Modelling self-organizing networks

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Cargese Fall School on Random Graphs
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Outline

Multidisciplinary research

Pure Mathematics:

- Graph Theory
- Random Structures and Algorithms
- Modelling

Applied Computer Science:

- ...

Social Science: for example,

- *Homophily, contagion and the decay of community structure in self-organizing networks* (PNAS paper!)
- *Social learning in a large, evolving network* (BlackBerry)

Multidisciplinary research

Applied Computer Science:

- *Utilizing big data for business-to-business matching and recommendation system* (ComLinked Corp., 2014-15)
- *A self-organizing dynamic network model increasing the efficiency of outdoor digital billboards* (KPM, 2014)
- *Exploiting Big Data for Customized Online News Recommendation System* (The Globe and Mail, 2014)
- *Personalized Mobile Recommender System* (BlackBerry, 2013-14)
- *Intelligent Rating System* (Mako, 2012-13)
- *Dynamic clustering and prediction of taxi service demand* (Winston, 2012)

Multidisciplinary research

Applied Computer Science (currently):

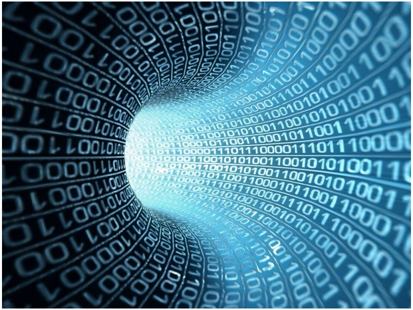
- *Web Visitor Engagement Measurement and Maximization* (The Globe and Mail, 2014-15)
- *Hypergraphs and their applications* (Tutte Institute for Mathematics and Computing)
- *Relationship Mapping Analytics for Fundraising and Sales Prospect Research* (Charter Press Ltd.)

Applied Computer Science (near future):

- *Network Modeling of Trust in Online Scientific Information Sources* (Bell Labs)
- ...

Outline

Big Data Era



Every human-technology interaction, or sensor network, generates new data points that can be viewed, based on the type of interaction, as a self-organizing network.

The web graph

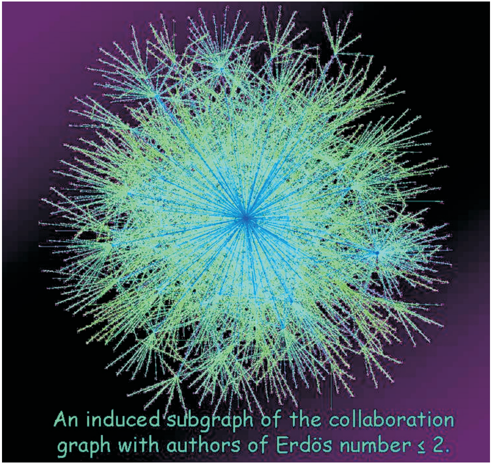
nodes: *web pages* edges: *hyperlinks*

Social networks

nodes: *people* edges: *social interaction (e.g. Facebook friendship)*

Social networks

nodes: *scientists* edges: *co-authorship*



Are these networks similar?

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Answer: Yes!

- large scale
- 'small world' property
  (e.g. low diameter of $O(\log n)$, high clustering coefficient)
- degree distribution
  (power-law, the number of nodes of degree k is proportional to $k^{-\gamma}$)
- bad expansion
- etc.

Why model self-organizing networks?

- uncover the generative mechanisms underlying self-organizing networks,
- models are a predictive tool,
- community detection,
- improving search engines (the web graph),
- spam and worm defense,
- nice mathematical challenges.

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(For example, PA model justifies “rich get richer” principle.)

A good graph model should...

- ...reproduce experimentally observed graph properties:
  - degree distribution follows a power law,
  - small average distance between nodes, (“small world”),
  - locally dense, globally sparse,
  - expansion properties (conductance),...
- ...include a credible model for agent behaviour guiding the formation of the link structure,
- ...agents should not need global knowledge of the network to determine their link environment.

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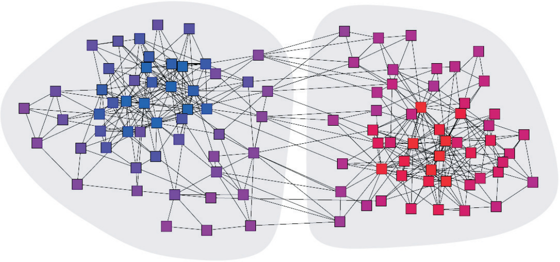
Common assumptions in the study of real-life networks

- Communities in a social network can be recognized as densely linked subgraphs.



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- Co-authors usually have similar research interests, etc.

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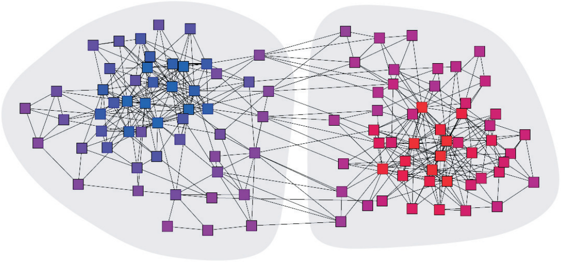

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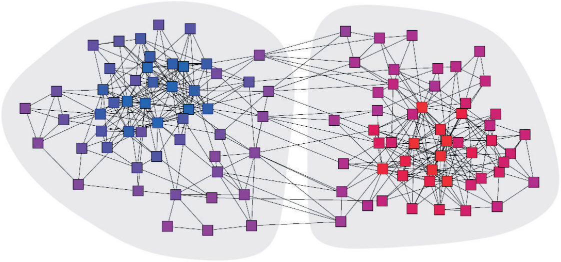

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Underlying metric

Such assumptions, commonly used in experimental and heuristic treatments of real-life networks, imply that there is an a priori “community structure” or “relatedness measure” of the nodes, which is reflected by the link structure of the graph.

