Spectra of sparse random graphs

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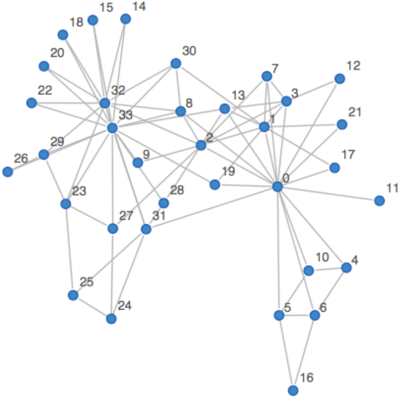
Framework

Take a finite, simple, non-oriented graph $G = (V, E)$.



Graph matrices

Natural matrices are associated to G.

They are matrices built from the local neighborhood of the vertices.

Adjacency matrix

The adjacency matrix is indexed by $V \times V$ and defined by

$$A_{xy} = \mathbf{1}(\{x, y\} \in E).$$

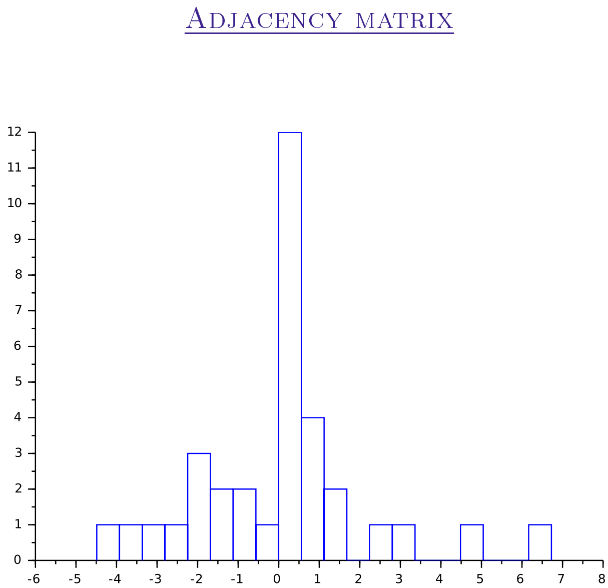
For integer $k \geqslant 0$,

$$A^k_{xy} = \text{nb of paths from } x \text{ to } y \text{ of length } k.$$

A is symmetric : it has real eigenvalues

$$\lambda_{|V|}(A) \leqslant \cdots \leqslant \lambda_1(A)$$

and an orthonormal basis of eigenvectors.


Adjacency matrix
12
11
10
9
8
7
6
5
4
3
2
1
0
-6
-5
-4
-3
-2
-1
0
1
2
3
4
5
6
7
8

Perron-Frobenius Theorem

Assume that the graph G is connected. Then A is irreducible: for any x, y in V, there exists k such that $A^k_{xy} > 0$.

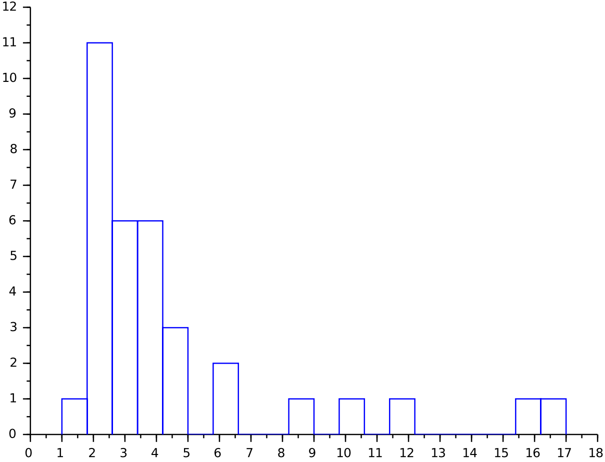
Then, the largest eigenvalue is positive and it is a simple eigenvalue. Its left and right eigenvector have positive coordinates.

Degree matrix

The degree matrix is the diagonal matrix indexed by $V \times V$ such that

$$D_{xx} = \deg(x) = \sum_y A_{yx}.$$

Degree



Incidence matrix

Define the set of oriented edges as

$$\vec{E} = \{(x,y) : \{x,y\} \in E\}$$

and the incidence matrix as the matrix on $\vec{E} \times V$

$$\nabla_{(xy),x} = 1, \quad \nabla_{(yx),x} = -1 \text{ and } \nabla_{e,x} = 0 \text{ otherwise.}$$

Observe for $x \neq y$

$$\begin{aligned}(\nabla^*\nabla)_{xx} &= \sum_e |\nabla_{e,x}|^2 = 2\deg(x).\\ (\nabla^*\nabla)_{xy} &= \sum_e \nabla_{e,x}\nabla_{e,y} = -2 \times \mathbf{1}(\{x,y\} \in E).\end{aligned}$$

$$\nabla^*\nabla = 2(D - A).$$

Hence, for any vector f,

$$2\langle (D-A)f, f\rangle = \langle \nabla f, \nabla f\rangle = \sum_{(x,y)\in\vec{E}} (f(x)-f(y))^2 \geqslant 0.$$

In other words,

$$D - A \geqslant 0.$$

We get

$$-\max_x \deg(x) \leqslant \lambda_{|V|}(A) \leqslant \cdots \leqslant \lambda_1(A) \leqslant \max_x \deg(x).$$

Markov transition matrix

The transition matrix of the simple random walk on G is

$$P_{xy} = \frac{A_{xy}}{\deg(x)}.$$

We have

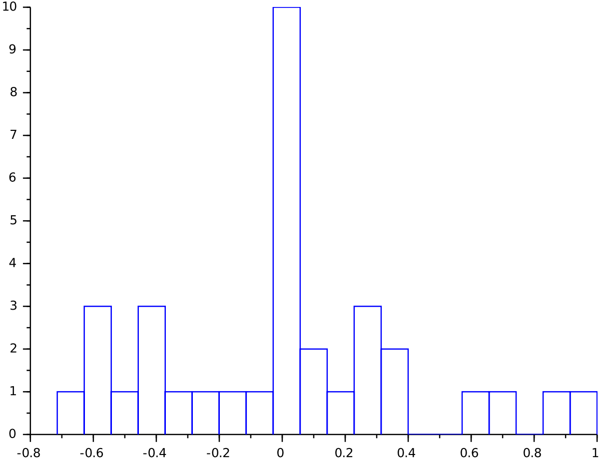
$$P = D^{-1}A.$$

P has real eigenvalues :

$$P = D^{-1}A = D^{-1/2}\left(D^{-1/2}AD^{-1/2}\right)D^{1/2}.$$

Google matrix : for $\alpha \in (0,1]$, $\alpha P + (1-\alpha)\mathbf{1}\mathbf{1}^*/|V|$.

Markov transition matrix



Markov transition matrix

Define the left vector

$$\nu(x) = \deg(x).$$

We have

$$\nu P = \nu.$$

ν is a left eigenvector with eigenvalue 1 and

$$\pi(x) = \frac{\nu(x)}{\sum_y \nu(y)} = \frac{\deg(x)}{2|E|}$$

is the invariant probability measure of the random walk.

Markov transition matrix

The symmetry

$$\pi(x)P_{xy} = \pi(y)P_{yx} = \frac{\mathbf{1}(\{x,y\} \in E)}{2|E|}$$

is called reversibility.

It asserts that the matrix P is symmetric in $L^2(\pi)$ with scalar product

$$\langle f, g \rangle_\pi = \sum_x \pi(x) f(x) g(x),$$

i.e. $\langle Pf, g \rangle_\pi = \langle f, Pg \rangle_\pi$.

It follows that P has real eigenvalues in $[-1, 1]$ and an orthonormal basis of eigenvectors in $L^2(\pi)$.

Laplacian matrix

$$L = D - A.$$

$-L$ is the infinitesimal generator of the countinuous time random walk:

$$\frac{d}{dt}\mathbb{E}^x f(X_t)\Big|_{t=0} = -Lf(x).$$

It is symmetric, $L \geqslant 0$ with eigenvalues in

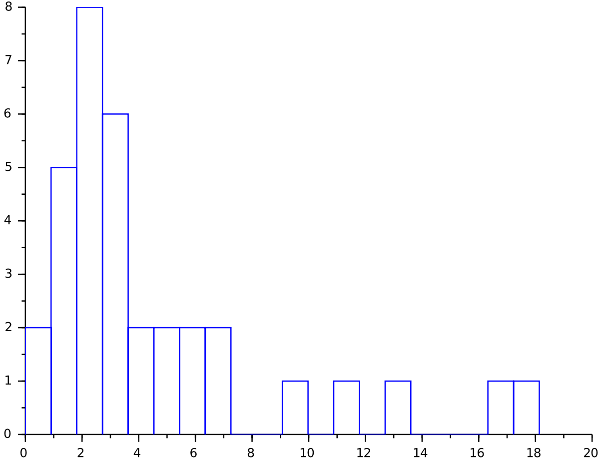
$$[0, 2\max_x \deg(x)].$$

Moreover

$$L\mathbf{1} = A\mathbf{1} - D\mathbf{1} = 0.$$

The invariant probability measure of the process is the uniform measure.

Laplacian matrix



Combinatorial Laplacian matrix

Matrix on $V \times V$,

$$D^{-1/2} L D^{-1/2} = D^{1/2}(I - P) D^{-1/2}.$$

It is symmetric and has eigenvalues in $[0, 2]$.

There are other interesting local matrices ...

Regular graphs

If G is d-regular, then $D = dI$ commutes with A : all these matrices have the same eigenspace decomposition.

Typical vs Extremal Eigenvalues

There are essentially two types of information encoded in the spectrum.

- PART II : the largest eigenvalues (and their eigenspaces) give some information on global graph properties (expansion, clustering, chromatic number, maximal cut, etc...),

- PART I : the typical eigenvalues give information on local graph properties (typical degree, partition function of spanning trees, matchings, percolation, etc...).

Large sparse random graphs

We will study the spectrum of classical random graphs in the regime :

- *Large*

$$|V| \to \infty.$$

- *Sparse / Dilute*

$$|E| = O(|V|).$$

Part I: Typical Eigenvalues

Spectral Measures

Eigenvalues

For $M \in M_n(\mathbb{R})$ is a symmetric matrix, we denote its real eigenvalues by

$$\lambda_n(M) \leqslant \ldots \leqslant \lambda_1(M).$$

Spectral measure

The spectral measure / empirical distribution of the eigenvalues / density of states is the probability measure on $\mathbb{R}$,

$$\mu_M = \frac{1}{n}\sum_{i=1}^{n} \delta_{\lambda_i(M)},$$

i.e. for any set $I \subset \mathbb{R}$

$$\mu_M(I) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(\lambda_i(M) \in I)$$

is the proportion of eigenvalues in I or equivalently, the probability that a typical eigenvalue is in I.

$$\int f d\mu_M = \frac{1}{n}\sum_{i=1}^{n} f(\lambda_i(M)).$$

Kirchoff Matrix-Tree Theorem

If G is a connected graph then the number of spanning trees of G is equal to

$$t(G) = \frac{1}{n} \prod_{\lambda_i \neq 0} \lambda_i,$$

where $\lambda_i = \lambda_i(L)$.

