A high-dimensional random graph process

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based on joint work with

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erdős-rényi random graphs

An Erdős-Rényi random graph $G(n, p)$ is a graph on n vertices. Each edge is present independently, with probability p.

Let $U_{i,j}$ be i.i.d. uniform $[0, 1]$ random variables for $1 \leq i < j \leq n$.

Join vertex i with vertex j if an only if $U_{i,j} < p$. This defines $G(n, p)$.

Moreover, $U_n = (U_{i,j})_{1 \leq i < j \leq n}$ defines a $G(n, p)$ for all p on the same probability space.

This random graph process has been studied thoroughly.

a random graph process

Let $X_n = (X_{i,j})_{1 \le i < j \le n}$ be independent standard normal vectors in $\mathbb{R}^d$.

For each unit vector $s \in S^{d-1}$ and $t \in \mathbb{R}$, define the graph $\Gamma(X_n, s, t)$ with vertex set $[n]$ and edge set

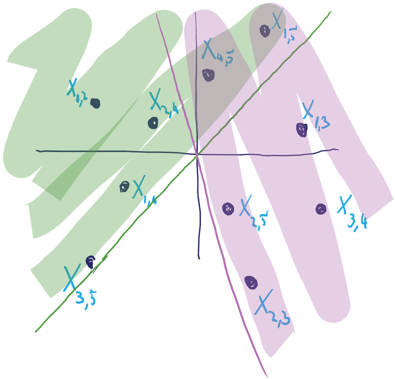
$$\{\{i,j\} : \langle X_{i,j}, s \rangle \ge t\}$$

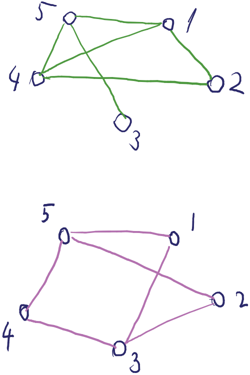
For any $s \in S^{d-1}$ and $t \in \mathbb{R}$, $\Gamma(X_n, s, t)$ is a $G(n, p)$, with $p = 1 - \Phi(t)$ where Φ is the distribution function of a standard normal random variable.

In particular, $\Gamma(X_n, s, 0)$ is a $G(n, 1/2)$ random graph.

We write $\Gamma(X_n, s)$ for $\Gamma(X_n, s, 0)$.

two graphs from the family. $t = 0$ $(p = 1/2)$





a random graph process

We study the random graph process

$$\mathcal{G}_{d,p}(X_n) = \left\{ \Gamma(X_n, s, \Phi^{-1}(1-p)) : s \in S^{d-1} \right\} .$$

$\mathcal{G}_{d,p}(X_n)$ is a stationary process of $G(n,p)$ random graphs, indexed by d-dimensional unit vectors.

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For larger d, the process becomes "richer".

How large does d have to be so that we find "atypical" behavior?

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How large does d have to be so that we find "atypical" behavior?

We study clique number, chromatic number (when $p = 1/2$) and connectivity (when $p \sim \log n/n$).

schläffli's lemma

The number of different ways of dichotomizing $N \geq d$ points in general position by half-spaces (with $\mathbf{0}$ on the boundary) in $\mathbb{R}^d$ equals

$$C(N, d) = 2 \sum_{k=0}^{d-1} \binom{N-1}{k} .$$

In particular, when $N = d$, all 2^N possible dichotomies of the N points are realizable by some linear half space.

The N points are shattered by half spaces.

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In particular, when $N = d$, all 2^N possible dichotomies of the N points are realizable by some linear half space.

The N points are shattered by half spaces.

If $d \geq \binom{n}{2}$, then with probability one, $\mathcal{G}_{d,1/2}(X_n)$ contains all $2^{\binom{n}{2}}$ graphs on n vertices.

clique number

Consider $p = 1/2$.

The clique number cl of a graph is the number of vertices of the largest clique.

Matula's theorem (1972): for any fixed $s \in S^{d-1}$, for any $\epsilon > 0$,

$$cl(X_n, s) \in \{\lfloor \omega - \epsilon \rfloor, \lfloor \omega + \epsilon \rfloor\}$$

with high probability, where

$$\omega = 2 \log_2 n - 2 \log_2 \log_2 n + 2 \log_2 e - 1.$$

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How large does d have to be so that for some $s \in S^{d-1}$, $cl(X_n, s)$ is much larger/smaller than ω?

clique number–subcritical

For what values of d do we find graphs with clique number much smaller than ω?

