $S \subset \Omega$ is a sphere, prove that $\displaystyle\int_S u$ makes sense

c) Prove that the (point) limit of a non increasing sequence of continuous functions is u. s. c.

d) Prove that, if $\mu$ is a finite positive compactly supported Borel measure, and if $E$ is the fundamental solution of $-\Delta$, then $-E * \mu$ is u. s. c.

e) Prove that a subharmonic distribution is necessarily given by a locally integrable u. s. c. function

f) Give an example of two different locally integrable u. s. c. functions, which are equal a. e.

g) Prove that, given a locally integrable u. s. c. function $u$, there is a minimal u. s. c. function $v$ which equals $u$ a. e. $g$ u. s. c. Equivalently, the function $v = \inf\{w\ ;\ w \text{ u. s. c.}, w = f \text{ p. p.}\}$ is u. s. c. and equals $u$ a. e. In addition, prove that the above $v$ is, in each compact $K \Subset \Omega$, the point limit of a non increasing sequence of continuous functions
Hint : consider $\displaystyle⨍_{B(x,1/n)} u$
We say that $v$ is *minimal*. This is motivated by the fact that, if $w = v$ a. e. and $w$ u. s. c., then $w \geq v$

h) Let $u$ be locally integrable u. s. c. and minimal. Prove that the following are equivalent:

   (i) $u$ is a subharmonic distribution

   (ii) If $B$ is a ball s. t. $\overline{B} \subset \Omega$, and if $v \in C(\overline{B})$ is harmonic in $B$ and satisfies $u \leq v$ on $\partial B$, then $u \leq v$ in $B$

   (iii) for $x \in \Omega$ and $0 < r < \operatorname{dist}(x, \partial\Omega)$, we have $u(x) \leq \displaystyle⨍_{S(x,r)} u$.

**2.44 Exercise.** Assume that $N = 2$ and that $\overline{\Omega} \cap \mathbb{R}^- = \{0\}$. Prove that $z \mapsto w(z) = -\operatorname{Re} \dfrac{1}{\ln z}$ is a local barrier at the origin.

**2.45 Exercise.** Prove that DPL is solvable in continuous domains in $\mathbb{R}^2$.

**2.46 Remark.** When $N \geq 3$, DPL need not be solvable in continuous domains. Lebesgue found an explicit counterexample, detailed in Courant and Hilbert, vol. II, pp. 303-305.

**2.47 Exercise.** Let $\Omega = \{x\ ;\ \rho < |x| < R\}$. Solve $\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{if } |x| = R \\ u = 1 & \text{if } |x| = \rho \end{cases}$.

**2.48 Exercise.** We discuss here simple aspects of the capacity. In particular, this will allow us to have more complicated examples of DPL without solution. However, we do not discuss here an iff theory. For such a tool (Wiener's criterion), see the references in Gilbarg and Trudinger.

Let $K \Subset \Omega$, with $\Omega$ Lipschitz. The *capacity* of $K$ with respect to $\Omega$ is

$$\operatorname{cap}(K) = \operatorname{cap}(K;\Omega) = \inf\left\{\int_\Omega |\nabla v|^2\ ;\ v \in H^1(\Omega) \cap C(\overline{\Omega}), v_{|\partial\Omega} = 0,\ v_{|K} = 1\right\}.$$

Prove that the capacity satisfies:

a) $\operatorname{cap}(K \cup L) \leq \operatorname{cap}(K) + \operatorname{cap}(L)$

b) If $L \subset K$, then $\operatorname{cap}(L) \leq \operatorname{cap}(K)$

c) If $\Omega \subset U$, with $U$ open, then $\operatorname{cap}(K;\Omega) \geq \operatorname{cap}(K;U)$

d) If $N \geq 2$, then the capacity of a point is zero. When $N = 1$, the capacity of a point is positive

e) The capacity of a compact contained in a hyperplane and having positive $\mathscr{H}^{N-1}$ measure is positive

f) The capacity of a compact contained in an $(N-2)$ plane is zero

g) $\text{cap}(K) = 0 \Longrightarrow \overset{\circ}{K} = \emptyset$

h) If $K_n \searrow K$, then cap $(K_n) \searrow$ cap $(K)$.

**2.49 Exercise.** Here, we discuss the connection between capacity and the solvability of DPL. Let $\omega \Subset U$ be two smooth bounded open sets. Set $\Omega = U \setminus \overline{\omega}$. Let $u$ solve $(P)\begin{cases} -\Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial U \\ u = 1 & \text{on } \partial\omega \end{cases}$.

a) Prove that $0 < u < 1$ dans $\omega$

b) For $0 < t < 1$, let $\Omega_t = \{0 < u < t\}$ and $\Sigma_t = \{u = t\}$. Let $t \in (0,1)$ be a regular value of $u$. Prove that, for such $t$, we have $\partial\Omega_t = \partial U \cup \Sigma_t$

c) Let $t \in (0,1)$ be a regular value of $u$. We consider, on $\Sigma_t$, the orientation induced by the outward normal to $\Omega_t$. Prove that $\displaystyle\int_{\Sigma_t} \frac{\partial u}{\partial \nu}$ does not depend on $t$

d) Prove that $\displaystyle\int |\nabla u|^2 < \infty$

e) Prove that $u$ is a weak solution of $(P)$, i. e., that $\displaystyle\int \nabla u \cdot \nabla v = 0$ for each $v \in H_0^1(\Omega)$.

f) Prove that $\displaystyle\int |\nabla u|^2 = \text{cap } (\overline{\omega}; U)$

g) Generalize the above results to the case where $\overline{\omega}$ is replaced by an arbitrary compact $K \Subset U$: if the problem $\begin{cases} -\Delta u = 0 & \text{in } U \setminus K \\ u = 0 & \text{on } \partial U \\ u = 1 & \text{on } K \end{cases}$ has a solution in $C(\overline{U})$, then $\displaystyle\int_{U \setminus K} |\nabla u|^2 = \text{cap } (K; U)$

h) Prove that, if cap $(K) = 0$, then the problem $\begin{cases} -\Delta u = 0 & \text{in } \omega \setminus K \\ u = 0 & \text{on } \partial\omega \\ u = 1 & \text{on } K \end{cases}$ has no solution.

**2.50 Exercise.** Prove that, in $\mathbb{R}^2$, the problem $\begin{cases} -\Delta u = 0 & \text{if } |x| > 1 \\ u(x) = 1 & \text{if } |x| = 1 \\ \lim\limits_{|x|\to\infty} u(x) = 0 & \end{cases}$

has no continuous solution.

**2.51 Exercise.** Let $g \in C(\mathbb{S}^1)$. If $\sum a_n e^{\imath n\theta}$ is the Fourier series of $g$, prove that the harmonic extension of $g$ is given by $u(z) = \sum\limits_{n\geq 0} a_n z^n + \sum\limits_{n<0} a_n \overline{z}^n$.

**2.52 Remark.** The above formula generalizes to dimension 3 (or higher). Instead of a Fourier series decomposition, the analog formula involves spherical harmonics (in this setting, this was discovered by Laplace). For a brief account of spherical harmonics (in 3D), see, e. g., Courant and Hilbert, vol. II, pp. 510-521.

# 3 Singular integrals

In this section, we discuss the properties of solutions of the Poisson equation $-\Delta u = f$ (PE, hereafter) in $\mathbb{R}^N$. Here, $f$ belongs to one of the spaces $L^p$ or $C^{0,\alpha}$. Throughout this section we assume that $f$ is compactly supported. Some notations: we let $L^p_c$ denote the space of compactly supported $L^p$ functions, and $L^p_K$ the space of $L^p$ functions supported in some compact $K$. Similar notations in the $C^\alpha$ context.

**3.1 Proposition.** *If $f \in L^p_c$ for some $1 \leq p \leq \infty$, then $u := E * f$ solves PE.*
*In particular, the same holds if $f \in C^\alpha_c$.*

*Proof.* This is true if $f \in C^\infty_c(\mathbb{R}^N)$. Note that $E \in L^1_{loc}$ (check!). Therefore, if $f_n \to f$ in $L^1_K$, then $(E * f_n)$ converges to $E * f$ in $L^1_{loc}$, and thus also in $\mathscr{D}'(\mathbb{R}^N)$. Using this argument, we find that, if $f \in L^p_c$, then $E * f \in L^p_{loc}$ and $-\Delta(E * f) = f$. $\square$

The two main results of this section are:

a) if $1 < p < \infty$ and if $f \in L^p$, then $u \in W^{2,p}_{loc}$ (Calderón-Zygmund's theorem)

b) if $0 < \alpha < 1$ and if $f \in C^\alpha$, then $u \in C^{2,\alpha}_{loc}$ (Korn's theorem)

These results are sharp in the following sense:

a) "loc" is necessary. E. g., if $f \in C_c^\infty(\mathbb{R}^N)$, it may happen that $|u(x)| \to \infty$ as $|x| \to \infty$, and thus $u$ may not belong to any global version of the above spaces.

b) if $\alpha = 0$ or $1$, respectively if $p = 1$ or $\infty$, the corresponding would be results are wrong.

