Rigidity percolation on random graphs

J. Barré + results from many other people

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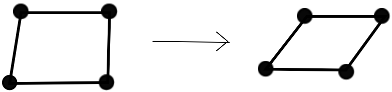
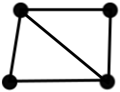
May 2014

Rigidity

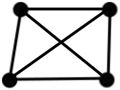
• Consider bars, which have a fixed length, linked together by "joints". Is the system rigid or floppy ?
Example in 2 dimensions; bar lengths are fixed, not the angles:



Floppy



Rigid, not overconstrained

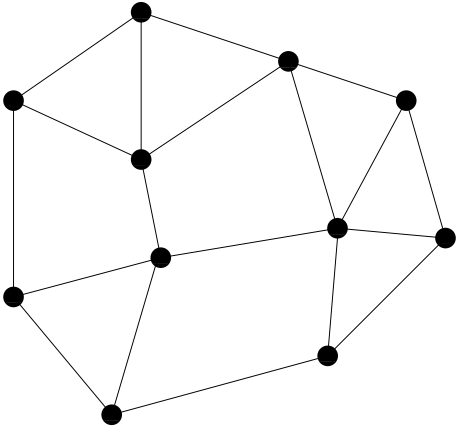


Rigid, overconstrained

Rigidity

- When there are only a few joints and bars, it is easy...

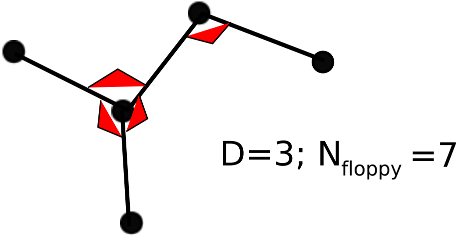
What about this network, with 11 sites?



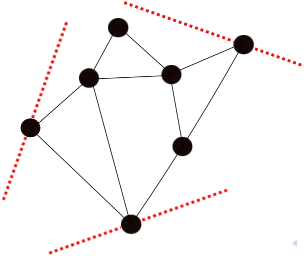
- Is it floppy? Rigid? How many floppy modes? Where?

Related problems

- Bond bending constraints: angles between two adjacent bonds have to be kept fixed ($D = 3$)



- Rigidity with "gliders": some joints constrained to move on a line



Constraint counting

Maxwell's idea: constraint counting

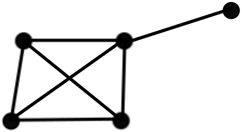
- each joint starts with 2 degrees of freedom
- each bar removes one degree of freedom

→ formula for the number of floppy modes (N joints, M bars):

$$N_{floppy} = 2N - M \text{ if } M < 2N - 3 \; ; \; N_{floppy} = 3 \text{ if } M \geq 2N - 3$$

- Counting redundant constraints:

$$N_{floppy} = 2N - M + N_{redundant}$$



N=5; M=7

$N_{redundant} = 1$

$N_{floppy} = 4$

From geometry to graph theory

• Power of constraint counting: replace a geometrical problem by a discrete, graph theoretical one.

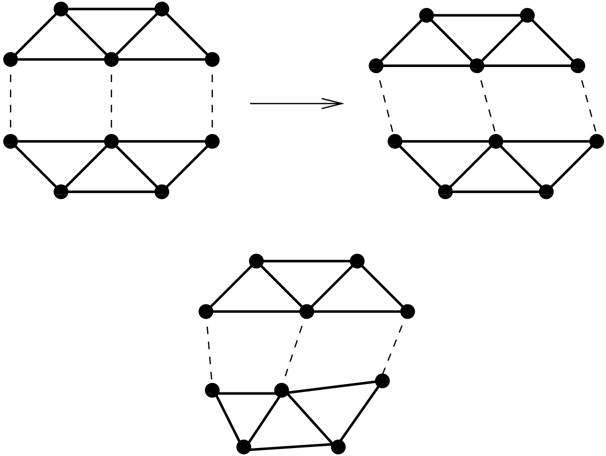
Question: is it possible to keep this desirable feature, correcting the approximations of constraint counting?

• Generic rigidity in 2D can be characterized in a purely graph theoretical way (Laman 1970):

G has no redundant constraint $\iff$ there is no subgraph with n vertices, m edges and $m > 2n - 3$.

$\rightarrow$ constraint counting on each subgraph

Generic rigidity

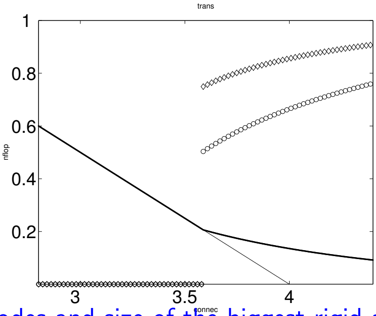


Top: a non generic realization; Laman theorem does not apply.
Bottom: a generic realization of the same graph.

Large networks: rigidity percolation

• Physical applications: very large networks (covalent glasses; proteins). Relevant question: Is there a macroscopic rigid cluster? → rigidity percolation (M. Thorpe).

• Example: Erdös-Rényi random graph $\mathcal{G}(n, c/n)$. Vary c; is there a threshold for a macroscopic rigid cluster? Yes, very sharp!



Number of floppy modes and size of the biggest rigid and stressed clusters, as functions of the mean connectivity

Note: Straight line at low connectivity = constraint counting; discontinuous transition.