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If $d \geq \binom{n}{2}$, all $X_{i,j}$ are shattered and for some $s \in S^{d-1}$, $\Gamma(X_n, s)$ has no edges $\implies cl(X_n, s) = 1$.

clique number–subcritical

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If $d = o(n^2/(\log n)^9)$, then whp.,

$$\min_{s \in S^{d-1}} cl(X_n, s) > \lfloor \omega - 3 \rfloor .$$

clique number, subcritical–proof

A "cap argument":

Let $k = \lfloor \omega - 3 \rfloor$ and let $N_k(s)$ be the number of cliques of size k in $\Gamma(X_n, s)$.

Let $\eta \in (0, 1]$ and let $\mathcal{C}_\eta$ be a minimal η-cover of S^{d-1}.
By a standard volume argument

$$|\mathcal{C}_\eta| \leq \left(\frac{4}{\eta}\right)^d \ .$$

We take $\eta = 1/n^2$.

clique number, subcritical–proof

$$
\begin{aligned}
&\mathbb{P}\left\{\exists s \in S^{d-1} : N_k(s) = 0\right\} \\
&= \mathbb{P}\left\{\exists s' \in \mathcal{C}_\eta \text{ and } \exists s \in S^{d-1} : \|s - s'\| \leq \eta : N_k(s) = 0\right\} \\
&\leq |\mathcal{C}_\eta| \mathbb{P}\left\{\exists s \in S^{d-1} : \|s - s_0\| \leq \eta : N_k(s) = 0\right\}
\end{aligned}
$$

clique number, subcritical–proof

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\end{aligned}$$

Key observation:

$$\bigcup_{s \in S^{d-1} : \|s - s_0\| \le \eta} \Gamma(X_n, s) \subset \Gamma(X_n, s_0) \cup E$$

where E is a set of $\text{Bin}\left(\binom{n}{2}, \frac{\eta\sqrt{d}}{\sqrt{2\pi}}\right)$ edges

clique number, subcritical–proof

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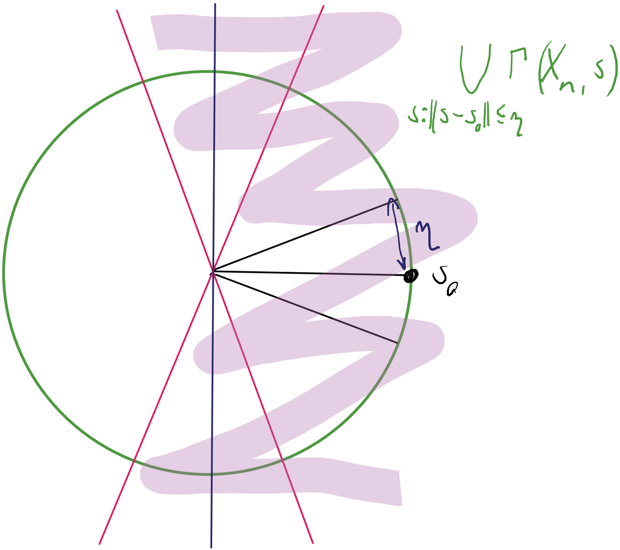
The probability of this is at most the probability that $G(n, 1/2 - 1/n)$ does not have any clique of size k.

union of graphs near s_0



clique number, subcritical–proof

Janson's inequality:

$$\mathbb{P}\{N_k = 0\} \leq \exp\left(\frac{-(\mathbb{E}N_k)^2}{\Delta}\right) ,$$

where $\mathbb{E}N_k = \binom{n}{k} p^{\binom{k}{2}}$ and

$$\Delta = \sum_{j=2}^{k} \binom{n}{k} \binom{k}{j} \binom{n-k}{k-j} p^{2\binom{k-j}{2} - \binom{j}{2} - 2j(k-j)} .$$

This implies

$$\mathbb{P}\{N_k = 0\} \leq \exp\left(\frac{-C'n^2}{(\log_2 n)^8}\right) .$$

clique number–supercritical

Very large cliques appear much earlier.

clique number–supercritical

Very large cliques appear much earlier.

Fix k arbitrary vertices. If $d \geq \binom{k}{2}$, then all $X_{i,j}$ corresponding to edges connecting these vertices are shattered.

In particular, the complete graph on these k vertices is present for some $s \in S^{d-1}$.