We give counterexamples in the exercise section.

## 3.1 Preliminaries to the Calderón-Zygmund theory

For $1 \le j, k \le N$, let $T : L^p_c \to \mathscr{D}'(\mathbb{R}^N)$, $Tf = \partial_j\partial_k(E * f)$.

**3.2 Lemma.** *If $f \in L^2_c$, then*

$$Tf = \mathscr{F}^{-1}\left( -\frac{\xi_j\xi_k}{|\xi|^2}\hat{f}\right). \tag{3.1}$$

*Consequently, $T$ has a (unique) continuous extension to $L^2(\mathbb{R}^N)$, given by the l. h. s. of (3.1).*
*In addition, $T$ is self adjoint in $L^2$, i. e.,*

$$\int Tf\ \overline{g} = \int f\ \overline{Tg}, \quad \forall\ f, g \in L^2. \tag{3.2}$$

*Proof.* The r. h. s. of (3.1) is continuous from $L^2$ into $L^2$ (and thus into $\mathscr{D}'(\mathbb{R}^N)$), by Plancherel's theorem. On the other hand, for fixed $K$, the l. h. s. of (3.1) is continuous from $L^2_K$ into $\mathscr{D}'(\mathbb{R}^N)$. Therefore, it suffices to prove (3.1) when $f \in C_c^\infty$. The proof will make a moderate use of properties of temperate distributions, for which we send to Hörmander. Since $(1 + |x|^2)^{-N}E \in L^1$, we have $E \in \mathscr{S}'$, and thus $\partial_j\partial_k E \in \mathscr{S}'$. It follows that $\widehat{Tf} = \widehat{\partial_j\partial_k E}\hat{f}$, and thus it suffices to prove that $\widehat{\partial_j\partial_k E} = -\dfrac{\xi_j\xi_k}{|\xi|^2}$. Write $E = E_1 + E_2$, with $E_1 = \Phi E$, $E_2 = (1 - \Phi)E$. Here, $\Phi \in C_c^\infty$ is s. t. $\Phi \equiv 1$ near the origin. Then $\widehat{\partial_j\partial_k E} = \widehat{\partial_j\partial_k E_1} + \widehat{\partial_j\partial_k E_2} \in C^\infty + L^2$. Indeed, $\partial_j\partial_k E$ is compactly supported, while $\partial_j\partial_k E_2 \in L^2$. On the other hand, we have

$-\Delta\partial_j\partial_k E = \partial_j\partial_k\delta$, so that $|\xi|^2\widehat{\partial_j\partial_k E} = -\xi_j\xi_k$. Using Schwartz's result on the structure of distributions supported at the origin, we find that

$$\widehat{\partial_j\partial_k E} = -\frac{\xi_j\xi_k}{|\xi|^2} + \sum_{\text{finite}} c_\alpha \partial^\alpha\delta.$$

The coefficients $c_\alpha$ are vanishing, since $\widehat{\partial_j\partial_k E} \in C^\infty + L^2$. This proves the first part of the lemma.

As for (3.2), it follows immediately from Plancherel's theorem, since we have:

$$\int Tf\ \overline{g} = (2\pi)^{-N}\int \widehat{Tf}\ \overline{\hat{g}} = (2\pi)^{-N}\int \frac{\xi_j\xi_k}{|\xi|^2}\hat{f}\ \overline{\hat{g}} = -(2\pi)^{-N}\int \hat{f}\,\overline{\frac{\xi_j\xi_k}{|\xi|^2}\hat{g}}$$
$$= (2\pi)^{-N}\int \hat{f}\ \overline{\widehat{Tg}} = \int f\ \overline{Tg}.$$

□

**3.3 Lemma.** *Let $f \in L^p_c$. If $x \notin$ supp $f$, then*

$$Tf(x) = \int K(x-y)f(y)\,dy, \tag{3.3}$$

*where* $K(x) = -\frac{1}{\sigma_N}\left(\frac{\delta_{j,k}}{|x|^N} - \frac{Nx_jx_k}{|x|^{N+2}}\right)$.
*In addition, $K$ satisfies*

$$|K(x-y) - K(x)| \leq \frac{C|y|}{|x|^{N+1}}, \qquad \text{if } |y| < 1/2|x|. \tag{3.4}$$

*Proof.* Let $K =$ supp $f$. If $\Omega$ is an open bounded set s. t. $\overline{\Omega}\cap K = \emptyset$, then the (point) derivatives of $E(x-y)f(y)$ w. r. t. $x$ satisfy

$$|\partial_x^\alpha(E(x-\cdot)f(\cdot))| \leq c_\alpha|f(\cdot)| \in L^1, \quad x \in \Omega.$$

Therefore, $E * f \in C^\infty(\Omega)$. In addition, we may differentiate twice under the integral sign and find that, in $\Omega$, the second order derivatives of $E * f$ are given by $\partial_j\partial_k(E*f) = \int \partial_j\partial_k E(x-y)f(y)\,dy$. This implies the first part of the lemma, since $\partial_j\partial_k E = K$.

In order to prove (3.4), we first note that $|\nabla K(z)| \leq C|z|^{-N-1}$. Consequently, if $x, y$ are s. t. $|y| < 1/2|x|$, then we have

$$|K(x-y) - K(x)| \leq |y| \sup_{z\in[x-y,x]} |\nabla K(z)| \leq C|y| \sup_{z\in[x-y,x]} |z|^{-N-1} \leq \frac{C|y|}{|x|^{N+1}}.$$

□

The next result is a special case of Marcinkiewicz' interpolation theorem. For its general form, we send to Stein and Weiss, pp. 183-205.

**3.4 Theorem** (Marcinkiewicz). *Let $1 < q < \infty$ and let $T : L^1 \cap L^q(\mathbb{R}^N) \to \mathfrak{M}$ be a linear application s. t.*

$$|\{|Tf| > t\}| \leq \frac{C_1 \|f\|_{L^1}}{t}, \quad \forall\ f \in L^1 \cap L^q, \forall\ t > 0 \tag{3.5}$$

*and*

$$|\{|Tf| > t\}| \leq \frac{C_q \|f\|_{L^q}^q}{t^q}, \quad \forall\ f \in L^1 \cap L^q, \forall\ t > 0. \tag{3.6}$$

*Then $T$ has a (unique) continuous extension from $L^p$ into $L^p$, for $1 < p < q$.*

*Equivalently, we have*

$$\|Tf\|_{L^p} \leq C \|f\|_{L^p}, \quad \forall\ f \in L^1 \cap L^q, \forall\ 1 < p < q. \tag{3.7}$$

*Proof.* Let $t > 0$ and let $f \in L^1 \cap L^q$. Split $f = f_1 + f_2$, with $f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| > t \\ 0, & \text{otherwise} \end{cases}$ and $f_2 = f - f_1$. Since $Tf = Tf_1 + Tf_2$, we have $|Tf| > t \Longrightarrow |Tf_1| > t/2$ or $|Tf_2| > t/2$. We find that

$$\begin{aligned} |\{|Tf| > t\}| &\leq |\{|Tf_1| > t/2\}| + |\{|Tf_2| > t/2\}| \\ &\leq \frac{2C_1}{t} \|f_1\|_{L^1} + \frac{2^q C_q^q}{t^q} \|f_2\|_{L^q}^q. \end{aligned} \tag{3.8}$$

On the other hand, by Fubini we have

$$\begin{aligned} p \int_0^\infty t^{p-1} |\{|Tf| > t\}|\, dt &= p \int_0^\infty t^{p-1} \int_{\{|Tf| > t\}} dx dt \\ &= \int_{\mathbb{R}^N} \int_0^{|Tf|} p t^{p-1}\, dt = \|Tf\|_{L^p}^p. \end{aligned}$$

Equivalently, if $F = F_f$ is the *distribution function* of $f$, defined by $F(\alpha) = |\{|f| > \alpha\}|$, $\alpha > 0$, then $\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} F_f(\alpha)\, d\alpha$.