In particular,

$$\frac{1}{n} \log t(G) = \int_{0^+}^{\infty} \log \lambda d\mu_L(\lambda) - \frac{1}{n} \log n.$$

Closed paths

For t integer, let

$$S_t \quad = \quad |\{\text{closed paths of length } t \text{ in } G\}|$$

We have

$$S_t = \mathrm{Tr}\{A^t\} = \sum_{i=1}^{n} \lambda_i(A)^t = n \int \lambda^t d\mu_A(\lambda).$$

In particular, for $z \in \mathbb{C}$, $\mathfrak{Im}(z) > 0$,

$$\frac{1}{n} \sum_{t \geqslant 0} \frac{S_t}{z^{t+1}} = \sum_{t \geqslant 0} \int \frac{\lambda^t}{z^{t+1}} d\mu_A = \int \frac{1}{z - \lambda} d\mu_A(\lambda)$$

is the Cauchy-Stieltjes transform of μ_A.

Return times

If X_t is the Markov chain with transition matrix P,

$$\frac{1}{n}\sum_{v=1}^{n}\mathbb{P}(X_t = v|X_0 = v) = \frac{1}{n}\mathrm{Tr}\{P^t\} = \int \lambda^t d\mu_P(\lambda).$$

Similarly, for $t > 0$ real, if X_t is the Markov process with generator L,

$$\frac{1}{n}\sum_{v=1}^{n}\mathbb{P}(X_t = v|X_0 = v) = \int e^{-t\lambda} d\mu_L(\lambda).$$

Spectral measure at a vector

Let $M \in M_n(\mathbb{R})$ be a symmetric matrix. Let $\psi_1, \ldots \psi_n$ be an orthonormal basis of eigenvectors :

$$M = \sum_k \lambda_k \psi_k \psi_k^*.$$

For $\phi \in \mathbb{R}^n$ with $\|\phi\|_2 = 1$, we define the probability measure,

$$\mu_M^\phi = \sum_{k=1}^n \langle \psi_k, \phi \rangle^2 \delta_{\lambda_k}.$$

We have

$$\int \lambda^k d\mu_M^\phi = \langle \phi, M^k \phi \rangle.$$

Spectral measure at a vector

We recover the spectral measure from the spatial average

$$\frac{1}{n}\sum_{x=1}^{n}\mu_M^{e_x} = \frac{1}{n}\sum_{x=1}^{n}\sum_{k=1}^{n}|\psi_k(x)|^2\delta_{\lambda_k} = \frac{1}{n}\sum_{k=1}^{n}\delta_{\lambda_k}\sum_{x=1}^{n}|\psi_k(x)|^2 = \mu_M.$$

While $\mu_M^{e_x}$ depends on the eigenvectors, its spatial average μ_M does not.

Spectral measure at a vector

This local notion of spectrum will be used to define the spectral of a possibly infinite graph.

We will restrict ourselves to the adjacency matrix and set

$$\mu_G := \mu_A \quad \textit{and} \quad \mu_G^{e_x} := \mu_A^{e_x}.$$

It works the same for P or L.

Adjacency operator

Let $G = (V, E)$ be a locally finite graph : for all $x \in V$,

$$\deg(x) = \sum_{y \in V} \mathbf{1}\left(\{x, y\} \in E\right) < \infty.$$

Let $\ell^2(V) = \left\{\psi : \sum_{x \in V} \psi(x)^2 < \infty\right\}$ and $\ell^2_c(V)$ as the subspace of vectors with finite support : i.e. the subspace spanned by finite linear combinations of $e_x, x \in V$.

Adjacency operator : defined for vectors $\psi \in \ell^2_c(V)$

$$A\psi(x) = \sum_{y : \{x, y\} \in E} \psi(y),$$

equivalently, with matrix notation :

$$A_{xy} = \langle e_x, Ae_y \rangle = \mathbf{1}(\{x, y\} \in E).$$

Adjacency operator

Under mild assumptions, A is essentially self-adjoint (e.g. for all $v \in V$, $\deg(v) \leqslant \theta$).

The spectral measure with vector $\psi \in \ell^2_c(V)$, $\|\psi\|_2 = 1$, is the probability measure μ_G^ψ on $\mathbb{R}$ such that

$$\forall k \geqslant 1, \quad \int \lambda^k d\mu_G^\psi = \langle \psi, A^k \psi \rangle.$$

As a consequence,

$$\int \lambda^k d\mu_G^{e_x} = \left| \{\text{closed paths of length } k \text{ starting from } x\} \right|.$$

Transitive graphs

If G is vertex-transitive (e.g. a Cayley graph associated to a transitive group Γ with a finite symmetric generating set $S \subset \Gamma$), the measure

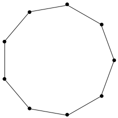
$$\mu_G := \mu_G^{e_x}$$

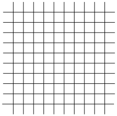
does not depend on x.

Plancherel measure, Kesten-von Neumann-Serre spectral measure.

(If G is finite, then the two definitions coincide).

Lattices





Cycle

$$\mu_{\mathbb{Z}/n\mathbb{Z}} = \frac{1}{n}\sum_{k=1}^{n} \delta_{2\cos\left(\frac{2\pi k}{n}\right)}.$$

Bi-infinite path

$$\mu_{\mathbb{Z}}(dx) = \frac{1}{\pi\sqrt{4-x^2}}\mathbf{1}_{|x|\leqslant 2}dx.$$

Regular lattice

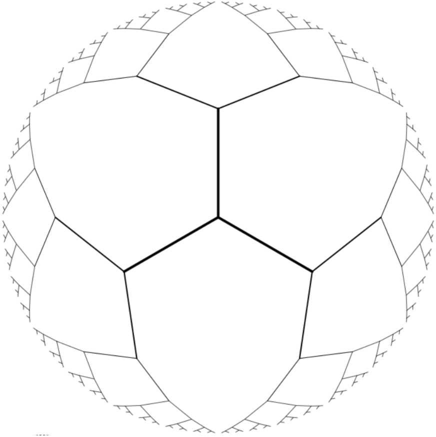
$$\mu_{\mathbb{Z}^d} = \mu_{\mathbb{Z}} * \cdots * \mu_{\mathbb{Z}}.$$

Infinite regular tree

$\mathbb{T}_d$ infinite d-regular tree

$$\mu_{\mathbb{T}_d}(dx) = \frac{d\sqrt{4(d-1) - x^2}}{2\pi(d^2 - x^2)} \mathbf{1}_{|x| \leqslant 2\sqrt{d-1}} dx.$$

Kesten (1959)

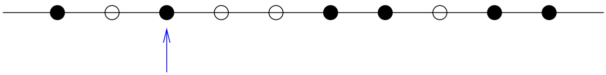


Lamplighter

Consider a vertex-transitive graph $G = (V, E)$ and a colored lamp in $L = \mathbb{Z}/n\mathbb{Z}$ on each vertex. A vertex of the lamplighter graph is

$$v = (\eta, x)$$

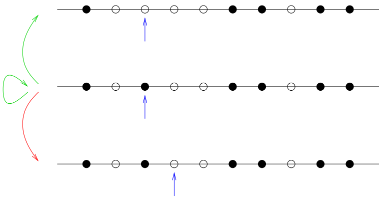
where $\eta : V \to L$ is the configuration of the lamps and $x \in V$ is the position of the walker.



Lamplighter

A switch edge (S) $\{v, v'\}$ is an edge between two vertices which differ only by the lamp at the position of the walker.

A walk edge (W) $\{v, v'\}$ is an edge s.t. $\eta = \eta'$, $\{x, y\} \in E$.



The WS lamplighter graph is the graph with edge set

$$\{\{v, v'\} : \{v, u\} \in W, \{u, v'\} \in S \text{ for some } u\}.$$

Similarly for SW and SWS graphs.

Let G_p be the site percolation with parameter $p \in [0,1]$ and $o \in V$.

Theorem (Lehner, Neuhauser and Woess (2008))
For $p = 1/n$, *we have*

$$\mu_{SW}(\cdot/n) = \mu_{WS}(\cdot/n) = \mu_{SWS}(\cdot/n^2) = \mathbb{E}\mu_{G_p}^{e_o}(\cdot).$$

For $G = \mathbb{Z}$, $n = 2$, for some explicit (ω_n),

$$\mu_{SW} = \sum_{n=0}^{\infty} \omega_n \sum_{k=1}^{n} \delta_{4\cos\left(\frac{\pi k}{(n+1)}\right)},$$

Grigorchuk and Żuk (2001)

Connectivity and homogeneity do not guarantee a density for the spectral measure !

Sketch of Proof

Let $\mu = \mathbb{E}\mu^{e_o}_{G_p}$ and $\nu = \mu_{WS}(\cdot/n)$. We compare moments.

Let W_k be the set of closed walks $\gamma = (\gamma_0, \cdots, \gamma_k)$ in G of length k starting at o.

$$\int \lambda^k d\mu^{e_o}_{G_p}(\lambda) = \sum_{\gamma \in W_k} \prod_{t=0}^{k} \mathbf{1}(\gamma_t \text{ is open}) = \sum_{\gamma \in W_k} \prod_{x \in V(\gamma)} \mathbf{1}(x \text{ is open})$$

$$\int \lambda^k d\mu(\lambda) = \sum_{\gamma \in W_k} p^{|V(\gamma)|}.$$

Sketch of Proof

The graph G is d-regular. If $S_t = (\eta_t, x_t)$ is a random walk on the WS-lampighter graph and $\varepsilon = (\underline{0}, o)$,

$$\int \lambda^k d\nu = d^k \mathbb{P}^\varepsilon(S_k = \varepsilon).$$

We have

$$\eta_t(x_t) = \eta_{t-1}(x_t) + \ell_t,$$

where ℓ_t is independent of (x_t, η_{t-1}) and uniform on $\mathbb{Z}/n\mathbb{Z}$.

For any $q \in \mathbb{Z}/n\mathbb{Z}$.

$$\mathbb{P}(\ell_t + q = 0) = \frac{1}{n} = p.$$

If τ_x is the last passage time of $(x_t)_{0 \leqslant t \leqslant k}$ at x,

$$\begin{aligned} \mathbb{P}^\varepsilon(S_k = \varepsilon) &= d^{-k} \sum_{\gamma \in W_k} \mathbb{P}(\forall x \in V(\gamma) : \eta_{\tau_x}(x) + \ell_{\tau_x} = 0) \\ &= d^{-k} \sum_{\gamma \in W_k} p^{|V(\gamma)|}. \end{aligned}$$

Random rooted graphs

So far : μ_G well defined for finite graphs and vertex-transitive graphs :

$$\mu_G = \mathbb{E}\mu_G^{e_o} = \begin{cases} \frac{1}{|V|}\sum_x \mu_G^{e_x} & \text{(finite)} \\ \mu_G^{e_x} & \text{(transitive)} \end{cases}$$

We want to extend the notion to a large class of "stationary" random graphs.