Results

• Physics literature: random graphs locally look like trees $\rightarrow$ heuristic computation possible for the threshold, number of floppy modes, etc... (C. Moukarzel, P. Duxbury, D. Jacobs, M. Thorpe 97-99)

• Pushing the heuristic computations further: obtain large deviation functions for the redundant constraints (O. Rivoire, JB 2006).

• A theorem for the threshold $c \simeq 3.588\ldots$: V. Kasiviswanathan, C. Moore and L. Théran, 2011.

Method: show that the threshold for rigidity percolation is the same as 2-orientability (is there a way to orient all edges of a graph such that no vertex has more than two incoming edges?)

Rigidity percolation with gliders

• Consider a structure with n_1 sites within gliders, n_2 free sites and m bars.

A Laman-type theorem (I. Streinu, L. Théran, 2010).
Difficulty: gliders "pin" the rigidity components to the plane
$\rightarrow$ Distinguish between free, partly pinned, and pinned rigid clusters

redundant constraint $\iff$ subgraph with

$$n_1' + 2n_2' - m' - \max(3 - n_1', 0) < 0$$

$\rightarrow$ A graph theoretical approach possible (under a genericity condition, as usual)

Rigidity percolation with gliders

• Erdös-Renyi random graph $\mathcal{G}(n, c/n)$, with $n = n_1 + n_2$
$n_1 = qn$, $n_2 = (1 - q)n$.
$q =$ proportion of sites with gliders

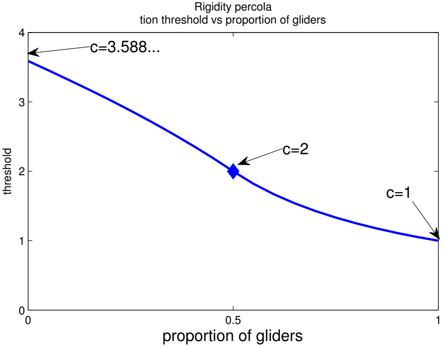
• $q = 1$: ordinary percolation = well known; continuous

• $q = 0$: rigidity percolation, discontinuous

• What happens in between?
Moukarzel 2003 (heuristic): some vertices are "pinned"
→ The transition remains discontinuous, and disappears when too many sites are pinned (physics jargon: first order transition and critical point)

Work in progress (with D. Mitsche and M. Lelarge)

- percolation threshold vs proportion of gliders

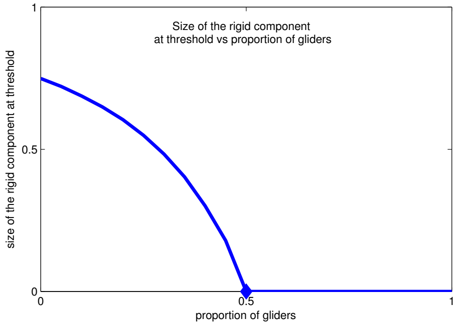


Conjecture:

- $c^\star = 1/q$ for $q \geq 1/2$
- $c^\star = \ldots$ (implicit expression) for $q > 1/2$

Work in progress (with D. Mitsche and M. Lelarge)

- Size of the largest component at threshold: jump for $q < 1/2$.



Conjecture:

- Continuous transition for $q \geq 1/2$: $\sim$ connectivity percolation
- Discontinuous transition for $q < 1/2$

Strategy

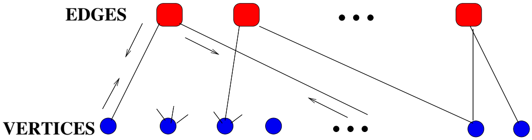
• A "tree-like" heuristic tells us what to expect

• Make the link with an orientability problem: uses density arguments (presence of small rigid components unlikely)

Sites on gliders: at most one incoming edge

Free sites: at most two incoming edges

• Analyze a message passing algorithm as in Lelarge 2012

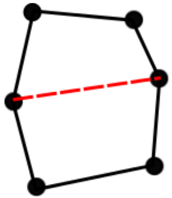


→ compute the probability distributions of messages

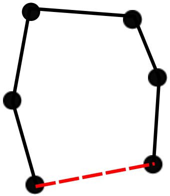
A more general framework

- Laman's theorem: no redundancy $\iff$ every subgraph with n vertices has $m \leq 2n - 3$ edges.

2= number of degrees of freedom of one point in 2D; 3=number of degrees of freedom of a solid body in 2D.
One may ask the same questions with $(k, l) \neq (2, 3)$!
$\rightarrow$ Graph theoretical concept of (k, l) sparsity ($l < 2k$): a graph is (k, l) sparse if every subgraph with n vertices has $m \leq kn - l$ edges



(1,0) sparsity



(1,1) sparsity

Physical meaning

Some (k, l) have a physical meaning

- $(k, l) = (2, 3)$: 2D bars-joints rigidity
- $(k, l) = (3, 3)$: 2D bodies-bars rigidity (more generally (k, k))
- $(k, l) = (1, 1)$: ordinary percolation
- $(k, l) = (2, 0)$: "2-orientability"
- gliders: interpolate between $k = 1$ and $k = 2$!

Remark: there is a large mathematical literature on this subject (graph theory, combinatorics, matroids theory...); not much on percolation however.

Conclusions

- A family of new percolation problems with an interesting physical meaning
- With gliders: interpolate between connectivity and rigidity percolation; a tricritical point (physics jargon again). Complete proof hopefully available soon…
- Physics literature: tree-like heuristics give access to much more detailed results (Large Deviation Cavity Method); could these be transformed into theorems? A general question, beyond rigidity.
- Much more difficult problem: on lattices…