For example, if $d \geq (9/2)(\log_2 n)^2$, then for some $s \in S^{d-1}$

$$cl(X_n, s) \geq 3 \log_2 n$$

clique number–supercritical

Using the second moment method we can do a little better:
if $d \geq 7\log^2 n / \log\log n$, then $cl(X_n, s) \geq 3\log_2 n$ for some $s \in S^{d-1}$.

clique number–supercritical

Using the second moment method we can do a little better:
if $d \geq 7 \log^2 n / \log \log n$, then $cl(X_n, s) \geq 3 \log_2 n$ for some $s \in S^{d-1}$.

This is essentially sharp:

For any $c > 2$ there exists $c' > 0$ such that if $d \leq c' \log^2 n / \log \log n$, then

$$\max_{s \in S^{d-1}} cl(X_n, s) \leq c \log_2 n \ .$$

Proof is by "cap argument".

clique number–results

(SUBCRITICAL; NECESSARY.) If $d = o(n^2/(\log n)^9)$, then

$$\min_{s \in S^{d-1}} cl(X_n, s) > \lfloor \omega - 3 \rfloor$$

(SUBCRITICAL; SUFFICIENT.) If $d \geq \binom{n}{2}$, then

$$\min_{s \in S^{d-1}} cl(X_n, s) = 1$$

(SUPERCRITICAL; NECESSARY.) For any $c > 2$ there exists $c' > 0$ such that if $d \leq c' \log^2 n / \log \log n$, then

$$\max_{s \in S^{d-1}} cl(X_n, s) \leq c \log_2 n$$

(SUPERCRITICAL; SUFFICIENT.) For any $c > 2$ and $c' > c^2/(2 \log 2)$, if $d \geq c' \log^2 n / \log \log n$, then

$$\max_{s \in S^{d-1}} cl(X_n, s) \geq c \log_2 n$$

chromatic number

A proper coloring of vertices of a graph is such that no pair of vertices joined by an edge share the same color.

The chromatic number $\chi(G)$ of G is the smallest number of colors for which a proper coloring of the graph exists.

We still assume $p = 1/2$.

For a fixed s, by a result of Bollobás (1988),

$$\frac{n}{2\log_2 n} \le \chi(\Gamma(X_n, s)) \le \frac{n}{2\log_2 n}(1 + o(1)) \qquad \text{whp.}$$

chromatic number–results

(SUBCRITICAL; NECESSARY.) If $d = o(n/(\log n)^3)$, then

$$\min_{s \in S^{d-1}} \chi(\Gamma(X_n, s)) \geq (1-\epsilon)n/(2\log_2 n).$$

(SUBCRITICAL; SUFFICIENT.) If $d \geq n \log_2 n/(1-2\epsilon)$, then

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(SUPERCRITICAL; NECESSARY.) If $d = o(n^2/(\log n)^6)$, then

$$\max_{s \in S^{d-1}} \chi(\Gamma(X_n, s)) \leq (1+\epsilon)n/(2\log_2 n).$$

(SUPERCRITICAL; SUFFICIENT.) If $d \geq (1/2)\left[(1+\epsilon)n/(2\log_2 n)\right]^2$, then

$$\max_{s \in S^{d-1}} \chi(\Gamma(X_n, s)) \geq (1+\epsilon)n/(2\log_2 n).$$

chromatic number–proof

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$$\max_{s \in S^{d-1}} \chi(\Gamma(X_n, s)) \geq (1+\epsilon)n/(2\log_2 n).$$

This is immediate because for some $s \in S^{d-1}$, there is a clique of size $(1+\epsilon)n/(2\log_2 n)$.

chromatic number–proof

(SUPERCRITICAL; NECESSARY.) If $d = o(n^2/(\log n)^6)$, then

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This follows by a "cap argument" combined with the high resilience of the chromatic number proved by Alon and Sudakov (2010):

With probability at least $1 - \exp(c_1 n^2/(\log n)^4)$, for every collection E of at most $c_2 \epsilon^2 n^2/(\log_2 n)^2$ edges, the chromatic number of $G(n, 1/2) \cup E$ is at most $(1+\epsilon)n/(2\log_2 n)$.

chromatic number–proof

(SUBCRITICAL; SUFFICIENT.) If $d \geq n \log_2 n/(1-2\epsilon)$, then

$$\min_{s \in S^{d-1}} \chi(\Gamma(X_n, s)) \leq (1-\epsilon)n/(2 \log_2 n).$$

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$$\min_{s \in S^{d-1}} \chi(\Gamma(X_n, s)) \leq (1-\epsilon)n/(2\log_2 n).$$

Partition the n vertices of $k = (1-\epsilon)n/(2\log_2 n)$ sets of n/k vertices.

χ is at least k if all k sets are independent.

Need to remove $k\binom{n/k}{2}$ edges.