For a random (unlabeled) connected rooted graph (G, o) with law ρ, we define

$$\mu_\rho := \mathbb{E}_\rho \mu_G^{e_o}.$$

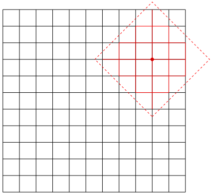
PART I: TYPICAL EIGENVALUES

Spectral measures and BS convergence

Benjamini-Schramm convergence

BS convergence of finite graph sequences = convergence of typical local neighborhood.

For integer k : $(G, o)_k$ is the rooted (connected) graph spanned by vertices at distance at most k from o.



$G_n = (V_n, E_n)$ has BS limit $\rho = \mathcal{L}((G, o))$ if for any integer k and unlabeled rooted graph g of diameter k,

$$\frac{1}{|V_n|} \sum_{x \in V_n} \mathbf{1}((G_n, x)_k = g) \to \mathbb{P}_\rho((G, o)_k = g).$$

BS limits

$G_n = \mathbb{Z}^d \cap [0,n]^d$ has BS limit ? $\delta_{(\mathbb{Z}^d,0)}$

$T_n = \mathbb{T}_3 \cap \{x : |x| \leqslant n\}$ has BS limit ?

BS LIMITS

Uniform d-regular graph : a.s. the limit is the (Dirac mass at) $\mathbb{T}_d$ rooted somewhere.

Erdős-Rényi graph, $\mathcal{G}(n, \alpha/n)$: a.s. the limit is the law of (T, o) where T is a Galton-Watson tree with offspring distribution $\mathrm{Poi}(\alpha)$.

Random graphs : many random graphs have random rooted trees as BS limit.

Unimodular graphs

Unimodular random rooted graphs : subclass which contains Cayley graphs and all BS limits of finite graphs.

A law ρ on (unlabeled) rooted graphs is unimodular if for any non-negative functions $f(G,x,y)$ invariant by graph-isomorphisms,

$$\mathbb{E}_\rho \sum_{x\in V} f(G,o,x) = \mathbb{E}_\rho \sum_{x\in V} f(G,x,o).$$

Benjamini/Schramm (2001), Aldous/Steele (2004)

Uniform rooting is unimodular

For finite G, $U(G)$ the law of $(G(o), o)$, where o is uniform on V and $G(o)$ is the c.c. of o, is unimodular

$$\begin{aligned}
\mathbb{E}_{U(G)} \sum_{x \in V} f(G, o, x) &= \frac{1}{|V|} \sum_{y} \sum_{x \in V(y)} f(G(y), y, x) \\
&= \frac{1}{|V|} \sum_{x} \sum_{y \in V(x)} f(G(y), y, x) \\
&= \frac{1}{|V|} \sum_{x} \sum_{y \in V(x)} f(G(x), y, x) \\
&= \mathbb{E}_{U(G)} \sum_{x \in V} f(G, x, o).
\end{aligned}$$

Continuity of spectral measure

Theorem

Let G_n be a sequence of finite graphs with BS-limit ρ. Then

$$d_{\mathrm{KS}}(\mu_{G_n}, \mu_\rho) = \sup_{t \in \mathbb{R}} |\mu_{G_n}(-\infty, t] - \mu_\rho(-\infty, t]| \to 0.$$

Consequently, for any real λ, $\mu_{G_n}(\{\lambda\}) \to \mu_\rho(\{\lambda\})$.

Veselić (2005), Thom (2008), Bordenave/Lelarge (2010), Abèrt/Thom/Viràg (2013)

Continuity of spectral measure

Corollary (Thom (2008))

Let G_n be a sequence of finite graphs with BS-limit ρ. Then

$$\mu_\rho(\{\lambda\}) > 0$$

implies that λ is a totally real algebraic integer.

Sketch of proof

Assume for simplicity that $\deg_{G_n}(x) \leqslant \theta$.

Weak convergence is easy :

$$\int \lambda^k d\mu_{G_n} = \frac{1}{|V_n|} \sum_{x \in V_n} |\{\text{closed paths of length } k \text{ starting from } x\}| .$$

is bounded by θ^k and it depends only on $(G_n, o)_k$.

Sketch of proof

Convergence in KS-distance = weak convergence + cv of atoms.

From $\liminf \mu_n(O) \geqslant \mu(O)$, $\limsup \mu_n(C) \leqslant \mu(C)$, we should prove that

$$\liminf \mu_{G_n}(\{\lambda\}) \geqslant \mu_\rho(\{\lambda\}).$$

Since

$$\liminf \mu_{G_n}((\lambda-\varepsilon, \lambda+\varepsilon)) \geqslant \mu_\rho((\lambda-\varepsilon, \lambda+\varepsilon)) \geqslant \mu_\rho(\{\lambda\}),$$

the theorem follows from

Lemma (Lück)

Let $\lambda \in \mathbb{R}$, $\theta > 0$. There exists a continuous function $\delta : \mathbb{R} \to [0, 1]$ with $\delta(0) = 0$ depending on (λ, θ) s.t. for any finite graph G with degrees bounded θ, $\varepsilon > 0$,

$$\mu_G((\lambda-\varepsilon, \lambda+\varepsilon)) \leqslant \mu_G(\{\lambda\}) + \delta(\varepsilon).$$

Sketch of proof

For $\lambda = 0$, $\varepsilon \in (0,1)$,

$$\mu_G((-\varepsilon,\varepsilon)) \leqslant \mu_G(\{0\}) + \frac{\log(\theta)}{\log(1/\varepsilon)}.$$

reads, with $n = |V|$, $k = |\{i : 0 < |\lambda_i| < \varepsilon\}|$,

$$k \leqslant n \frac{\log(\theta)}{\log(1/\varepsilon)}.$$

We observe

$$\prod_{i:\lambda_i \neq 0} \lambda_i \in \mathbb{Z}\backslash\{0\}.$$

Hence

$$1 \leqslant \prod_{\lambda_i \neq 0} |\lambda_i| = \prod_{0<|\lambda_i|<\varepsilon} |\lambda_i| \prod_{|\lambda_i| \geqslant \varepsilon} |\lambda_i| \leqslant \varepsilon^k \theta^n.$$

Kesten-McKay Law

Theorem

Fix integer $d \geqslant 2$. If G_n has BS limit $\mathbb{T}_d$, then for any $I \subset \mathbb{R}$,

$$\mu_{G_n}(I) \to \mu_{\mathbb{T}_d}(I),$$

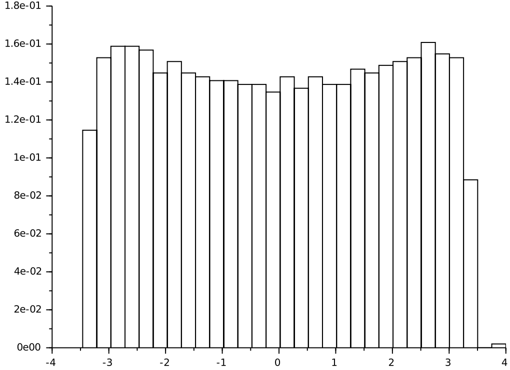
where

$$\mu_{\mathbb{T}_d}(dx) = \frac{d}{2\pi} \frac{\sqrt{4(d-1) - x^2}}{d^2 - x^2} \mathbf{1}_{|x| \leqslant 2\sqrt{d-1}} dx.$$

We have $\mu_{KM}(I\sqrt{d}) \to \mu_{sc}(I)$, the semi-circular distribution, when $d \to \infty$.

Kesten-McKay Law

Take $d = 4$, $n = 2000$ and G a uniformly sampled d-regular graph.



Theorem

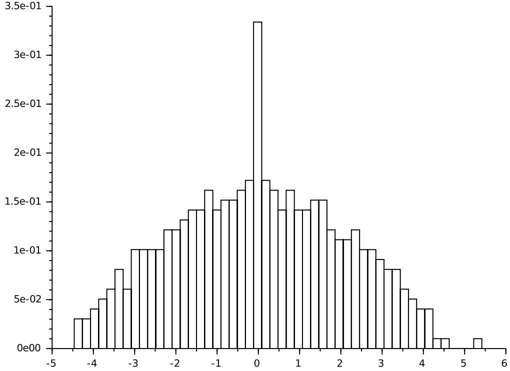
Fix $\alpha > 0$. Let G_n be an Erdős-Rényi graph with parameter $p = \alpha/n$. Then, with probability one, for any interval $I \subset \mathbb{R}$,

$$\mu_{G_n}(I) \to \mu_\rho(I).$$

where ρ is the law of a Galton-Watson tree with $\mathrm{Poi}(\alpha)$ offspring distribution.

Erdős-Rényi

Histogram of eigenvalues for $\alpha = 4$ and $n = 500$.



Erdős-Rényi

There is no explicit expression for μ_ρ.

Let $\Lambda = \{\lambda_i, i \geqslant 1\}$, be the atoms of μ_ρ, i.e.

$$\Lambda = \{\lambda : \mu_\rho(\{\lambda\}) > 0\}.$$

Λ is the set totally real algebraic integers and

$$\sum_{\lambda \in \Lambda} \mu_\rho(\{\lambda\}) < 1$$

if and only if $\alpha > 1$.

Also, $\mu_\rho(\{0\})$ has a closed-form expression.

Bordenave/Lelarge/Salez (2012), Salez (2013), Bordenave/Virág/Sen (2014)

Part I: Typical Eigenvalues

Spectral percolation

Regularity of the spectral measure

Any probability measure on $\mathbb{R}$ can be decomposed as

$$\mu = \mu_{pp} + \mu_c = \mu_{pp} + \mu_{ac} + \mu_{sc}.$$

For $|V| = \infty$, the decompositions of

$$\mu_G^{e_o} \quad \text{and} \quad \mu_\rho = \mathbb{E}\mu_G^{e_o}$$

reveal deep information on the graph.

In the context of random Schrödinger operators, called quantum percolation, *De Gennes, Lafore, Millot (1959)*.

Resolution of the identity

For finite graphs, the decomposition

$$A = \sum_k \lambda_k \psi_k \psi_k^*$$

induces a projection-valued measure, for Borel $I \subset \mathbb{R}$,

$$E(I) = \sum_k \mathbf{1}(\lambda_k \in I) \psi_k \psi_k^*.$$

$E(\{\lambda\})$ is the orthogonal projection on the vector space of λ-eigenvectors and

$$\mu_G^{\psi}(I) = \langle E(I)\psi, \psi \rangle = \|E(I)\psi\|_2^2.$$

This p.v.m. exists also for infinite graphs.