It follows that

$$\|Tf\|_{L^p}^p = p \int t^{p-1} |\{|Tf| > t\}| \leq 2pC_1 \int t^{p-2} \|f_1\|_{L^1} + 2^q p C_q^q \int t^{p-q-1} \|f_2\|_{L^q}^q.$$

We next note that, with $f_1$ and $f_2$ as above, we have $F_{f_1}(\alpha) = \begin{cases} F_f(\alpha), & \text{if } \alpha \geq t \\ F_f(t), & \text{if } \alpha < t \end{cases}$.

Similarly, we have $F_{f_1 2}(\alpha) = \begin{cases} 0, & \text{if } \alpha \geq t \\ F(\alpha) - F(t), & \text{if } \alpha < t \end{cases}$. We find that

$$\|f_1\|_{L^1} = tF(t) + \int_t^\infty F(\alpha)\, d\alpha, \ \|f_2\|_{L^q}^q = q\int_0^t \alpha^{q-1}F(\alpha)\, d\alpha - t^qF(t). \quad (3.9)$$

By combining (3.8) with (3.9), we find, via Fubini's theorem:

$$\|Tf\|_{L^p}^p \leq p\left(\frac{2C_1}{p-1} + \frac{2^qC_q^q}{q-p}\right)\|f\|_{L^p}^p.$$

□

**3.5 Corollary.** *Let $1 < q < \infty$ and let $T : L^1 \cap L^q(\mathbb{R}^N) \to \mathscr{D}'(\mathbb{R}^N)$ be a linear continuous application from $L^1 \cap L^q$ (endowed with the $L^q$ norm) into $L^q$. If, in addition, $T$ satisfies (3.5), then $T$ has a (unique) continuous extension from $L^p$ into $L^p$ for $1 < p < q$.*

*Proof.* It suffices to check (3.6). This is done via Tchebychev's inequality:

$$|\{|Tf| > t\}| \leq \frac{\|Tf\|_{L^q}^q}{t^q} \leq \frac{C_q\|f\|_{L^q}^q}{t^q}.$$

□

**3.6 Theorem** (Calderón-Zygmund decomposition; first form)**.** *Let $f \in L^1(\mathbb{R}^N)$ and let $t > 0$. Then there is a sequence $(C_n)$ of disjoint cubes s. t.*

*a)* $|f(x)| \leq t$ *a. e. in* $\mathbb{R}^N \setminus \left(\bigcup_n C_n\right)$

*b) for each n,* $C^{-1}t \leq \fint_{C_n} |f(x)|\, dx \leq Ct$

*c)* $\sum_n |C_n| \leq \frac{C\|f\|_{L^1}}{t}$.

Here, $C$ depends only on $N$. The proof relies on the following

**3.7 Theorem** (Lebesgue's differentiability theorem)**.** *Let* $f \in L^1_{loc}(\mathbb{R}^N)$*. Then, for a. e.* $x \in \mathbb{R}^N$*, we have* $\displaystyle\lim_{x\in C,\,|C|\to 0} \fint_C f(y)\,dy = f(x)$*.*
*Here,* $C$ *denotes a cube, and the limit is taken as the size of the cube tends to* 0*.*

For a proof, see, e. g., Evans and Gariepy. We will come back to this in the exercise section, and prove that Lebesgue's differentiability theorem can be derived from a special case of the maximal function theorem.

*Proof.* Let $l > 0$ be s. t. $l^N > \dfrac{\|f\|_{L^1}}{t}$. We cover $\mathbb{R}^N$ with a grid of disjoint cubes of size $l$. Let $\mathscr{F}_1$ denote the family of these cubes. We cut each such cube in $2^N$ equal cubes. Let $\mathscr{F}_2$ be the family of these smaller cubes. We keep on cutting, and obtain inductively the families $\mathscr{F}_j$, $j \geq 1$.

We next start by throwing away all the cubes in $\mathscr{F}_1$. Let now $j = 2$. We keep a cube $C \in \mathscr{F}_2$ iff $\displaystyle\fint_C |f(x)|\,dx > t$. Inductively, for $j \geq 2$, we keep a cube $C \in \mathscr{F}_j$ iff all his ancestors were thrown and $\displaystyle\fint_C |f(x)|\,dx > t$. Let $\mathscr{F} = (C_n)$ be the (countable) family of the (disjoint) cubes which are kept, and set $A = \displaystyle\bigcup_n C_n$.

If $x \notin A$, then all the cubes containing $x$ have been thrown away. Thus $|f(x)| \leq t$ a. e. $\mathbb{R}^N \setminus A$, by Lebesgue's differentiability theorem. Let now $C \in \mathscr{F}$. Then $C \in \mathscr{F}_j$ for some $j \geq 2$. The (unique) cube $Q \in \mathcal{F}_{j-1}$ containing $C$ has been thrown away, so that

$$\fint_C |f(x)|\,dx \leq \frac{1}{|C|}\int_Q |f(x)|\,dx = 2^N \fint_Q |f(x)|\,dx \leq 2^N t.$$

Thus b) holds with $C = 2^N$.

Finally, c) follows from

$$\|f\|_{L^1} \geq \sum_n \int_{C_n} |f(x)|\,dx \geq C^{-1}\sum_n |C_n| t.$$

□

**3.8 Theorem** (Calderón-Zygmund decomposition, second form)**.** *Let* $f \in L^1(\mathbb{R}^N)$*. Let* $t > 0$ *and let* $(C_n)$ *be associated to* $f$ *and* $t$ *as in the previous theorem. Then* $f = g + \displaystyle\sum_n h_n$*, where:*

*a)* $g \in L^1$, $|g| \le Ct$ *a. e. and* $g = f$ *in* $\mathbb{R}^N \setminus \left( \bigcup_n C_n \right)$

*b)* $\operatorname{supp} h_n \subset C_n$

*c) for each* $n$*, we have* $\displaystyle\int_{C_n} h_n(x)\, dx = 0$

*d) for each* $n$*, we have* $\displaystyle\fint_{C_n} |h_n(x)|\, dx \le Ct$

*e)* $\displaystyle\|g\|_{L^1} + \sum_n \|h_n\|_{L^1} \le C\|f\|_{L^1}$.

*Proof.* Let $g(x) = \begin{cases} f(x), & \text{if } x \notin A \\ \displaystyle\fint_{C_n} f(y)\, dy, & \text{if } x \in C_n \end{cases}$ and $h_n(x) = f(x) - \displaystyle\fint_{C_n} f(y)\, dy$ if $x \in C_n$ ($h_n$ is extended with the value 0 outside $C_n$). It is straightforward that these functions have all the desired properties. □

## 3.2 The Calderón-Zygmund theorem

We start by stating and proving the case we are interested in. We will see later that we have more for our money. Recall that $Tf = \partial_j \partial_k E * f$.

**3.9 Theorem.** *[Calderón-Zygmund] The following hold:*

*a)* $T$ *satisfies*

$$|\{|Tf| > t\}| \le \frac{C\|f\|_{L^1}}{t}, \quad f \in L^2_c,\ t > 0. \tag{3.10}$$

*b) For* $1 < p < \infty$*, the operator* $T$*, initially defined in* $L^p_c$*, has a unique continuous extension from* $L^p$ *into* $L^p$*.*

*Proof.* The proof goes as follows:

a) we first prove that $T$ satisfies (3.10); this will be a consequence of the second version of the Calderón-Zygmund decomposition

b) the Marcinkiewicz interpolation theorem, combined with the continuity of $T$ from $L^2$ into $L^2$, implies b) when $1 < p < 2$

c) the case $2 < p < \infty$ is obtained using b) and the selfadjointness of $T$ in $L^2$.

We start by proving (3.10). Let $t > 0$. Consider the Calderón-Zygmund decomposition of $f$ at height $t$ : $f = g + \sum h_n$. Note that $g \in L^2$. Indeed, we have $|g| \le Ct$ a. e., so that

$$\int g^2 \le Ct \int |g| \le Ct\|f\|_{L^1}. \tag{3.11}$$

Next, we prove that the series $\sum h_n$ converges in $L^2$. Observing that the functions $h_n$ are mutually orthogonal in $L^2$, we are bound to prove that $\sum \|h_n\|^2_{L^2} < \infty$. Using the fact that

$$\|h_n\|_{L^2} = \left\| f - \fint_{C_n} f \right\|_{L^2(C_n)} \le \|f\|_{L^2(C_n)},$$

we find that $\sum \|h_n\|^2_{L^2} \le \sum \|f\|^2_{L^2(C_n)} \le \|f\|^2_{L^2}$. This implies that the series $\sum h_n$ converges.