Such graph appears with probability one if $d \geq k\binom{n/k}{2}$.

chromatic number–proof

(SUBCRITICAL; NECESSARY.) If $d = o(n/(\log n)^3)$, then

$$\min_{s \in S^{d-1}} \chi(\Gamma(X_n, s)) \geq (1-\epsilon)n/(2\log_2 n).$$

By a classical result of Shamir and Spencer (1987), for any fixed s,

$$|\chi(\Gamma(X_n, s)) - \mathbb{E}(\chi(\Gamma(X_n, s)))| = O_p(n^{1/2}) .$$

It follows from the bounded differences inequality if we consider $\chi(\Gamma(X_n, s))$ as a function of $Y_{i,s} = (\mathbb{1}_{\{\langle X_{i,j}, s\rangle \geq 0\}})_{j=1,\ldots,i-1} \in \{0,1\}^{i-1}$ for $i = 2, \ldots, n$.

bounded differences inequality

If f is such that

$$|f(x_1, \ldots, x_n) - f(x_1, \ldots, x_i', \ldots, x_n)| \leq 1$$

and $X_1, \ldots, X_n$ are independent, then $Z = f(X_1, \ldots, X_n)$ satisfies

$$\exp\left(\lambda(Z - \mathbb{E}Z)\right) \leq \exp(n\lambda^2/8)$$

and

$$\mathbb{P}\{|Z - \mathbb{E}Z| > t\} \leq 2e^{-2t^2/n} .$$

chromatic number–proof

It suffices to prove that

$$\mathbb{E} \sup_{s \in S^{d-1}} |\chi(\Gamma(X_n, s)) - \mathbb{E}\chi(\Gamma(X_n, s))| \le 4\sqrt{nd \log n} \ .$$

This can be done by a Vapnik-Chervonenkis-style symmetrization combined with the bounded differences inequality.

connectivity

Here we consider $p = c \log n / n$.

Erdős and Rényi (1960) proved that whp. for $c < 1$, the graph is disconnected and for $c > 1$ it is connected.

Two questions:

- if $c < 1$, for what values of d do connected graphs appear in $\mathcal{G}_{d,p}(X_n)$?
- if $c > 1$, for what values of d do disconnected graphs appear in $\mathcal{G}_{d,p}(X_n)$?

connectivity–results

Recall $t = \Phi^{-1}(1-p)$.

(SUBCRITICAL; NECESSARY.) If $c < 1$ and $d = O(n^{1-c-\epsilon})$, then for all $s \in S^{d-1}$, $\Gamma(X_n, s, t)$ is disconnected.

(SUBCRITICAL; SUFFICIENT.) If $d \geq Cn\sqrt{\log n}$, then there exists an $s \in S^{d-1}$ such that $\Gamma(X_n, s, t)$ is connected.

connectivity–results

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(SUPERCRITICAL; NECESSARY.) If $c > 1$ and $d \leq (1 - \epsilon)(c - 1) \log n / \log \log n$, then for all $s \in S^{d-1}$, $\Gamma(X_n, s, t)$ is connected.

(SUPERCRITICAL; SUFFICIENT.) If $c > 1$ and $d \geq (2 + \epsilon)(c - 1) \log n / \log \log n$, then for some $s \in S^{d-1}$, $\Gamma(X_n, s, t)$ is disconnected.

connectivity–proofs

(SUBCRITICAL; NECESSARY.) If $c < 1$ and $d = O(n^{1-c-\epsilon})$, then for all $s \in S^{d-1}$, $\Gamma(X_n, s, t)$ is disconnected.

connectivity–proofs

(SUBCRITICAL; NECESSARY.) If $c < 1$ and $d = O(n^{1-c-\epsilon})$, then for all $s \in S^{d-1}$, $\Gamma(X_n, s, t)$ is disconnected.

We prove that for all $s \in S^{d-1}$, $\Gamma(X_n, s, t)$ contains an isolated vertex.

"Cap argument" together with a sharp estimate for the number N of isolated vertices in $G(n, c \log n/n)$.

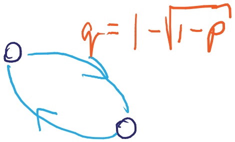
$$\mathbb{P}\{N = 0\} \leq \exp(-n^{-(1-c-\epsilon/2)}) \ .$$

bound for isolated vertices

<u>O'Connell's argument</u>

N = # of isolated vertices in $\mathcal{G}(n,p)$

M = # of vertices with no incoming or outgoing arc.



I = # of vertices with no incoming arc.