Localization/Delocalization of eigenvectors

What are the nature of the probability vectors,

$$\big(|\psi_k(x)|^2, x \in V\big) \quad ?$$

Localization is related to the atomic part of $\mu_G^{e_x}$

$$\mu_G^{e_x}(\{\lambda\}) = \|E(\{\lambda\})e_x\|_2^2.$$

Delocalization is related to the continuous part of $\mu_G^{e_x}$. If

$$\sum_{\lambda_k \in I} |\psi_k(x)|^2 = \mu_G^{e_x}(I) \leqslant c|I|,$$

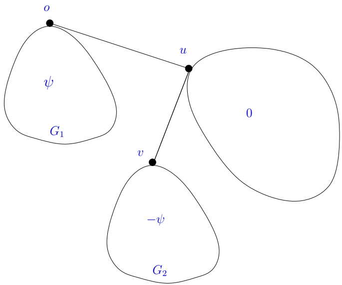
then $|\psi_k(x)|^2 \leqslant c|I|$ for all λ_k in I.

Atoms

Finite pending graphs create atoms (e.g. percolation graphs) *Kirkpatrick/Eggarter (1972).*

If $G_1 \simeq G_2$ and $A_{G_1}\psi = \lambda\psi$, $\|\psi\|_2 = 1/\sqrt{2}$, then

$$\mu_G^{e_o}(\{\lambda\}) = \|E(\{\lambda\})e_o\|_2^2 \geqslant \langle\varphi, e_o\rangle^2 = \psi(o)^2.$$



$\varphi = \psi_{|G_1} - \psi_{|G_2}$

Warning : recall lamplighter graphs !!

Random rooted trees

Topological end of a rooted tree : semi-infinite self-avoiding path starting from the root.

Theorem

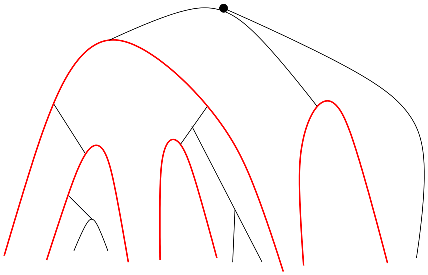
Let (T, o) be a unimodular tree with law ρ. If, with positive probability, T has 2 or more topological ends then μ_ρ has a continuous part.

0 end : finite trees.
1 end ?
2 ends : $\mathbb{Z}$.
∞ ends : all others, e.g. supercritical Galton-Watson trees.

Bordenave/Virág/Sen (2014)

Invariant Line Ensemble

Let (T, o) be a unimodular tree with law ρ.



An invariant line ensemble L is a subset of non intersecting doubly infinite lines in T which does not depend on the choice of the root o.

$\mathbb{P}(o \in L)$ is the density of the invariant line ensemble.

Theorem

Let (T, o) *be a unimodular tree with law* ρ.

If L *is an invariant line ensemble of* (T, o) *then the total mass of atoms of* μ_ρ *is bounded above by* $\mathbb{P}(o \notin L)$.

Moreover, for each real λ,

$$\mu_\rho(\{\lambda\}) \leqslant \mathbb{P}(o \notin L)\mu_{\rho'}(\{\lambda\})$$

where, if $\mathbb{P}(o \notin L) > 0$, ρ' *is the law of the rooted tree* $(T \backslash L, o)$ *conditioned on the root* $o \notin L$.

Invariant Line Ensemble

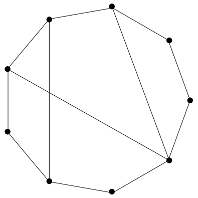
There are explicit lower bounds on the density $\mathbb{P}(o \in L)$.

For example, if (T, o) is a unimodular random tree, there exists an invariant line ensemble L such that

$$\mathbb{P}\left(o \in L\right) \geqslant \frac{1}{6} \frac{(\mathbb{E}\deg_T(o) - 2)_+^2}{\mathbb{E}\deg_T(o)^2}.$$

Watts-Strogatz random graph

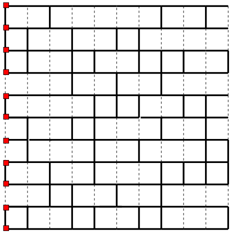
G_n is obtained by superposing the graphs of $\mathbb{Z}/n\mathbb{Z}$ + Erdős-Rényi graph $\mathcal{G}(n, \alpha/n)$.



Then μ_{G_n} converges and it is continuous.

Proof by an example : vertical percolation

Consider the following $n \times n$ graph.



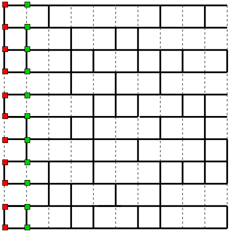
S = eigenspace associated to eigenvalue λ.
R = vector space spanned by red vertices.

$$\dim(S \cap R^{\perp}) \geqslant \dim(S) - \dim(R) = \dim(S) - n.$$

Proof by an example : vertical percolation

If $f \in S \cap R^{\perp}$, we write

$$0 = (A - \lambda) f(x) = \sum_{y \sim x} f(y).$$



For x red vertex, we get that f is also 0 on the green vertices.

By iteration, $S \cap R^{\perp} = \emptyset$ and

$$n^2 \mu_G(\{\lambda\}) = \dim(S) \leqslant n = o(n^2).$$

Other questions

Works also for supercritical percolation on $\mathbb{Z}^2$ (other method).

No criterion for existence of ac part in $\mu_\rho = \mathbb{E}_\rho \mu_G^{e_o}$.

The same questions for $\mu_G^{e_o}$ are essentially open, *Keller (2013), Bordenave (2014)*.

Their are finite volume versions of these questions.

Quantum percolation on a regular tree

Consider T_p, the bond percolation on $\mathbb{T}_d$ with parameter p.

Then, for any $0 < p < 1$, $\mathbb{E}\mu_{T_p}^{e_o}$ has dense atomic part on its support $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.

For all $p > p_0$, conditioned on non-extinction, $\mu_{T_p}^{e_o}$ has non-trivial ac part.

Bordenave (2014)

Part II: Extremal Eigenvalues

Convergence to Equilibrium

Spectral gap

Take a connected graph on n vertices.

The spectral gap

$$\min_{\lambda \neq 0} \lambda(L)$$

$$1 - \max_{\lambda \neq 1} \lambda(P)$$

is closely related to the rate convergence of the Markov chain/process.

For simplicity we only consider L.

Spectral gap

Let X_t be the Markov process with generator $-L$,

$$P_t^x = e^{-tL} e_x$$

is the probability distribution of X_t given $X_0 = x$.

Let $\lambda_1 = 0 < \lambda_2 \leqslant \cdots \leqslant \lambda_n$ the eigenvalues of L and $\psi_1 = \mathbf{1}/\sqrt{n}, \ldots, \psi_n$ an orthogonal basis of eigenvectors.

From the spectral theorem

$$\begin{aligned} e^{-tL} &= \sum_{i=1}^{n} e^{-t\lambda_i} \psi_i \psi_i^* \\ P_t^x &= \frac{\mathbf{1}}{n} + \sum_{i=2}^{n} e^{-t\lambda_i} \psi_i(x) \psi_i \end{aligned}$$

Spectral gap

Recall that $\Pi = \mathbf{1}/n$ is the invariant distribution. We get

$$\|P_t^x - \Pi\|_2^2 = \sum_{i=2}^{n} e^{-2t\lambda_i} |\psi_i(x)|^2 \leqslant e^{-2\lambda_2 t}.$$

Recall

$$\|x\|_2 \leqslant \sum_i |x_i| \leqslant \sqrt{n}\|x\|_2.$$

So,

$$|\psi_2(x)| e^{-\lambda_2 t} \leqslant 2\|P_t^x - \Pi\|_{TV} \leqslant \sqrt{n} e^{-\lambda_2 t}.$$

where the total variation norm is

$$\|\mu - \nu\|_{TV} = \frac{1}{2} \sum_x |\mu(x) - \nu(x)|.$$

Spectral gap

The mixing time of a Markov process is usually defined as

$$\tau = \inf_{t>0} \max_x \|P_t^x - \Pi\|_{TV} \leqslant \frac{1}{2}.$$

$$\frac{\max_x |\psi_2(x)|}{\lambda_2} \leqslant \tau \leqslant \frac{\log n}{2\lambda_2}.$$

(Note that $\max_x |\psi_2(x)| \geqslant 1/\sqrt{n}$).

There are similar developments for reversible Markov chains.

Levin/Peres/Wilmer (2009)

Part II: Extremal eigenvalues

Expanders

Let

$$1 = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant -1$$

be the eigenvalues of P.

Set

$$\lambda_\star = \max_{i \neq 1} |\lambda_i|.$$

Theorem

If G connected,

$$\mathrm{diam}(G) \leqslant \left\lceil \frac{\log (2|E|)}{\log (1/|\lambda_\star|)} \right\rceil.$$

Proof

Since

$$P = D^{-1}X = D^{-1/2}(D^{-1/2}AD^{-1/2})D^{1/2},$$

the λ_i 's are also the eigenvalues of S with $S = D^{-1/2}AD^{-1/2}$.

Since $P\mathbf{1} = \mathbf{1}$,

$$\psi_1 = \frac{D^{1/2}\mathbf{1}}{\sqrt{2|E|}}$$

is the normalized eigenvector of S associated to $\lambda_1 = 1$.

$$S^t = \psi_1\psi_1^* + \sum_{k\geqslant 2} \lambda_k^t \psi_k \psi_k^*.$$

Hence, from Cauchy-Schwartz

$$\begin{aligned}
(S^t)_{xy} &\geqslant \psi_1(x)\psi_1(y) - \lambda_\star^t \sum_{k\geqslant 2} |\psi_k(x)||\psi_k(y)| \\
&\geqslant \psi_1(x)\psi_1(y) - \lambda_\star^t \sqrt{\sum_{k\geqslant 2} |\psi_k(x)|^2} \sqrt{\sum_{k\geqslant 2} |\psi_k(y)|^2}.
\end{aligned}$$

Proof

Since

$$\sum_{k \geqslant 2} |\psi_k(x)|^2 = 1 - \psi_1(x)^2 < 1;$$

We find

$$(S^t)_{xy} > \psi_1(x)\psi_1(y) - \lambda_\star^t.$$

This is positive if

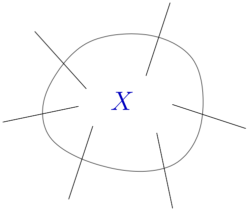
$$t > \frac{\log\left(\psi_1(x)\psi(y)\right)}{\log|\lambda_\star|} = \frac{\log\left(2|E|/\sqrt{\deg(x)\deg(y)}\right)}{\log\left(1/|\lambda_\star|\right)}.$$

Cheeger's Constant

For $X \subset V$, define

$$\mathrm{vol}(X) = \sum_{x \in X} \deg(x).$$

$$\mathrm{area}(\partial X) = \sum_{x \in X, y \in X^c} \mathbf{1}(xy \in E).$$



Isoperimetric / Expansion constant :

$$h(G) = \min_{X \subset V} \frac{\mathrm{area}(\partial X)}{\min\left(\mathrm{vol}(X), \mathrm{vol}(X^c)\right)}.$$

Cheeger's Inequality

Again

$$1 = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_n \geqslant -1$$

be the eigenvalues of P.