Consequently, we may write $Tf = T(g + \sum h_n) = Tg + T\sum h_n$. We have

$$|\{|Tf| > t\}| \le |\{|Tg| > t/2\}| + |\{|T\sum h_n| > t/2\}|. \tag{3.12}$$

On the other hand, (3.11) implies that that

$$|\{|Tg| > t/2\}| \le \frac{C}{(t/2)^2}\|g\|^2_{L^2} \le \frac{C}{t}\|f\|_{L^1}. \tag{3.13}$$

Let now $C_n^*$ be the cube concentric with $C_n$ and having twice the size of $C_n$. If we set $B := \mathbb{R}^N \setminus (\bigcup C_n^*)$, then we have

$$\begin{aligned} |\{|T\sum h_n| > t/2\}| &\le \left|\bigcup C_n^*\right| + \left|\{x \in B\ ;\ \sum |Th_n| > t/2\}\right| \\ &\le C\sum |C_n| + \frac{C}{t}\sum \|Th_n\|_{L^1(B)} \end{aligned} \tag{3.14}$$

Let $\overline{x}_n$, respectively $l_n$ be the center, respectively the size of $C_n$. For $x \in B$, we have

$$Th_n(x) = \int K(x-y)h_n(y)\,dy = \int [K(x-y) - K(x-\overline{x}_n)]h_n(y)\,dy,$$

so that

$$|Th_n(x)| \le \frac{C}{|x-\overline{x}_n|^{N+1}} \int |y-\overline{x}_n||h_n|\,dy \le \frac{Cl_n}{|x-\overline{x}_n|^{N+1}}\|h_n\|_{L^1}.$$

By integrating the above inequality and summing over $n$, we obtain

$$\sum \|Th_n\|_{L^1(B)} \le C \sum \|h_n\|_{L^1} \le C\|f\|_{L^1}; \tag{3.15}$$

the last inequality in the above line follows from the properties of the Calderón-Zygmund decomposition.

We complete the proof of (3.10) by combining (3.12), (3.13), (3.14) and (3.15) with the fact that $\sum |C_n| \le \dfrac{C}{t}\|f\|_{L^1}$.

We now know that $T$ satisfies the hypotheses of Marcinkiewicz' theorem with $q = 2$. Therefore, $T$ has a continuous exension $\tilde{T}$ from $L^p$ into $L^p$ if $1 < p < 2$. We actually have $\tilde{T}f = Tf$ when $f \in L^p_c$. Proof of this: we approximate $f$ in $L^p$ with a sequence $(f_n) \subset L^2_K$ (for a suitable compact $K$). Then $(Tf_n)$ converges to $Tf$ in $\mathscr{D}'(\mathbb{R}^N)$ and converges to $\tilde{T}f$ in $L^p$, whence $Tf = \tilde{T}f$. The proof of the case $1 < p < 2$ is complete.

Let now $2 < p < \infty$. It suffices to prove that $\|Tf\|_{L^p} \le C\|f\|_{L^p}$ when $f \in L^p \cap L^2$; if this is established, then we complete the proof of this case as in the previous paragraph. Let $1 < q < 2$ be the conjugate of $p$. If $f \in L^p \cap L^2$, then

$$\begin{aligned}\|Tf\|_{L^p} &= \sup_{g\in L^q\ ;\ \|g\|_{L^q}\le 1} \int Tf\ \overline{g} = \sup_{g\in L^q\cap L^2\ ;\ \|g\|_{L^q}\le 1} \int Tf\ \overline{g} \\ &= \sup_{g\in L^q\cap L^2\ ;\ \|g\|_{L^q}\le 1} \int f\,\overline{Tg} \le C\|f\|_{L^p};\end{aligned} \tag{3.16}$$

here, we rely on the continuity of $T$ in $L^q$. □

If we take a closer look at the above proof, we see that we actually established the following

**3.10 Theorem** (Calderón-Zygmund)**.** *Let $T : L^2_c(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ be a linear operator. Assume that, for some $K \in L^\infty_{loc}(\mathbb{R}^{2N} \setminus \Delta_{\mathbb{R}^N\times\mathbb{R}^N})$, we have:*
*(H1) $T$ is continuous from $L^2_c$ into $L^2$*
*(H2) for $x \notin \operatorname{supp} f$, $Tf(x) = \displaystyle\int K(x-y)f(y)\,dy$*
*(H3) for $|y| < 1/2|x|$, $|K(x-y) - K(x)| \le \dfrac{C|y|}{|x|^{N+1}}$.*
*Then $T$ is continuous from $L^p_c$ into $L^p$ for $1 < p < 2$.*
*If, in addition,*
*(H4) $T$ is symmetric in $L^2$,*
*then $T$ is continuous from $L^p_c$ into $L^p$ for $1 < p < \infty$.*

Note that $(H4)$ holds if $\widehat{Tf} = m\hat{f}$, $\forall\, f \in L^2_c$, where $m$ is real and bounded.

**3.11 Corollary.** *If* $-\Delta u = f$ *in* $\mathscr{D}'(\mathbb{R}^N)$, *with* $f \in L^p_{loc}$ *for some* $1 < p < \infty$, *then* $u \in W^{2,p}_{loc}$.

*Proof.* Let $K \Subset \mathbb{R}^N$. Let $\Phi \in C_c^\infty$ be s. t. $\Phi = 1$ in an open neighborhood $\Omega$ of $K$. Set $g = \Phi f \in L^p_c$. Then $v = E * g$ satisfies $-\Delta v = f$ in $\Omega$. We find that $u - v \in C^\infty(\Omega)$.

On the other hand, we have $v \in W^{2,p}_{loc}$. Indeed, let $L \Subset \mathbb{R}^N$ and let $\Psi \in C_c^\infty$ be s. t. $\Psi = 1$ in a neighborhood of $L - K$. Then $v = E * g = (\Psi E) * g$ in $L$, and the function $(\Psi E) * f$ belongs to $W^{1,p}$, as convolution of a map in $W^{1,1}$ and of a map in $L^p$. Finally, by the Calderón-Zygmund theorem, the second order derivatives of $v$ are in $L^p$. □

**3.12 Exercise.** Prove that the convolution of a map $F$ in $W^{1,1}$ and of a map $f$ in $L^p$ is in $W^{1,p}$.
Does this apply to $F = \Psi E$, with $\Psi \in C_c^\infty(\mathbb{R}^N)$?

## 3.3 Applications to the local theory

We are now in position apply the $L^p$ theory in order to obtain local estimates. (A subsequent section will be devoted to global estimates, which require additional ingredients.)

### 3.3.1 The first eigenvalue of $-\Delta$

Recall the Poincaré inequality

$$\int_\Omega u^2 \le C \int_\Omega |\nabla u|^2, \quad \forall\, u \in H^1_0(\Omega). \tag{3.17}$$

The best constant $C$ in the above inequality is given by the formula

$$C^{-1} = \lambda_1(\Omega) := \inf \left\{ \int_\Omega |\nabla u|^2 / \int_\Omega u^2 \ ; \ u \in H^1_0(\Omega) \setminus \{0\} \right\}. \tag{3.18}$$

**3.13 Proposition.** *a) In (3.18), the infimum is actually a minimum.*

*b) If* $u$ *achieves the minimum in (3.18), then* $u$ *is solution of*

$$-\Delta u = \lambda_1(\Omega) u. \tag{3.19}$$

*Conversely, if* $u \in H^1_0(\Omega)$ *solves (3.19), then either* $u = 0$ *or* $u$ *achieves the minimum in (3.18).*

*c) If $v \in H_0^1(\Omega) \setminus \{0\}$ satisfies*

$$-\Delta v = \lambda v, \tag{3.20}$$

*then $\lambda \geq \lambda_1(\Omega)$. Equivalently, $\lambda_1(\Omega)$ is the least eigenvalue of $-\Delta$ in $H_0^1(\Omega)$.*

*d) If $v$ satisfies (3.20), then $v \in C^\infty(\Omega)$.*

*In the remaining statements, we assume that $\Omega$ is connected.*

*e) If $u \neq 0$ satisfies (3.19), then $u$ is of constant sign in $\Omega$.*

*f) If (3.20) has a non trivial solution which does not change sign, then $\lambda = \lambda_1(\Omega)$.*

*g) The solutions of (3.19) are collinear.*

In agreement with the above result, $\lambda_1(\Omega)$ is called the first eigenvalue of $\Omega$. A positive solution of (3.19) is "the" first eigenfunction of $-\Delta$.