Then

$I \sim \mathrm{Bin}(n, (1-q)^{n-1})$

$M \sim \mathrm{Bin}(I, (1-q)^{n-I})$

Use Chernoff bounds.

connectivity–proofs

(SUBCRITICAL; SUFFICIENT.) If $d \geq Cn\sqrt{\log n}$, then there exists an $s \in S^{d-1}$ such that $\Gamma(X_n, s, t)$ is connected.

connectivity–proofs

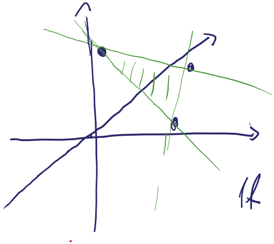
(SUBCRITICAL; SUFFICIENT.) If $d \geq Cn\sqrt{\log n}$, then there exists an $s \in S^{d-1}$ such that $\Gamma(X_n, s, t)$ is connected.

This bound is probably loose. We prove much more:

For every spanning tree of K_n, there exists $s \in S^{d-1}$ such that $\Gamma(X_n, s, t)$ contains the spanning tree.

We show that for any k, if $d \geq Ck\Phi^{-1}(1-p)$, then whp. k i.i.d. standard normal vectors are shattered by half spaces of the form $\{x : \langle x, s \rangle \geq t\}$.

shattering by half planes



Affine span of $x_1, \dots, x_k$:

$$S = \left\{ \sum_{i=1}^{k} c_i x_i : \sum_{i=1}^{k} c_i = 1 \right\}$$

If $\min_{y \in S} \|y\| > t$ then $\{x_1, \dots, x_k\}$ is shattered by the class of half spaces of the form $\{x : \langle x, s \rangle \geq t\}$, $s \in S^{d-1}$

distance of the affine span from the origin

$$\min_{y:\sum y_i=1} \left\| \sum_{i=1}^{k} y_i X_i \right\|^2 \geq \frac{1}{k} \min_{y:|y|^2=1} \left\| \sum_{i=1}^{k} y_i X_i \right\|^2 = \frac{1}{k} \left(\min_{y:|y|^2=1} \|Xy\| \right)^2$$

where X is the $d \times k$ matrix with columns $X_1, \ldots, X_k$.

This is just the square of the least singular value of X.

By Rudelson and Vershinin (2009), the least singular value is at least $\Omega(\sqrt{d} - \sqrt{k-1})$. In particular,

$$\mathbb{P}\left\{ \min_{y:\sum y_i=1} \left\| \sum_{i=1}^{k} y_i X_i \right\| \leq c_1 \sqrt{\frac{d}{k}} \right\} < 2e^{-c_2 d} \ .$$

Note that $t = \Phi^{-1}(1-p) \leq \sqrt{2\log(1/p)} \sim \sqrt{2\log n}$.

connectivity–proofs

(SUPERCRITICAL; NECESSARY.) If $c > 1$ and $d \leq (1-\epsilon)(c-1)\log n/\log\log n$, then for all $s \in S^{d-1}$, $\Gamma(X_n, s, t)$ is connected.

"Cap" argument—with careful covering estimate + standard estimates for the probability that $G(n, c\log n/n)$ is disconnected.

connectivity–proofs

(SUPERCRITICAL; SUFFICIENT.) If $c > 1$ and $d \geq (2+\epsilon)(c-1)\log n / \log\log n$, then for some $s \in S^{d-1}$, $\Gamma(X_n, s, t)$ is disconnected.

connectivity–proofs

(SUPERCRITICAL; SUFFICIENT.) If $c > 1$ and $d \geq (2+\epsilon)(c-1)\log n/\log\log n$, then for some $s \in S^{d-1}$, $\Gamma(X_n, s, t)$ is disconnected.

Second moment method.

Let $\theta = (\log n)^{-1/(2+\epsilon)}$.
Let $\mathcal{P}$ be a maximal set such that for all $s, s' \in \mathcal{P}$, $\langle s, s' \rangle \leq \cos\theta$. Then

$$|\mathcal{P}| \geq \frac{d}{16}\theta^{-(d-1)} .$$

Use the second moment method to prove that whp.,

$$\sum_{s\in\mathcal{P}} \text{number of isolated vertices in } \Gamma(X_n, s, t) > 0$$

questions

• Tighter bounds? Especially the subcritical, sufficient part for connectivity.

• Other properties? Giant component.

• More general model: $\binom{n}{2}$ i.i.d. points, class of sets.

• Inhomogeneous random graphs? Distribution of $\|X_{i,j}\|$ may depend on weights of vertices i and j.