$1 - \lambda_2$ is the spectral gap of P.

Theorem

$$\frac{h(G)^2}{2} \leqslant 1 - \lambda_2 \leqslant 2h(G).$$

Proof (easy half)

The λ_i 's are also the eigenvalues of S with $S = D^{-1/2} A D^{-1/2}$.

$\chi = D^{1/2}\mathbf{1}$ is the eigenvector of S associated to $\lambda_1 = 1$.

From Courant-Fisher variational formula,

$$\lambda_2 = \max_{g : \langle g, \chi \rangle = 0} \frac{\langle Sg, g \rangle}{\|g\|_2^2}.$$

Or equivalently,

$$1 - \lambda_2 = \min_{g : \langle g, \chi \rangle = 0} \frac{\langle (I - S)g, g \rangle}{\|g\|_2^2}.$$

Proof (easy half)

Recall, for the incidence matrix,

$$I - S = D^{-1/2}(D-A)D^{-1/2} = D^{-1/2}\frac{\nabla^*\nabla}{2}D^{-1/2}$$

Set $\pi(x) = \deg(x) = (D\mathbf{1})(x)$ and $f = D^{-1/2}g$,

$$1 - \lambda_2 = \min_{f:\langle f,\pi\rangle = 0} \frac{\sum_{x\sim y}(f(x)-f(y))^2}{\sum_x \deg(x) f(x)^2}.$$

Let X be such that

$$h(G) = \frac{\operatorname{area}(\partial X)}{\min\left(\operatorname{vol}(X), \operatorname{vol}(X^c)\right)}.$$

We take

$$f(x) = \frac{\mathbf{1}(x \in X)}{\operatorname{vol}(X)} - \frac{\mathbf{1}(x \notin X)}{\operatorname{vol}(X^c)}.$$

Proof (easy half)

We have

$$\langle f, \pi \rangle = \sum_{x \in X} \frac{\deg(x)}{\mathrm{vol}(X)} - \sum_{x \in X^c} \frac{\deg(x)}{\mathrm{vol}(X^c)} = 0,$$

and

$$\begin{aligned} 1 - \lambda_2 &\leqslant \frac{\sum_{x \sim y} (f(x) - f(y))^2}{\sum_x \deg(x) f(x)^2} \\ &= 2\mathrm{area}(\partial X) \frac{(1/\mathrm{vol}(X) - 1/\mathrm{vol}(X^c))^2}{1/\mathrm{vol}(X) + 1/\mathrm{vol}(X^c)} \\ &\leqslant 2 \frac{\mathrm{area}(\partial X)}{\min(\mathrm{vol}(X), \mathrm{vol}(X^c))} \\ &\leqslant 2h(G). \end{aligned}$$

Random graphs are expanders

Consider the configuration model with degree sequence $d_1, \cdots, d_n$ such that

$$\min_i d_i \geqslant 3 \qquad \text{and} \qquad \sum_i d_i \leqslant n^{5/4}.$$

Then, with high probability,

$$h(G) \geqslant 0.01.$$

Abdullah/Cooper/Frieze (2012)

Part II: Extremal Eigenvalues

Outliers

BS convergence

Theorem

Take A, L or P. Let G_n be a sequence of graphs on n vertices with BS limit ρ. Then for any $k = o(n)$,

$$\lambda_k \geqslant b + o(1) \quad \text{and} \quad \lambda_{n-k} \leqslant a + o(1).$$

where $[a, b]$ is the convex hull of the support of $\mu_\rho = \mathbb{E}_\rho \mu_G^{e_o}$ (with the corresponding operator).

$|a| \vee b$ is the spectral radius of the operator.

Proof

We know already that

$$d_{\mathrm{KS}}(\mu_{G_n}, \mu_\rho) = \sup_{t\in\mathbb{R}} |\mu_{G_n}(-\infty, t] - \mu_\rho(-\infty, t]| \to 0.$$

Hence, for $I = (b-\varepsilon, \infty)$,

$$\lim \mu_{G_n}(I) = \mu_\rho(I) = \eta > 0.$$

In words : the nb of eigenvalues larger than $b-\varepsilon$ is at least $n(\eta + o(1)) \gg k$.

We get that for n large enough, $\lambda_k \geqslant b - \varepsilon$.

Outliers

Assume G_n has BS limit ρ.

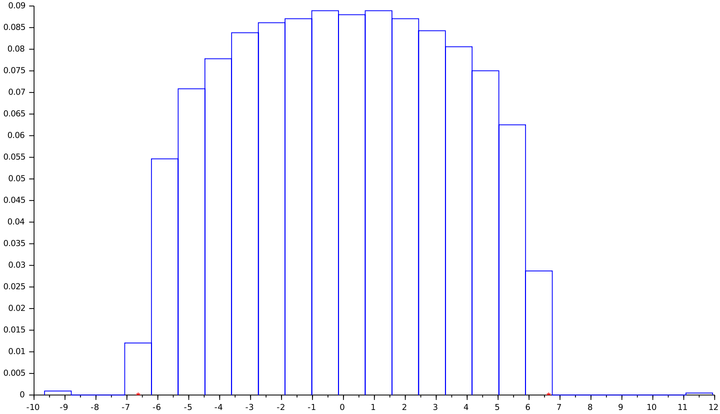
Eigenvalues/Eigenvectors of G_n outside the support of μ_ρ contain a global information on G_n : they are not seen in the local limit.

e.g. $\lambda_1 = -\lambda_n$ equivalent to G bipartite.

Spectral clustering try to exploit this information (usually low rank).

Outliers

A large locally tree-like 12-regular graph.



PART II: EXTREMAL EIGENVALUES

Regular graphs

Alon-Boppana bound

Theorem
If G is a d-regular graph on n vertices, then $\lambda_1(A) = d$ and

$$\lambda_2(A) \geqslant 2\sqrt{d-1} - \frac{c_d}{\log n}.$$

Since $P = A/d$,

$$1 - \lambda_2(P) \leqslant 1 - 2\frac{\sqrt{d-1}}{d} + o(1).$$

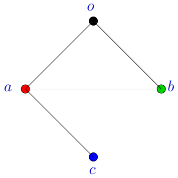
Cover and Universal covering tree

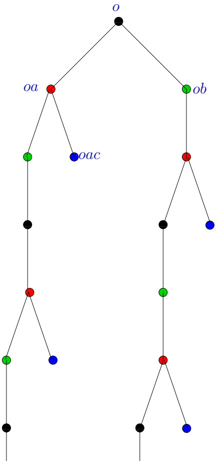
Assume G is connected.

A graph C is a covering graph of G if there is a surjective function $f : V_C \to V_G$ which is a local isomorphism (1-neighborhood is mapped bijectly).

The universal covering of G is a covering which is a tree (unique up to isomorphism). It covers any covering of G.

A construction of $T = (V_T, E_T)$ *:* take $o \in G$. V_T is the set of all non-backtracking paths $(v_0, \cdots, v_k)$ starting from $o = v_0$ $(v_{i-1} \neq v_{i+1})$. Two paths share an edge if one is the largest prefix of the other.





Sketch of Proof of Alon-Boppana

Weaker result on $\lambda_\star = \max_{i \geqslant 2} |\lambda_i| = \lambda_2 \vee (-\lambda_n)$.

$\mathbb{T}_d$ is the universal covering tree of G.

Hence, the nb of closed walks starting from x in G of length k is at least the nb of closed walks starting from the root in $\mathbb{T}_d$ of length k:

$$\mathrm{Tr}(A^k) = \sum_j \lambda_j^k = n \int \lambda^k d\mu_G \geqslant n \int \lambda^k d\mu_{\mathbb{T}_d}$$

$2\sqrt{d-1}$ is the spectral radius of the adjacency operator of $\mathbb{T}_d$ (Kesten) : for k even,

$$\int \lambda^k d\mu_{\mathbb{T}_d} \geqslant \frac{c}{k^{3/2}} \left(2\sqrt{d-1}\right)^k.$$

Sketch of Proof

For even k,

$$\mathrm{Tr}(A^k) = \sum_j \lambda_j^k \leqslant d^k + n\lambda_\star^k.$$

So finally,

$$\frac{c}{k^{3/2}}\left(2\sqrt{d-1}\right)^k \leqslant \frac{d^k}{n} + \lambda_\star^k.$$

Take $k = \log_d n$.

Replacing $\lambda_\star$ *by* λ_2 *requires another strategy (without trace).*

Ramanujan graphs

Let G be a d-regular graph on n vertices. Consider its adjacency matrix A.

$\lambda_n = -d$ is equivalent to G bipartite.

The largest non-trivial eigenvalue is

$$\lambda_\star = \max_i \{|\lambda_i| : |\lambda_i| \neq d\}.$$

G is Ramanujan if

$$\lambda_\star \leqslant 2\sqrt{d-1}.$$

They are the best possible expanders.

Existence of Ramanujan graphs

Sequence of (bipartite) Ramanujan graphs $G_1, G_2, \cdots$, with $|V(G_n)|$ growing to infinity, are known to exist when

- $d = q + 1$ with $q = p^k$ and p prime number *Lubotzky, Phillips, Sarnak (1988), Morgenstern (1994).*

- any $d \geqslant 3$, *Marcus, Spielman, Srivastava (2013).*

Alon's Conjecture (1986)

Theorem (Friedman (2007))
Fix integer $d \geqslant 3$. Let G_n is a sequence of uniformly distributed d-regular graphs on n vertices, then with high probability,

$$\lambda_2 = 2\sqrt{d-1} + o(1) = -\lambda_n.$$

Most regular graphs are nearly Ramanujan !!

Hashimoto's non-backtracking matrix

Oriented edge set :

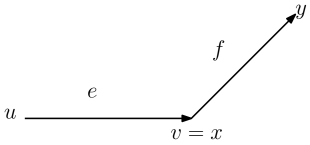
$$\vec{E} = \{(u,v) : \{u,v\} \in E\},$$

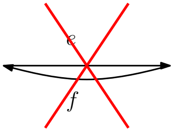
hence, $m = |\vec{E}| = 2|E|$.

If $e = uv, f = xy$ are in $\vec{E}$,

$$B_{ef} = \mathbf{1}(v = x)\mathbf{1}(u \neq y),$$

defines a $|\vec{E}| \times |\vec{E}|$ non-symmetric matrix on the oriented edges.