*Proof.* a) Let $(u_n)$ be a minimizing sequence for (3.18). We may assume that $\|u_n\|_{L^2} = 1$, $\forall\ n$. Then $(u_n)$ is bounded in $H_0^1(\Omega)$. Up to a subsequence (still denoted $(u_n)$), $u_n$ converges *-weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, to some $u \in H_0^1(\Omega)$. Thus $\|u\|_{L^2} = 1$ and $\displaystyle\int |\nabla u|^2 \leq \liminf \int |\nabla u_n|^2$. It follows that $u$ achieves the minimum in (3.18).
b) Let $f(t) = \displaystyle\int |\nabla(u+tv)|^2 - \lambda_1(\Omega) \int (u+tv)^2$ ($t \in \mathbb{R}$, $v \in H_0^1(\Omega)$). Then $f(t) \geq 0$, with equality when $t = 0$. We find that $f'(0) = 0$, which implies that

$$\int \nabla u \cdot \nabla v = \lambda_1(\Omega) \int uv, \quad \forall\ v \in H_0^1(\Omega). \tag{3.21}$$

By letting $v \in C_c^\infty(\Omega)$ in (3.21), we find that $u$ is indeed solution of (3.19). Conversely, if $u$ satisfies (3.19), then (3.21) holds when $v \in C_c^\infty(\Omega)$, and thus, by density, for $v \in H_0^1(\Omega)$ (check!). If we take $v = u$ in (3.21), we find that that $\displaystyle\int |\nabla u|^2 = \lambda_1(\Omega) \int u^2$.
c) As above, the identity $\displaystyle\int \nabla v \cdot \nabla w = \lambda \int vw$ holds for each $w \in H_0^1(\Omega)$ (check!). With $w = v$, we find that $\displaystyle\int |\nabla v|^2 = \lambda \int v^2$, so that $\lambda \geq \lambda_1(\Omega)$.
d) If $v \in H_0^1(\Omega)$ satisfies (3.20), then $v \in H^2_{loc}$ (since $-\Delta v \in L^2$). Therefore, we have $-\Delta \partial_j \partial_k v = \lambda \partial_j \partial_k v \in L^2_{loc}$, which in turn yields $\partial_j \partial_k v \in H^2_{loc}$. We find that $v \in H^4_{loc}$. Inductively, we obtain $v \in H^{2n}_{loc}$, $n \in \mathbb{N}$. In particular,

$v \in W^{n,1}_{loc}$ for each $n$. Since $W^{N+n,1}_{loc} \hookrightarrow C^n$ (by Sobolev embeddings), we find that $v \in C^\infty$.

e) By Theorem 3.16 and Lemma 3.20 below, we find that $v := |u|$ is a minimizer in (3.18). Indeed, we have $|u| \in H^1_0(\Omega)$ and $|\nabla u| = |\nabla |u||$ a. e. We find that $v \in C^\infty(\Omega)$ satisfies (3.19). The maximum principle implies that $v > 0$ everywhere; it follows that $u$ does not vanish, so that $u$ has constant sign.

f) If $u \neq 0$ solves (3.19), then $\displaystyle\int \nabla u \cdot \nabla v = \lambda_1(\Omega) \int uv = \lambda \int uv$, whence the conclusion.

g) Let $u, v$ be non trivial solutions of (3.19). If $x_0 \in \Omega$ is s. t. $v(x_0) \neq 0$, then there is some $\mu \in \mathbb{R}$ s. t. $(u - \mu v)(x_0) = 0$. We find that $u = \mu v$. □

**3.14 Corollary.** *If* $\Omega = \prod (a_i, b_i)$, *then* $\lambda_1(\Omega) = \pi^2 \sum \dfrac{1}{(b_i - a_i)^2}$.

*Proof.* The function $u = \prod \sin\left(\dfrac{\pi(x_i - a_i)}{b_i - a_i}\right)$ belongs to $H^1_0(\Omega)$, is $> 0$ in $\Omega$ and satisfies $-\Delta u = \pi^2 \sum \dfrac{1}{(b_i - a_i)^2} u$. □

**3.15 Remark.** The argument used in the proof of d) and leading to $v \in C^\infty$ is a "self improving" regularity argument reminiscent of $x' = x \Longrightarrow x \in C^\infty$. This kind of argument, aka "bootstrap", will be often encountered in the sequel and will be omitted.

**3.16 Theorem** (Chain rule of de la Vallée Poussin)**.** *Let* $\Phi$ *be Lipschitz (so that* $\Phi$ *has an usual derivative at each* $t \in \mathbb{R} \setminus A(\Phi)$, *where* $A(\Phi)$ *is a null Borel set). Let* $u \in W^{1,1}_{loc}$. *Then*

*a) If* $A$ *is a null Borel set, then* $\nabla u = 0$ *a. e. in the set* $[u \in A]$

*b)* $\Phi \circ u \in W^{1,1}_{loc}(\Omega)$ *and* $\nabla(\Phi \circ u)(x) = \Phi'(u)\nabla u(x)$ *a. e., with the convention that the r. h. s. is 0 if* $u(x) \in A(\Phi)$.

*Proof.* In b), we may assume that $\Phi(0) = 0$. We leave as an exercise the fact that, if $\Phi \in C^1$ is Lipschitz, and if $u \in W^{1,1}_{loc}$, then $\Phi \circ u \in W^{1,1}_{loc}(\Omega)$ and $\nabla(\Phi \circ u)(x) = \Phi'(u)\nabla u(x)$ (this is done by approximation, starting with the case where $u$ is smooth). This is the "usual" chain rule.

In a), we start with the case where $A$ is reduced to a single point, say $A = \{0\}$. Let $\Phi \in C^\infty_c(\mathbb{R})$ be s. t. $\Phi'(0) = 1$. By the usual chain rule applied to $t \mapsto \varepsilon\Phi(t/\varepsilon)$, we find that

$$\int \varepsilon\Phi(u/\varepsilon)\partial_j \psi = -\int \Phi'(u/\varepsilon)\partial_j u\, \psi, \quad \forall\ \psi \in C^\infty_c(\Omega).$$

By letting $\varepsilon \to 0$, we find $\nabla u = 0$ a. e. in the set $[u = 0]$.

It follows that a) holds when $\{0\}$ is replaced by a countable set.

Let now $f$ be the characteristic function of an interval $I$ and $\Phi(t) := \int_0^t f(s)\,ds$. Consider a sequence $(f_n)$ of continuous functions s. t. $|f_n| \le 1$ and $f_n(t) \to f(t)$ (possibly except when $t$ is an endpoint of $I$). By dominated convergence and using the fact that the usual chain rule holds for $t \mapsto \int_0^t f_n(s)\,ds$, we find that the following chain rule holds for $\Phi$:

$$\int \Phi(u)\partial_j \psi = -\int f(u)\partial_j u\,\psi, \quad \forall\ \psi \in C_c^\infty(\Omega). \tag{3.22}$$

By linearity, the chain rule holds for $t \mapsto \int_0^t f(s)\,ds$, where $f$ is the characteristic function of the union of a finite number of intervals. By countable additivity, the same holds if $f$ is the characteristic function of an open set.

Now comes the key argument: a) holds. Indeed, consider a non increasing sequence $(U_n)$ of open sets s. t. $|U_n| \to 0$ and $A \subset U_n$. Let $f_n$ be the characteristic function of $U_n$ and let $\Phi_n(t) = \int_0^t f_n(s)\,ds$. By passing to the limits in the identity

$$\int \Phi_n(u)\partial_j \psi = -\int f_n(u)\partial_j u\,\psi, \quad \forall\ \psi \in C_c^\infty(\Omega),$$

we find (by dominated convergence) that

$$0 = -\int_{f^{-1}(\cap U_n)} \partial_j u\,\psi, \quad \forall\ \psi \in C_c^\infty(\Omega).$$

We obtain that $\nabla u = 0$ a. e. in the set $f^{-1}(\cap U_n)$, and therefore the same holds in $f^{-1}(A)$.

Let now $\mathfrak{B}$ denote the set of bounded Borel functions. Then $\Phi$ Lipschitz and $\Phi(0) = 0$ is the same as $\Phi(t) = \int_0^s f(s)\,ds$ for some $f \in \mathfrak{B}$. In adition, we have $\Phi' = f$ outside some null Borel set $A(\Phi)$. Consider

$$\mathfrak{C} := \{f \in \mathfrak{B};\ \text{the chain rule (3.22) holds for } t \mapsto \Phi(t) := \int_0^t f(s)\,ds\}.$$

By a), if $f \in \mathfrak{C}$ and if $g = f$ a. e., then $g \in \mathfrak{C}$. We already know that $\mathfrak{C}$ contains characteristic functions of open sets. By copying the proof of a) and by using a), we find that $\mathfrak{C}$ contains characteristic functions of Borel sets. By linearity, $\mathfrak{C}$ contains step Borel functions. Since these functions are dense in $\mathfrak{B}$ (with the uniform norm), we easily find by dominated convergence that $\mathfrak{C} = \mathfrak{B}$. $\square$

**3.17 Corollary.** *Let $u \in W^{1,1}_{loc}$ and $a \in \mathbb{R}$. Then*

$$\partial_j |u-a| = \begin{cases} \partial_j u, & \text{in the set } [u \geq a] \\ -\partial_j u, & \text{in the set } [u < a] \end{cases} \quad a.\ e.$$

**3.18 Corollary.** *Let $\Phi$ be Lipschitz. (If $\Omega$ has infinite volume, we also assume that $\Phi(0) = 0$.) If $u \in W^{1,p}(\Omega)$, then $\Phi \circ u \in W^{1,p}(\Omega)$ and $|\nabla(\Phi \circ u)| \leq C(\Phi)|\nabla u|$, where $C(\Phi)$ is the Lipschitz constant of $\Phi$.*

The next result goes beyond the scope of these notes. In full generality, it will not be used or proved here. For a proof (which uses the chain rule), we send to the paper of Marcus and Mizel. What we will use is a special case (Lemma 3.20), which we will prove.