Perron eigenvalue

Complex eigenvalues, $m = 2|E|$,

$$\mu_1 \geqslant |\mu_2| \geqslant \cdots \geqslant |\mu_m|.$$

A non-backtracking path $(v_1 \dots v_n)$ is a path such that $v_{i-1} \neq v_{i+1}$.

$$B^{\ell}_{ef} = \text{nb of NB paths from } e \text{ to } f \text{ of length } \ell + 1.$$

If G is connected and $|E| > |V|$ then B is irreducible and

$$\mu_1 = \lim_{\ell \to \infty} \|B^{\ell} \delta_e\|_1^{1/\ell} = \text{growth rate of the universal cover of } G.$$

Ihara-Bass' Identity

With $Q = D - I$,

$$\det(z - B) \quad = \quad (z^2 - 1)^{|E|-|V|} \det(z^2 - Az + Q)$$

If G is d-regular, then $Q = (d-1)I$ and

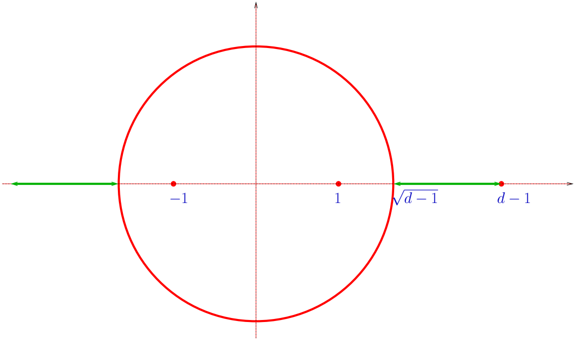
$$\sigma(B) = \{\pm 1\} \cup \left\{ \mu : \mu^2 - \lambda\mu + (d-1) = 0 \text{ with } \lambda \in \sigma(A) \right\}.$$

Kotani & Sunada (2000), Angel, Friedman & Hoory (2007), Terras (2011), . . .

Non-Backtracking matrix of regular graphs

For a d-regular graph, $\mu_1 = d - 1$,

- ⋆ Alon-Boppana bound : $\max_{k \neq 1} \mathfrak{Re}(\mu_k) \geqslant \sqrt{\mu_1} - o(1)$.
- ⋆ Ramanujan (non bipartite) : $|\mu_k| = \sqrt{\mu_1}$ for $k = 2, \ldots, n$.
- ⋆ Friedman's thm : $|\mu_2| \leqslant \sqrt{\mu_1} + o(1)$ if G random uniform.



Ihara-Bass Formula

Theorem (Ihara-Bass Formula)

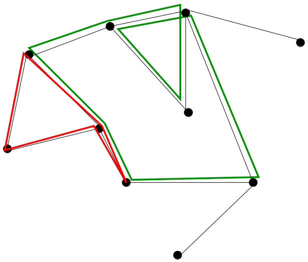
Let ζ_G *be the Ihara's zeta function. We have*

$$\frac{1}{\zeta_G(z)} = \det(I - Bz) = (1 - z^2)^{|E|-|V|} \det(I - Az + Qz^2).$$

The poles of the zeta function are the reciprocal of eigenvalues of B.

Ihara's Zeta Function (1966)

A closed non-backtracking walk without tail $p = (v_1, \cdots, v_n)$ is a closed path such that $v_{i-1} \neq v_{i+1} \bmod(n)$.



A closed non-backtracking walk without tail is prime if it cannot be written as $p = (q, q, \cdots, q)$ with q closed non-backtracking walk .

Ihara's Zeta Function (1966)

If N_ℓ is the number of closed non-backtracking paths without tails of length ℓ in G and $|z|$ small,

$$\zeta_G(z) = \exp\left(\sum_\ell \frac{N_\ell}{\ell} z^\ell\right) = \prod_{p\,:\,\text{prime}} \left(1 - z^{|p|}\right)^{-1}.$$

Stark & Terras draw a parallel between Riemann hypothesis and Ramanujan property.

Sketch of proof of Ihara-Bass Identity

$$\det(I_m - Bz) = (1 - z^2)^{|E|-|V|} \det(I_n - Az + Qz^2).$$

Introduce the matrices

$$\begin{aligned} J &: \mathbb{R}^{\vec{E}} \to \mathbb{R}^{\vec{E}} && Je_{(x,y)} = e_{(y,x)} \\ S &: \mathbb{R}^{\vec{E}} \to \mathbb{R}^{V} && Se_{(x,y)} = e_x \\ T &: \mathbb{R}^{\vec{E}} \to \mathbb{R}^{V} && Te_{(x,y)} = e_y. \end{aligned}$$

$J^2 = I_m$ and J has $m/2 = |E|$ eigenvalues equal to 1 and -1.

We have

$$\begin{aligned} &SJ = T && A = ST^* \\ &D = Q + I = SS^* = TT^* && B + J = T^* S. \end{aligned}$$

Sketch of proof of Ihara-Bass Identity

We check the identity

$$\begin{pmatrix} I_n & 0 \\ T^* & I_m \end{pmatrix} \begin{pmatrix} (1-z^2)I_n & zS \\ 0 & I_m - zB \end{pmatrix} = \begin{pmatrix} I_n - zA + z^2 Q & zS \\ 0 & I_m + zJ \end{pmatrix} \begin{pmatrix} I_n & 0 \\ T^* - zS^* & I_m \end{pmatrix}$$

Take determinant and observe,

$$\det(I_m + zJ) = (1+z)^{m/2}(1-z)^{m/2} = (1-z^2)^{|E|}.$$

Part II: Extremal Eigenvalues

Sketch of proof of Friedman's Theorem

Alon's conjecture (1986)

Theorem (Friedman (2007))

Fix integer $d \geqslant 3$. Let G_n is a sequence of uniformly distributed d-regular graphs on n vertices, then with high probability,

$$\lambda_2 = 2\sqrt{d-1} + o(1) = -\lambda_n.$$

We should prove $\lambda_2 \vee |\lambda_n| \leqslant 2\sqrt{d-1} + o(1)$.

Trace method

If A is the adjacency matrix of G_n we would like to prove for even k,

$$d^k + \lambda_2^k + \lambda_n^k \leqslant \mathrm{Tr}(A^k) \overset{?}{\leqslant} d^k + n\Big(2\sqrt{d-1} + o(1)\Big)^k.$$

No real hope to do better since, for any $\varepsilon > 0$,

$$\mathrm{Tr}(A^k) = n \int \lambda^k d\mu_A \geqslant cn\Big(2\sqrt{d-1} - \varepsilon\Big)^k,$$

with $c = \mu_A(2\sqrt{d-1} - \varepsilon, \infty) = \mu_{\mathbb{T}_d}(2\sqrt{d-1} - \varepsilon, \infty) + o(1) > 0.$

Then,

$$\lambda_2^k \leqslant n\Big(2\sqrt{d-1} + o(1)\Big)^k.$$

or

$$\lambda_2 \leqslant n^{1/k}\Big(2\sqrt{d-1} + o(1)\Big).$$

If $k \gg \log n$ then

$$n^{1/k} = 1 + o(1),$$

and Friedman's Theorem follows.

It is wiser to project orthogonally on $\mathbf{1}^{\perp}$:

$$\mathrm{Tr}(A^k) - d^k = \mathrm{Tr}\left(A - \frac{d}{n}\mathbf{1}\mathbf{1}^*\right)^k \overset{?}{\leqslant} n\Big(2\sqrt{d-1} + o(1)\Big)^k.$$

Trace method

For a first moment estimate, we would aim at

$$\mathbb{E}\mathrm{Tr}(A^k) - d^k = \mathbb{E}\mathrm{Tr}\left(A - \frac{d}{n}\mathbf{1}\mathbf{1}^*\right)^k \overset{?}{\leqslant} n\left(2\sqrt{d-1} + o(1)\right)^k$$

for $k \gg \log n$.

This is wrong !

The probability that the graph contains K_{d+1} as subgraph is at least n^{-c}. On this event $\lambda_2 = d$. Hence, for even $k \gg \log n$,

$$\mathbb{E}\mathrm{Tr}\left(A - \frac{d}{n}\mathbf{1}\mathbf{1}^*\right)^k \geqslant n^{-c} d^k \gg n\left(2\sqrt{d-1} + o(1)\right)^k.$$

Subgraphs which have polynomially small probability compromise the first moment method. Called Tangles.

Strategy

1. Use B instead of A : $|\mu_2| \leqslant \sqrt{d-1} + o(1)$.

2. Remove the tangles.

3. Project on $\mathbf{1}^{\perp}$.

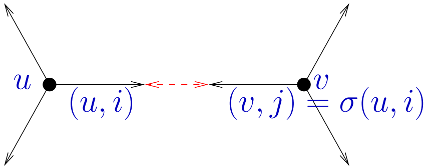
4. Use the trace method / first moment method to evaluate the remainder terms.

Bordenave/Massoulié/Lelarge (2015), Bordenave (2015)

Configuration model

The oriented edge set $\vec{E}$, $|\vec{E}| = m = nd$ is written as

$$\vec{E} = \{(u,i) : 1 \leqslant u \leqslant n, 1 \leqslant i \leqslant d\}.$$



A matching σ on $\vec{E}$ defines a multi-graph with adjacency matrix

$$A = Q^* M Q,$$

where, $M : \mathbb{R}^{\vec{E}} \to \mathbb{R}^{\vec{E}}$, $Q : \mathbb{R}^{V} \to \mathbb{R}^{\vec{E}}$,

$$M_{ef} = \mathbf{1}(\sigma(e) = f) = M_{fe} \quad \text{and} \quad Q_{eu} = \mathbf{1}(e_1 = u).$$

M is the permutation matrix associated to σ.

The non-backtracking matrix with $f = (u, i)$,

$$B_{ef} = \mathbf{1}(\sigma(e) = (u, j) \text{ for some } j \neq i).$$

can be written as

$$B = MN$$

where

$$N_{ef} = \mathbf{1}(e_1 = f_1, e \neq f) = N_{fe}.$$

We have

$$M\mathbf{1} = \mathbf{1} \quad \text{and} \quad N\mathbf{1} = (d-1)\mathbf{1}.$$

Hence,

$$B\mathbf{1} = B^*\mathbf{1} = (d-1)\mathbf{1}.$$

If $B\psi = \mu\psi$, $\mu \neq d-1$, we deduce

$$\mu\langle \mathbf{1}, \psi\rangle = \langle \mathbf{1}, B\psi\rangle = \langle B^*\mathbf{1}, \psi\rangle = (d-1)\langle \mathbf{1}, \psi\rangle.$$

For any integer ℓ, the second largest eigenvalue of B is thus bounded by

$$|\mu_2|^\ell \leqslant \max_{x:\langle \mathbf{1}, x\rangle = 0} \frac{\|B^\ell x\|_2}{\|x\|_2}.$$

We prove if σ is a uniform random matching that with high probability

$$\max_{x:\langle \mathbf{1}, x\rangle = 0} \frac{\|B^\ell x\|_2}{\|x\|_2} \leqslant (\log n)^c (d-1)^{\ell/2}.$$

with $\ell \simeq \log n$. The theorem follows with

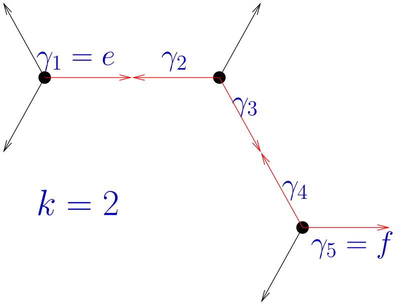
$$\varepsilon = O(\log\log n / \log n).$$

Path decomposition

Recall $M_{ef} = \mathbf{1}(\sigma(e) = f)$, $N_{ef} = \mathbf{1}(e_1 = f_1, e \neq f)$

$$B^k_{ef} = \left((MN)^k\right)_{ef} = \sum_{\gamma \in \Gamma^k_{ef}} \prod_{s=1}^{k} M_{\gamma_{2s-1}\gamma_{2s}},$$

where Γ^k_{ef} is the set of paths $\gamma = (\gamma_1, \ldots, \gamma_{2k+1})$ such that $\gamma_1 = e$, $\gamma_{2k+1} = f$ and $N_{\gamma_{2s},\gamma_{2s+1}} = 1$.



Path decomposition

$$B_{ef}^{k} = \sum_{\gamma \in \Gamma_{ef}^{k}} \prod_{s=1}^{k} M_{\gamma_{2s-1}\gamma_{2s}},$$

The set of paths Γ_{ef}^{k} is independent of σ : combinatorial part.

The summand is the probabilistic part.

Path decomposition

$$B^k_{ef} = \left((MN)^k\right)_{ef} = \sum_{\gamma \in \Gamma^k_{ef}} \prod_{s=1}^{k} M_{\gamma_{2s-1}\gamma_{2s}},$$

The projection of M on $\mathbf{1}^\perp$ is

$$\underline{M} = M - \frac{\mathbf{1}\mathbf{1}^*}{m}.$$

Hence, if $\langle x, \mathbf{1} \rangle = 0$, we get

$$B^k x = \underline{B}^k x,$$

where $\underline{B} = \underline{M}N$ and

$$\underline{B}^k_{ef} = \left((\underline{M}N)^k\right)_{ef} = \sum_{\gamma \in \Gamma^k_{ef}} \prod_{s=1}^{k} \underline{M}_{\gamma_{2s-1}\gamma_{2s}},$$

However, due to the presence of tangles, we will reduce the sum before doing the projection.

Tangles

A multi-graph (or a path) is tangle-free if it contains at most one cycle.

A multi-graph (or a path) is ℓ-tangle-free if all vertices have at most at most one cycle in their ℓ-neighborhood.

We denote by F_{ef}^k the subset of tangle-free paths Γ_{ef}^k.

Observe that F_{ef}^k *is much smaller than* Γ_{ef}^k.

Path decomposition

Assume that $G = G(\sigma)$ is ℓ-tangle-free. Then, for $0 \leqslant k \leqslant \ell$,

$$B^k = B^{(k)},$$

where

$$(B^{(k)})_{ef} = \sum_{\gamma \in F^k_{ef}} \prod_{s=1}^{k} M_{\gamma_{2s-1}\gamma_{2s}}.$$

For $0 \leqslant k \leqslant \ell$, we define the "projected" matrix

$$(\underline{B}^{(k)})_{ef} = \sum_{\gamma \in F^k_{ef}} \prod_{s=1}^{k} \underline{M}_{\gamma_{2s-1}\gamma_{2s}}.$$

Beware that $\underline{B}^k \neq \underline{B}^{(k)}$ and a priori $B^{(k)}x \neq \underline{B}^{(k)}x$ for $\langle x, \mathbf{1}\rangle = 0$. This is only approximately true !

$$(B^{(\ell)})_{ef} = (\underline{B}^{(\ell)})_{ef} + \sum_{\gamma \in F^{\ell}_{ef}} \sum_{k=1}^{\ell} \prod_{s=1}^{k-1} \underline{M}_{\gamma_{2s-1}\gamma_{2s}} \left(\frac{1}{m}\right) \prod_{k+1}^{\ell} M_{\gamma_{2s-1}\gamma_{2s}},$$

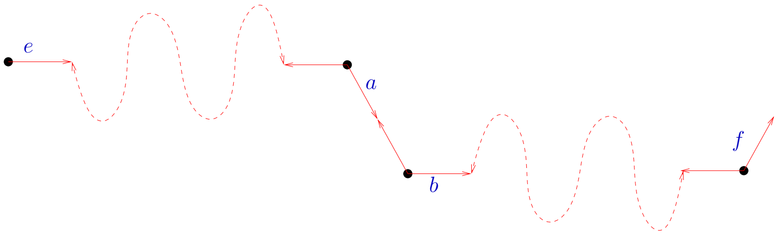
which follows from the identity,

$$\prod_{s=1}^{\ell} x_s = \prod_{s=1}^{\ell} y_s + \sum_{k=1}^{\ell} \prod_{s=1}^{k-1} y_s (x_k - y_k) \prod_{k+1}^{\ell} x_s.$$

Path decomposition

An path $\gamma \in F^{\ell}_{ef}$ can be decomposed as the union of

$$\gamma' \in F^{k-1}_{ea}, \quad \gamma'' \in F^{1}_{ab} \quad \text{and} \quad \gamma''' \in F^{\ell-k}_{bf}.$$



Path decomposition

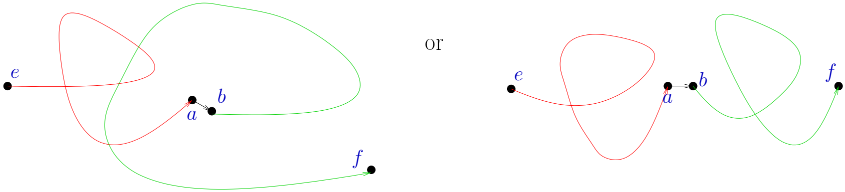
Set

$$K = (d-1)\mathbf{1}\mathbf{1}^* - N$$

$K_{ef} \in \{d-1, d-2\}$ is the cardinal of Γ^1_{ef}.

$$\sum_{\gamma \in F^{\ell}_{ef}} \prod_{s=1}^{k-1} \underline{M}_{\gamma_{2s-1}\gamma_{2s}} \prod_{k+1}^{\ell} M_{\gamma_{2s-1}\gamma_{2s}} = \left(\underline{B}^{(k-1)} K B^{(\ell-k)}\right)_{ef} - \left(R^{(\ell)}_k\right)_{ef}$$

where $\left(R^{(\ell)}_k\right)_{ef}$ counts the extra paths :



So finally, $K = (d-1)\mathbf{1}\mathbf{1}^* - N$,

$$\begin{aligned}
B^{(\ell)} &= \underline{B}^{(\ell)} + \frac{1}{m}\sum_{k=1}^{\ell} \underline{B}^{(k-1)} K B^{(\ell-k)} - \frac{1}{m}\sum_{k=1}^{\ell} R_k^{(\ell)} \\
&= \underline{B}^{(\ell)} + \frac{d-1}{m}\sum_{k=1}^{\ell} \underline{B}^{(k-1)} \mathbf{1}\mathbf{1}^* B^{(\ell-k)} - \frac{1}{m}\sum_{k=1}^{\ell} \underline{B}^{(k-1)} N B^{(\ell-k)} \\
&\quad - \frac{1}{m}\sum_{k=1}^{\ell} R_k^{(\ell)}.
\end{aligned}$$

Hence, if $\langle x, \mathbf{1}\rangle = 0$, since $\mathbf{1}^* B^{(\ell-k)} = (d-1)^{\ell-k}\mathbf{1}^*$,

$$B^{(\ell)} x = \underline{B}^{(\ell)} x - \frac{1}{m}\sum_{k=1}^{\ell} \underline{B}^{(k-1)} N B^{(\ell-k)} x - \frac{1}{m}\sum_{k=1}^{\ell} R_k^{(\ell)} x.$$

We arrive at

$$\max_{x:\langle \mathbf{1},x\rangle=0} \frac{\|B^{\ell}x\|_2}{\|x\|_2} \leqslant \|\underline{B}^{(\ell)}\| + \frac{1}{m}\sum_{k=0}^{\ell-1}(d-1)^{\ell-k}\|\underline{B}^{(k)}\| + \frac{1}{m}\sum_{k=1}^{\ell}\|R_k^{(\ell)}\|.$$

where $\|S\| = \max_{x:\|x\|_2=1}\|Sx\|_2$ is the operator norm.

This inequality holds if $G(\sigma)$ is ℓ tangle-free : for random σ, ok with $\ell = 0.1\log_{d-1}(n)$.

$$\max_{x:\langle \mathbf{1},x\rangle=0} \frac{\|B^\ell x\|_2}{\|x\|_2} \leqslant \|\underline{B}^{(\ell)}\| + \frac{1}{m}\sum_{k=0}^{\ell-1}(d-1)^{\ell-k}\|\underline{B}^{(k)}\| + \frac{1}{m}\sum_{k=1}^{\ell}\|R_k^{(\ell)}\|.$$

Our aim is then to prove that w.h.p.

$\|\underline{B}^{(\ell)}\| \leqslant (\log n)^c (d-1)^{\ell/2}$ and $\|R_k^{(\ell)}\| \leqslant (\log n)^c (d-1)^{\ell-k/2}$

By estimating, for $S = \underline{B}^{(\ell)}$ or $S = R_k^{(\ell)}$.

$$\mathbb{E}\|S\|^{2k} \leqslant \mathbb{E}\mathrm{Tr}(SS^*)^k.$$

with $k \simeq \log n/(\log\log n)$: on the overall paths of length $2\ell k \gg \log n$.

Trace method

For $S = \underline{B}^{(\ell)}$,

$$\mathbb{E}\|S\|^{2k} \leqslant \mathbb{E}\mathrm{Tr}(SS^*)^k \leqslant \left(\sqrt{d-1} + o(1)\right)^{2k\ell},$$

with $k \simeq \log n / (\log \log n)$.

The combinatorial part of the proof is made possible thanks to the tangle-free reduction.

The probabilistic part relies on an estimate of the type

$$\left| \mathbb{E} \prod_{t=1}^{t} \left(M_{\gamma_{2t-1}, \gamma_{2t}} - \frac{1}{m} \right) \right| \leqslant c \left(\frac{1}{m} \right)^{a} \left(\frac{4t}{\sqrt{m}} \right)^{a_1},$$

where a is the nb of visited edges $\{e, f\}$ and a_1 is the nb of edges visited exactly once.

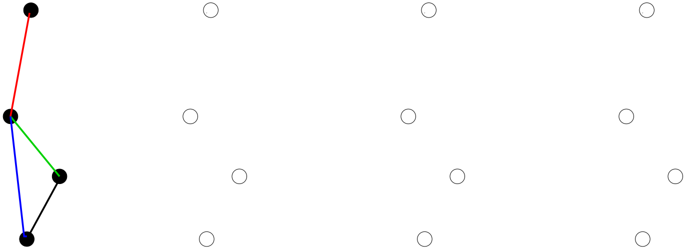
Part II: Extremal Eigenvalues

Random n-Lifts

Graph Lift/Cover

A graph C is a covering graph of G if there is a surjective function $f : V_C \to V_G$ which is a local isomorphism (1-neighborhood is mapped bijectly).