**3.19 Theorem** (Marcus, Mizel)**.** *Let $\Phi$ be a Lipschitz function. (If $\Omega$ has infinite volume, we also assume that $\Phi(0) = 0$.) Let $1 \leq p < \infty$. Then the map $W^{1,p}(\Omega) \ni u \mapsto \Phi \circ u \in W^{1,p}(\Omega)$ is continuous.*
*In particular, if $u \in W^{1,p}_0(\Omega)$ and $\Phi(0) = 0$, then $\Phi \circ u \in W^{1,p}_0(\Omega)$.*

Just a word about the second assertion, whose proof is easy: take $(u_n) \subset C^\infty_c(\Omega)$ s. t. $u_n \to u$ in $W^{1,p}$. Then $\Phi \circ u_n \to \Phi \circ u$ in $W^{1,p}$, and we are reduced to proving that $\Phi \circ u_n \in W^{1,p}_0(\Omega)$. This follows immediately by regularization, using the fact that $\Phi \circ u_n$ is compactly supported.

**3.20 Lemma.** *Let $1 \leq p < \infty$. Then $W^{1,p}(\Omega) \ni u \mapsto |u| \in W^{1,p}(\Omega)$ is continuous.*
*In particular, if $u \in W^{1,p}_0(\Omega)$, then $|u| \in W^{1,p}_0(\Omega)$.*

*Proof.* As above, the second assertion is obtained from the first one.

In order to prove the first assertion, it suffices to check that, if $u_n \to u$ in $W^{1,p}$ then, possibly after passing to a subsequence, $|u_n| \to |u|$ in $W^{1,p}$. This amounts to $\nabla u_n \to \nabla u$ in $L^p$ (possibly up to a subsequence). Since, up to a subsequence, we have $u_n \to u$ a. e. and $|\nabla u_n| \leq f$ with $f \in L^p$, the conclusions follows by combining the chain rule and the dominated convergence. $\square$

**3.21 Exercise.** Let $\Phi$ be Lipschitz. Assume that $\Phi(0) = 0$ and that $\Phi$ is piecewise $C^1$. Let $1 \leq p < \infty$. Prove that $W^{1,p}(\Omega) \ni u \mapsto \Phi(u) \in W^{1,p}(\Omega)$ is continuous.
In particular, if $u \in W^{1,p}_0(\Omega)$, then $\Phi(u) \in W^{1,p}_0(\Omega)$.

### 3.3.2 (Sub)critical semilinear equations

We examine here the equation

$$-\Delta u = f(u), \tag{3.23}$$

where $u \in H^1_{loc}$ and $f \in C^\infty$. With obvious adaptations, the remarks we make in this section also apply to $-\Delta u = f(x,u)$.

Assume first that $N \geq 3$. Then $u \in H^1_{loc} \implies u \in L^{2N/(N-2)}_{loc}$. It follows that (3.23) makes sense (in $\mathscr{D}'(\Omega)$) if

$$|f(t)| \leq C(1+|t|^{2N/(N-2)}). \tag{3.24}$$

Indeed, in this case we have $f(u) \in L^1_{loc}$. Similarly, if $N = 2$, then (3.23) makes sense if

$$|f(t)| \leq C(1+|t|^p) \quad \text{for some } p < \infty. \tag{3.25}$$

The question we address here is whether the equation (3.23) combined with the growth conditions (3.24) or (3.25) yields better regularity then merely $u \in H^1_{loc}$, say if we have $u \in C^\infty$.

**3.22 Exercise.** Assume that either $N \geq 3$ and $|f(t)| \leq C(1+|t|^p)$, where $p < 2N/(N-2)$, or that $N = 2$ and that (3.25) holds. Prove that, if $u \in H^1_{loc}$ satisfies (3.23), then $u \in C^\infty$. [Hint: bootstrap.]

In some sense, (3.24) is optimal, at least when $N \geq 3$:

**3.23 Example.** Let $N \geq 3$ and set $u(x) = |x|^{-\alpha}$, with $0 < \alpha < \dfrac{N}{2} - 1$, $\alpha \neq N-2$. Then $u \in H^1_{loc}$ and $-\Delta u \sim |x|^{-\alpha-2} \sim u^p$, with $p = 1 + \dfrac{2}{\alpha} \in ((N+2)/(N-2), \infty)$.

Thus, if $N \geq 3$ and if we want to obtain that $u \in C^\infty$, we cannot go beyond the (3.24). Consider the growth condition

$$|f(t)| \leq C(1+|t|^p). \tag{3.26}$$

**3.24 Definition.** If $N \geq 3$ and $p < (N+2)/(N-2)$, then we say that $f$ is subcritical (same if $N = 2$ and $p < \infty$). By the Exercise 3.22, if $f$ is subcritical and $u \in H^1_{loc}$, then $u$ is smooth.

If $N \geq 3$ and (3.26) is satisfied with $p = (N+2)/(N-2)$, then we say that $f$ is critical.

When $N = 2$, critical growth is more subtle, and is not discussed here.

A first result asserts that the exponential growth is, in two dimensions, kind of subcritical.

**3.25 Proposition.** *If $N = 2$, $|f(t)| \le Ce^{k|t|}$ and $u \in H^1_{loc}$ solves (3.23), then $u \in C^\infty$.*

*Proof.* We rely on Trudinger's inequality: if $u \in W^{1,N}_{loc}(\Omega)$, then for each ball $B \Subset \Omega$ there is some $c > 0$ (depending on $u$ and on the ball) s. t. $\displaystyle\int_B \exp(c|u|^{N/(N-1)}) < \infty$ (Gilbarg and Trudinger, Theorem 7.15, p. 162). In 2D, we find that $u \in H^1_{loc} \Longrightarrow f(u) \in L^p_{loc}$ for $1 < p < \infty$. It follows that $u \in W^{2,p}_{loc}$ for $1 < p < \infty$. In particular, $u \in L^\infty_{loc}$, by the Sobolev embeddings. Next, we have $-\Delta \partial_j u = f'(u)\partial_j u \in L^p_{loc}$ for $1 < p < \infty$, so that $u \in W^{3,p}_{loc}$ for $1 < p < \infty$. We next bootstrap, using the formula for the derivatives of $f(u)$. $\square$

In order to explain the subtleness of the critical case, let us consider an example: in $R^3$, the equation $-\Delta u = u^5$ is critical. In this case $u \in H^1_{loc} \Longrightarrow u^5 \in L^{6/5}_{loc} \Longrightarrow -\Delta u \in L^{6/5}_{loc} \Longrightarrow u \in W^{2,6/5}_{loc} \Longrightarrow u \in L^6_{loc}$. Back to the starting point.

In order to treat the critical case, we rely on

**3.26 Lemma.** *Let $N \ge 3$. If $u \in H^1_{loc}$ solves*

$$-\Delta u = a(x)u + b(x), \qquad \textit{where } a \in L^{N/2}_{loc} \textit{ and } b \in L^\infty_{loc}, \tag{3.27}$$

*then $u \in L^q_{loc}$ for each $q < \infty$.*

*Proof.* Idea (Moser): multiply (3.27) by $u^\alpha$ ($\alpha > 1$ to be fixed later) and integrate by parts in order to prove the implication $u \in L^q_{loc} \Longrightarrow u \in L^{\gamma q}_{loc}$ for some $\gamma > 1$. Since, from the beginning, we know that $u \in L^{2N/(N-2)}_{loc}$, we end up with $u \in L^q_{loc}$ for each $q < \infty$.

Several points are to be fixed: first, since $u$ is only locally in $H^1$, there is no way to integrate by parts in the whole $\Omega$. Thus $u^\alpha$ has to be multiplied by a cutoff function. Second, if $u \in H^1_{loc}$, then we need not have $u^\alpha$ in $H^1_{loc}$. Thus $u^\alpha$ has to be truncated. Finally, obtaining estimates requires dealing with positive quantities. Thus we will work with the positive and negative part $u_+$ and $u_-$ of $u$.

The plan is the following: we prove that, if $u \in L^q_{loc}$ for some $q > 2$, then $u_+ \in L^{Nq/(N-2)}_{loc}$.

Let $k \in \mathbb{N}^*$. Let $u_k = \begin{cases} u, & \text{if } 0 < u < k \\ k, & \text{if } u \geq k \\ 0, & \text{if } u \leq 0 \end{cases}$. If $x_0 \in \Omega$, let $\eta \in C_c^\infty(\Omega)$ be s. t. $\eta = 1$ near $x$; this $\eta$ will be fixed later.

Let $\beta = q-2$. We multiply (3.27) by $\eta^2 u u_k^\beta$. In view of the chain rule, this map belongs to $\in H^1$ and is compactly supported. The chain rule, combined the Leibniz rule and the fact that

$$\int \nabla u \cdot \nabla v = \int f(u)v, \quad \forall\ v \in H^1(\Omega), v \text{ compactly supported,}$$

(check this fact!) implies that

$$I_1 + I_2 + I_3 = J_1 + J_2, \tag{3.28}$$

where

$$I_1 = \int \eta^2 |\nabla u|^2 u_k^\beta,\ I_2 = \beta \int u u_k^{\beta-1} \eta^2 \nabla u \cdot \nabla u_k,\ I_3 = 2 \int \eta u u_k^\beta \nabla u \cdot \nabla \eta,$$

$$J_1 = \int \eta^2 a u^2 u_k^\beta,\ J_2 = \int \eta^2 b u u_k^\beta.$$

On the one hand, we have

$$|I_3| \leq \varepsilon \int \eta^2 |\nabla u|^2 u_k^\beta + \frac{1}{\varepsilon} \int |\nabla \eta|^2 u^2 u_k^\beta = \varepsilon I_1 + \frac{1}{\varepsilon} L.$$

By taking $\varepsilon = 1/2$, we find that

$$I_1 + I_2 \leq C(|J_1| + |J_2| + L). \tag{3.29}$$

On the other hand, let $v_k = \eta u u_k^{\beta/2}$. Since $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, we have

$$\int |\nabla v_k|^2 \leq 3(I_1 + K + L); \tag{3.30}$$

here,

$$K = \left(\frac{\beta}{2}\right)^2 \int \eta^2 u^2 u_k^{\beta-2} |\nabla u_k|^2. \tag{3.31}$$

Using the fact that $\nabla u_k = 0$ a. e. in the complement of the set $[0 < u < k]$, we have $I_2 = \beta \int u_k^\beta \eta^2 |\nabla u_k|^2$ et $K = \left(\frac{\beta}{2}\right)^2 \int \eta^2 u_k^\beta |\nabla u_k|^2$ . We find that $K = cI_2$, with $c > 0$. By comparing (3.29) to (3.30), we find that

$$\int |\nabla v_k|^2 \leq C(|J_1| + |J_2| + L).$$

The Sobolev embedding implies that

$$\|v_k\|^2_{L^{2N/(N-2)}} \leq C(|J_1| + |J_2| + L). \tag{3.32}$$

It remains to estimate the integrals in (3.32). Hölder's inequality yields

$$|J_1| \leq \int |a| v_k^2 \leq \|a\|_{L^{N/2}(\text{supp }\eta)} \|v_k\|^2_{L^{2N/(N-2)}}.$$

Now comes the key point: we choose $\eta$. If the support of $\eta$ is sufficiently small, then we have $C|J_1| \leq 1/2\|v_k\|^2_{L^{2N/(N-2)}}$.

On the other hand, we have $|u_k| \leq |u|$, so that

$$|J_2| \leq C\|u\|^{q-1}_{L^{q-1}(\text{supp }\eta)} \quad \text{and} \quad L \leq C\|u\|^q_{L^q(\text{supp }\eta)}.$$

For appropriate $\eta$, we find thus

$$\|v_k\|^2_{L^{2N/(N-2)}} \leq C(\|u\|^{q-1}_{L^{q-1}(\text{supp }\eta)} + \|u\|^q_{L^q(\text{supp }\eta)}).$$

If we let $k \to \infty$, then we find that $\eta(u_+)^{q/2} \in L^{2N/(N-2)}$ (i. e., that $u_+ \in L^{Nq/(N-2)}$ near $x$) provided that $u \in L^q_{loc}$. $\square$

**3.27 Proposition.** *If $N \geq 3$ and $f$ is critical, then $H^1_{loc}$ solutions of* (3.23) *are $C^\infty$.*

*Proof.* Write $f(t) = g(t)t + b$, where $b = f(0) \in L^\infty$ and $g(t) = \dfrac{f(t) - f(0)}{t}$ satisfies $|g(t)| \leq C(1 + |t|^{4/(N-2)})$. Then (3.23) rewrites as $-\Delta u = a(x)u + b$, with $a(x) = f(u(x))$. Since $u \in L^{2N/(N-2)}_{loc}$, we find that $a \in L^{N/2}_{loc}$. By the previous lemma, we have $u \in L^q_{loc}$ for each $q < \infty$, so that $f(u) \in L^q_{loc}$ for each $q < \infty$. As in the proof of Proposition 3.25, this implies that $u \in C^\infty$. $\square$

### 3.3.3 Standard estimates

This is a ubiquitous type of results. We will prove here just one. In the sequel, they will be all left to the reader.

**3.28 Proposition.** *Let $K$, $L$ be two compact subsets of $\Omega$ s. t. $K \subset \overset{\circ}{L}$. If $-\Delta u = f$ in $\overset{\circ}{L}$ and $1 < p < \infty$, then*

$$\|u\|_{W^{2,p}(K)} \leq C(\|f\|_{L^p(L)} + \|u\|_{L^1(L)}).$$

Before going into the details of the proof, let us explain the philosophy of such estimates (the same applies to all standard estimates involving the Laplace equation): a) the estimate has to be true for the special solution $u = E * f$ of $-\Delta u = f$; b) the r. h. s. has to contain not only $f$, but also a control of $u$ (think of $u = 1$, $f = 0$!).

*Proof.* Let $\zeta \in C_c^\infty(\overset{\circ}{L})$ be s. t. $\zeta = 1$ near $K$. Set $g = \zeta f$ and let $v = E * g$. Then $\|v\|_{W^{2,p}(L)} \leq C\|g\|_{L^p} \leq C\|f\|_{L^p(L)}$. Let now $w = u - v$, which is harmonic in a neighborhood $M$ of $K$. Let $0 < r <$dist$(K, \mathbb{R}^N \setminus M)$. If $x \in K$, then $\displaystyle\int_{r/2}^{r}\int_{S(x,s)} |u|\, ds \leq C\|w\|_{L^1(L)}$, and thus we may find some $s = s(x) \in (r/2, r)$ s. t. $\displaystyle\int_{S(x,s)} |w| \leq C\|w\|_{L^1(L)}$. In $B(x, s)$, we may compute $w$ via the Poisson fromula. If we differentiate under the integral sign in the Poisson formula, we find that $|\partial^\alpha w| \leq C\|w\|_{L^1(L)}$ in $B(x, r/4)$, with $C$ independent of $x$. Finally, we have $\|w\|_{W^{2,p}(B(x,r/4))} \leq C\|w\|_{L^1(L)}$, so that $\|w\|_{W^{2,p}(K)} \leq C\|w\|_{L^1(L)}$.

We find that

$$\|u\|_{W^{2,p}(K)} \leq \|v\|_{W^{2,p}(K)} + \|w\|_{W^{2,p}(K)} \leq C(\|f\|_{L^p(L)} + \|v\|_{L^1(L)} + \|u\|_{L^1(L)}).$$

We complete the proof by noting that $\|v\|_{L^1(L)} \leq C\|v\|_{L^p(L)} \leq C\|v\|_{W^{2,p}(L)} \leq C\|f\|_{L^p(L)}$. □

## 3.4 Exercises

**3.29 Exercise** (Hilbert transform)**.** For a compactly supported distribution $f$ in $\mathbb{R}$, we define its Hilbert transform through the formula $Tu =$v. p. $\dfrac{1}{x} * u$. Prove that $T$ is continuous from $L_c^p$ into $L^p$, $1 < p < \infty$

**3.30 Exercise** (Regularity for $\overline{\partial}$)**.** Let, in $\mathbb{R}^2$, $\overline{\partial} := \dfrac{1}{2}\partial_1 + \dfrac{1}{2}\imath\partial_2$. Prove that, for $1 < p < \infty$, $\overline{\partial}\, u \in L^p_{loc} \Longrightarrow u \in W^{1,p}_{loc}$.