C is a n-cover of G if $|f^{-1}(x)| = n$ for all $x \in V_G$.



The n-lift can encoded by a permutation σ_e on each edge $e \in V_G$.

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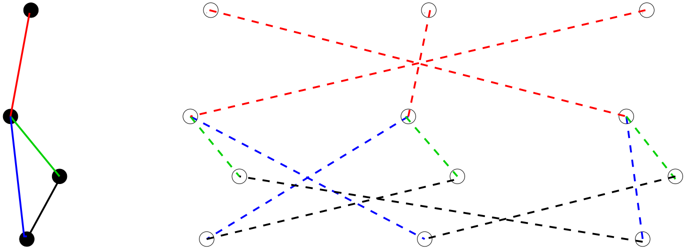

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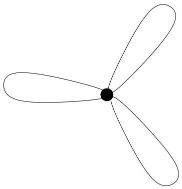
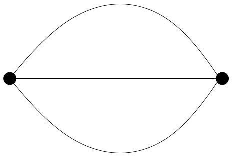
Graph Lift/Cover





BS limit

Let G_n is a uniformly random n-lift of G. Then, as $n \to \infty$, what it is the BS-limit of G ?

The universal covering tree of G rooted uniformly.

New eigenvalues

Let $G = (V, E)$ be a base graph and $G_n = (V_n, E_n)$ a n-lift of G,

$$V_n = \{(x, i) : x \in V, i \in [n]\}.$$

We consider for example, the adjacency matrices A and A_n of G and G_n.

Define the vector space

$$H = \left\{ f \in \mathbb{R}^{V_n} : f(x, i) = f(x, j) \right\} = \operatorname{span}(\chi_x, \, x \in V),$$

where $\chi_x(y, i) = \mathbf{1}(x = y)$.

We have

$$A_n H \subset H$$

and A_n restricted to H is A.

New eigenvalues

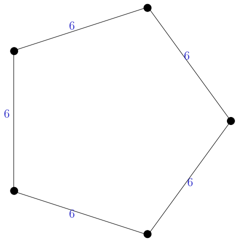
The eigenvalues of A are also eigenvalues of A_n (counting multiplicities).

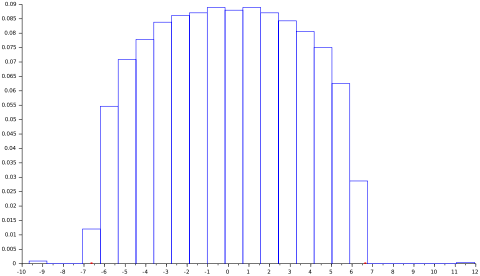
The other eigenvalues of A are called new eigenvalues. They are the eigenvalues of the matrix A restricted to $H^\perp$.

The largest new eigenvalue is

$$\lambda_n^\star := \max\{|\lambda| : \lambda \text{ new eigenvalue of } A_n\}.$$

New eigenvalues





Generalized Alon's conjecture

Let G_n is a uniformly random n-lift of G. Then, as $n \to \infty$, with high probability,

$$\lambda_n^\star \leqslant \rho + o(1),$$

where ρ is the spectral radius of the adjacency operator of the universal covering tree of G.

The converse $\lambda_n^\star \geqslant \rho + o(1)$ follows from the BS-limit (and also from a generalized Alon-Boppana bound).

Generalized Alon's conjecture

This should hold for any reasonable local operator : $A, P, L, B, \ldots$.

This is proved for non-backtracking operator B, *Friedman, Kohler (2014), Bordenave (2015)*. For B, $\rho = \sqrt{\mu_1}$ where μ_1 is the growth rate of the universal cover *Angel, Friedman, Hoory (2007)*.

The bound $\lambda_n^\star \leqslant \sqrt{3}\rho + o(1)$ is known, *Puder (2012)*.

This is a been used for exact reconstruction of the base graph *Brito, Dumitriu, Ganguly, Hoffman, Tran (2015)*.

Part II: Extremal Eigenvalues

Stochastic Block Model

Stochastic Block Model

Consider a set of labels $\{1, \cdots, r\}$ and assign label $\sigma_n(v)$ to vertex v. We assume that

$$\pi_n(i) = \frac{1}{n} \sum_{v=1}^{n} \mathbf{1}(\sigma_n(v) = i) = \pi(i) + O(n^{-\varepsilon}),$$

for some probability vector π.

If $\sigma(u) = i, \sigma(v) = j$, the edge $\{u, v\}$ is present independently with probability

$$\frac{W_{ij}}{n} \wedge 1,$$

where W is a symmetric matrix.

(Inhomogeneous random graph, Chung-Lu random graph, ...)

Stochastic Block Model

If $\sigma(v) = j$, mean number of label i neighbors is

$$\pi(i)W_{ij} + O(1/n).$$

Mean progeny matrix

$$M = \mathrm{diag}(\pi)W.$$

We assume that the average degree is homogeneous, for all $1 \leqslant j \leqslant r$,

$$\sum_{i=1}^{r} M_{ij} = \alpha > 1.$$

Assume that M is strongly irreducible and we order its real eigenvalues

$$\alpha = \rho_1 > |\rho_2| \geqslant \cdots \geqslant |\rho_r|.$$

Stochastic Block Model

If $r = 1$, we retrieve $\mathcal{G}(n, \alpha/n)$.

Model used in community detection. Notably for $r = 2$,

$$\pi = \left(\frac{1}{2}, \frac{1}{2}\right)$$

and, with $a > b$,

$$W = \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

Then

$$\rho_1 = \alpha = \frac{a+b}{2} \quad \text{and} \quad \rho_2 = \frac{a-b}{2}.$$

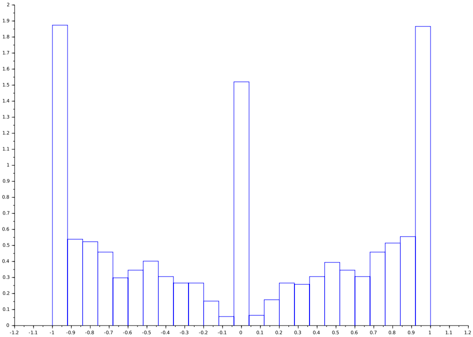
BS limit

The BS limit of SBM is a multi-type Galton-Watson tree with $\mathrm{Poi}(W_{ij})$ offspring distribution and the root has label i with proba $\pi(i)$.

The growth rate of the random tree conditionned on non-extinction is a.s. α, i.e. the expected number of offsprings.

Transition Matrix

Transition matrix P in an Erdős-Rényi graph $\mathcal{G}(n, \alpha/n)$, $n = 2000$, $\alpha = 1.5$.



Classical local operators

The spectral measure of Galton-Watson tree with Poisson offspring distribution has full support : $\mathbb{R}$ for A, $[-1, 1]$ for P and $\mathbb{R}_+$ for L.

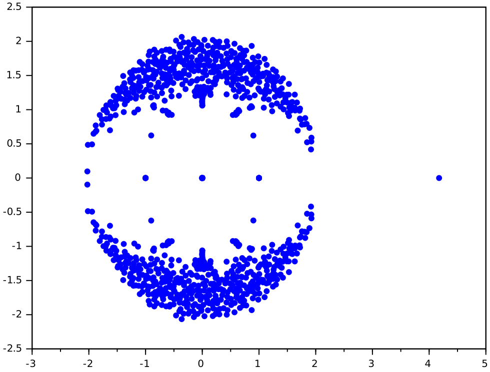
This is due to high degree vertices (for A) and long line segments for P, L.

No outliers : the extremal eigenvalues are related to small subgraphs and not to global graph properties.

Various regularization have been proposed to solve this issue. Including the non-backtracking matrix,
Krzakala/Moore/Mossel/Neeman/Sly/Zdeborová/Zhang (2013).

Simulation for Erdős-Rényi Graph

Eigenvalues of B for an Erdős-Rényi graph $\mathcal{G}(n, \alpha/n)$ with $n = 500$ and $\alpha = 4$.



Erdős-Rényi Graph

$$\mu_1 \geqslant |\mu_2| \geqslant \dots.$$

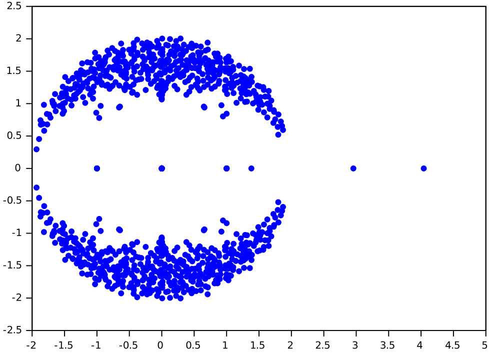
Theorem
Let $\alpha > 1$ *and* G *with distribution* $\mathcal{G}(n, \alpha/n)$. *With high probability,*

$$\begin{aligned} \mu_1 &= \alpha + o(1) \\ |\mu_2| &\leqslant \sqrt{\alpha} + o(1). \end{aligned}$$

Bordenave/Massoulié/Lelarge (2015)

Stochastic Block Model

$$n = 500, \quad r = 2, \quad a = 7, \quad b = 1, \quad \rho_1 = 4, \quad \rho_2 = 3.$$



Let $1 \leqslant r_0 \leqslant r$ be such that

$$\alpha = \rho_1 > |\rho_2| \geqslant \cdots \geqslant |\rho_{r_0}| > \sqrt{\rho}_1 \geqslant |\rho_{r_0+1}| \geqslant \cdots \geqslant |\rho_r|.$$

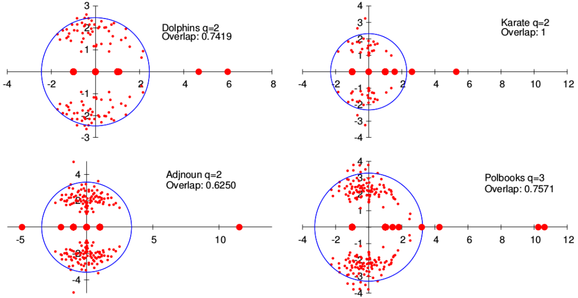
Theorem

Let $\alpha > 1$ and G a stochastic block model as above. With high probability, up to reordering the eigenvalues of B,

$$\begin{aligned} \mu_k &= \rho_k + o(1) && \text{if } 1 \leqslant k \leqslant r_0 \\ |\mu_k| &\leqslant \sqrt{\alpha} + o(1) && \text{if } k > r_0. \end{aligned}$$

+ a description of the eigenvectors of λ_k, $1 \leqslant k \leqslant r_0$, if the μ_k are distinct, In particular, they are asymptotically orthogonal.

Spectral redemption : *eigenvalues/eigenvectors such that* $|\mu_k| > \sqrt{\mu_1}$ *should contain relevant global information on the graph.*



Krzakala/Moore/Mossel/Neeman/Sly/Zdeborová/Zhang (2013)

Conference : Spectrum of Random Graphs
January 4-8, 2016
Luminy - CIRM

Thank you for your attention !