**3.31 Exercise** (Riesz transforms)**.** This exercise relies on the Fourier transform of temperate distributions, see, e. g., Hörmander.

We define the Riesz transforms through the formulae $R_j u = \mathscr{F}^{-1}\left(\dfrac{\xi_j}{|\xi_j|}\hat{u}\right)$, $j = 1, \ldots, N$. Here, $u \in L^2(\mathbb{R}^N)$. Alternatively, we have $R_j u = K_j * u$, where $K_j = \mathscr{F}^{-1}\left(\dfrac{\xi_j}{|\xi_j|}\right)$.

a) Let $u \in L^1(\mathbb{R}^N)$ be radial. Prove that its Fourier transform is radial, too

b) The same holds if $u \in L^2$

c) Let $N/2 < \alpha < N$. Let $u_\alpha(x) = \dfrac{1}{|x|^\alpha}$. Prove that $\mathscr{F}(u_\alpha) = C_\alpha u_{N-\alpha}$
Hint: prove first that $\mathscr{F}(u_\alpha)$ is a radial function. Prove next that $\mathscr{F}(u_\alpha)$ is homogeneous of degree $\alpha - N$

d) Prove that $C_\alpha = \dfrac{\pi^{N/2} 2^{N-\alpha} \Gamma((N-\alpha)/2)}{\Gamma(\alpha/2)}$

e) For $N \geq 2$, determine the Fourier transform of $\partial_j u_{N-1}$. Obtain that $K_j \in C^\infty(\mathbb{R}^N \setminus \{0\})$ satisfies the assumptions of the Calderón-Zygmund theorem (general form)

f) Derive the following: the Riesz transforms map continuously $L^p \cap L^2$ (endowed with the $L^p$ norm) into $L^p$, for $1 < p < \infty$

g) What becomes the Riesz transform when $N = 1$?

**3.32 Exercise.** We prove here that, in the limiting cases uncovered by the Calderón-Zygmund or Korn theorems, there is no good regularity result. We also prove that, in general, one cannot replace local regularity conclusions by global ones (basically, this is due to the fact that $u$ may behave badly at infinity).

a) Let $K, L \Subset \mathbb{R}^N$. Assume that the linear continuous operator $S : L^p_c \to \mathfrak{M}$ sends $L^p_K$ into $L^p_{loc}$ (i. e., we have $f \in L^p_K \implies \Phi S f \in L^p$, for each $\Phi \in C^\infty_c(\mathbb{R}^N)$). Prove that $\Phi S$ is continuous from $L^p_K$ into $L^p_L$, provided supp $\Phi \subset L$

b) By starting with $f = \rho_\varepsilon$, prove that $f \in L^1_c \not\Longrightarrow u \in W^{2,1}_{loc}$

c) (Weierstrass' example) Let $v(x) = (x_1^2 - x_2^2) \ln |x|$. Prove that $-\Delta v = f \in C(\mathbb{R}^N)$, but that $-\Delta u = f$ has no $C^2_{loc}$ solution

d) Adapt the following example and find an example of Lipschitz function $f$ s. t. $-\Delta u = f$ has no solution in $C^{2,1}_{loc}$

e) Prove that, if $f \in L^\infty_c$, then the equation $-\Delta u = f$ need not have a solution in $W^{2,\infty}_{loc}$

f) Prove that, if $u$ is harmonic in $\mathbb{R}^N$ and if $|u(x)| \leq C(1 + |x|)^m$, then $u$ is a polynomial of degree at most $m$

g) Prove that, if $N = 2$, then there is some $f \in C_c^\alpha$ s. t. the equation $-\Delta u = f$ has no solution in $C^{2,\alpha}$
Hint: find first $f \in C_c^\alpha$ s. t. $u \notin C^{2,\alpha}$, then use the previous question

h) If $N \geq 3$, prove that $f \in C_c^\alpha \Longrightarrow u \in C^{2,\alpha}$.

i) Let $1 < p < \infty$ and let $N = 2$. Prove that, if $f \in L_c^p$, then we need not have $u \in W^{2,p}$

j) If $N \geq 3$ and $p \leq N/(N-2)$, prove that $f \in L_c^p \not\Longrightarrow u \in W^{2,p}$

k) If $N \geq 3$ and $p > N/(N-2)$, prove that $f \in L_c^p \Longrightarrow u \in W^{2,p}$.

**3.33 Exercise.** We try to discuss here to what extent the would be implications (*) $f \in L_c^1 \Longrightarrow \partial_j \partial_k u \in L_{loc}^1$ and (**) $f \in L_c^\infty \Longrightarrow \partial_j \partial_k u \in L_{loc}^\infty$ are wrong. If (*) where true, then we would have, via the Sobolev embeddings, $\nabla u \in L_{loc}^{N/(N-1)}$ and, if $N = 2$, $u \in L_{loc}^\infty$. Similarly, (**), if true, would imply $\nabla u \in Lip_{loc}$.

In this exercise, we give a qualitative form to the fact that the last implications are almost true. This is expressed in the context of the BMO (=bounded mean oscillations) space of John et Nirenberg, defined as follows:

$$f \in \text{BMO} \Longleftrightarrow \exists\, K = K(f) \text{ s. t. } \fint_C \left| f - \fint_C f \right| \leq K, \text{ for each cube } C \subset \mathbb{R}^N.$$

For a thorough study of BMO, we refer to Stein, Chapter IV. We mention here, without proof, the *John-Nirenberg inequality*: there is some $c = c(N)$ s. t. $\exp(cf/K) \in L_{loc}^1$. In particular, $f \in \text{BMO} \Longrightarrow f \in L_{loc}^p$ for $p < \infty$.

a) If $m \in \mathbb{R}$, prove that $\displaystyle\fint_C \left| f - \fint_C f \right| \leq 2 \fint_C |f - m|$. Consequently, in order to prove that $f \in$ BMO, it suffices to prove that, for each cube $C$, there is some $m$ s. t. $\displaystyle\fint_C |f - m| \leq K$

b) Prove that $x \mapsto \ln |x| \in$ BMO

c) If $f \in L_c^\infty$, prove that $\partial_j \partial_k u \in$ BMO

d) If $f \in L_c^\infty$, prove that $|\nabla u(x) - \nabla u(y)| \leq C|x - y|(1 + |\ln |x - y||)$

e) If $f \in L_c^1$, prove that $\nabla u \in W_{loc}^{1,q}$ for $q < N/(N-1)$. The same holds if $f$ is a compactly supported measure

f) If $f \in L_c^1$ and $N = 2$, prove that $u \in$ BMO. The same holds if $f$ is a compactly supported measure.

**3.34 Exercise.** We discuss here various consequences of the maximal function theorem. Recall that, if $u \in L^1_{loc}(\mathbb{R}^N)$, then the *maximal function* of $u$ at $x$ is defined as $\mathscr{M}u(x) = \sup_{x\in C} \fint_C |u(y)|\, dy$. Here, the supremum is taken over the cubes (or balls) containing $x$. The maximal function is a Borel function. Recall the following special case of the Hardy-Littlewood maximal theorem: if $u \in L^1$, then there is some $K = K(N)$ s. t. $|\{\mathscr{M}u(x) > t\}| \leq \dfrac{K\|u\|_{L^1}}{t}$, $\forall\, t > 0$. For a proof, see the first pages in Stein.

a) Let $\varphi$ be integrable, radial, non increasing. Prove that $|u * \varphi_\varepsilon(x)| \leq \|\varphi\|_{L^1}\mathscr{M}u(x)$, $\forall\, x$, $\forall\, \varepsilon > 0$
Hint: start with a step function

b) If $\rho$ is a mollifier, prove that there is some $C = C(\rho)$ s. t. $|u * \rho_\varepsilon(x)| \leq C\mathscr{M}u(x)$, $\forall\, x$, $\forall\, \varepsilon > 0$

c) If $u \in L^1_{loc}$, prove that $u * \rho_\varepsilon \to u$ a. e. as $\varepsilon \to 0$
Hint: start with the case where $u \in C_c^\infty(\mathbb{R}^N)$. For the general case, use the converse to the dominated convergence theorem

d) (Lebesgue's differentiation theorem) If $u \in L^1_{loc}(\mathbb{R}^N)$, prove that

$$\lim_{x\in C,\, |C|\to 0} \fint_C |u(y) - u(x)|\, dy = 0 \quad \text{ for a. e. } x.$$

In particular, we have $\lim_{x\in C,\, |C|\to 0} \fint_C u(y)\, dy = u(x)$ for a. e. $x$.