The network is a visible manifestation of an underlying hidden reality.

Spatial graph models

- Nodes correspond to points in a (high-dimensional) feature space.
- The metric distance between nodes is a measure of "closeness."
- The edge generation is influenced by the position and relative distance of the nodes.

This gives a basis for reverse engineering: given a graph, and assuming a spatial model, it is possible to estimate the distribution of nodes in the feature space from information contained in the graph structure.

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Outline

1. Introduction
2. Spatial Preferred Attachment (SPA) Model
3. Future work

Spatial Preferred Attachment (SPA) Model

- Nodes are points in *Euclidean space* (randomly and uniformly distributed).

  We let S be the unit hypercube in $\mathbb{R}^m$, equipped with the torus metric derived from any of the L_p norms. This means that for any two points x and y in S,

$$d(x,y) = \min \left\{ ||x - y + u||_p \ : \ u \in \{-1, 0, 1\}^m \right\}.$$

Spatial Preferred Attachment (SPA) Model

- Nodes are points in *Euclidean space* (randomly and uniformly distributed).
- Each node has a *“sphere of influence”* centered at the node. The size is determined by the *in-degree* of the node.

$$|S(v,t)| = \frac{A_1 \deg^{-}(v,t) + A_2}{t}$$

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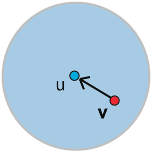

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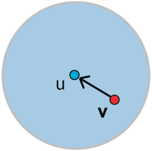
- If v falls into the sphere of influence u, it will link to u with probability p.



Spatial Preferred Attachment (SPA) Model

There are at least three features that distinguish the SPA model from previous models:

- A new node can choose its links purely based on *local* information.
- Since a new node links to each visible node independently, the out-degree is not a constant nor chosen according to a pre-determined distribution, but arises naturally from the model.
- The varying size of the influence regions allows for the occasional *long links*, edges between nodes that are spaced far apart. (This implies a certain "small world" property.)

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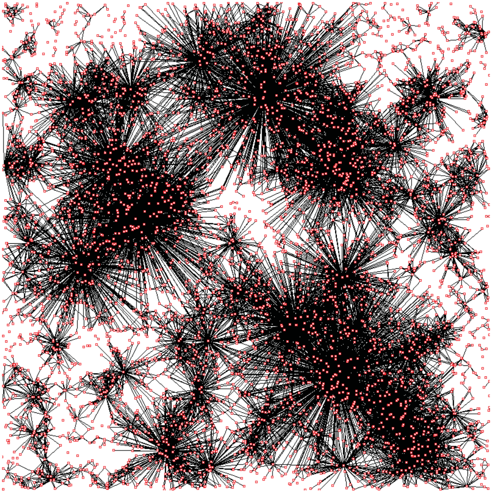

A simulation of the SPA model on the unit square with $t = 5,000$ and $p = 1$

Degree distribution

Power law with exponent $x = 1 + \frac{1}{p}$.

Theorem (Aiello, Bonato, Cooper, Janssen, Prałat)

A.a.s.

$$N(0,t) = (1+o(1))\frac{t}{1+p},$$

and for all k satisfying $1 \le k \le \left(\frac{t}{\log^8 t}\right)^{\frac{p}{4p+2}}$,

$$N(k,t) = (1+o(1))\frac{p^k}{1+p+kp} t \prod_{j=0}^{k-1} \frac{j}{1+p+jp}.$$

(The differential equations method is used.)

A little taste of DEs method

Definition

A **martingale** is a sequence $X_0, X_1, \dots$ of random variables defined on the random process such that

$$\mathbb{E}(X_{n+1} \mid X_0, X_1, \dots, X_n) = X_n.$$

In most applications, the martingale satisfies the property that $\mathbb{E}(X_{n+1} \mid X_0, X_1, \dots, X_n) = \mathbb{E}(X_{n+1} \mid X_n) = X_n$.

Example

Toss a coin n times. Let S_n be the difference between the number of heads and the number of tails after n tosses.

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Theorem (Hoeffding-Azuma inequality)

Let $X_0, X_1, \ldots$ be a martingale. Suppose that there exist constants $c_k > 0$ such that

$$|X_k - X_{k-1}| \leq c_k$$

for each $k \leq n$. Then, for every $t > 0$,

$$\begin{aligned} \mathbb{P}(X_n \geq \mathbb{E}X_n + t) &\leq \exp\left(-\frac{t^2}{2\sum_{k=1}^n c_k^2}\right), \\ \mathbb{P}(X_n \leq \mathbb{E}X_n - t) &\leq \exp\left(-\frac{t^2}{2\sum_{k=1}^n c_k^2}\right). \end{aligned}$$

A little taste of DEs method

$$\mathbb{E}(N(0,t+1) - N(0,t) \mid N(0,t)) = 1 - \frac{N(0,t)pA_2}{t}$$

We first transform $N(0,t)$ into something close to a martingale. It provides some insight if we define real function $f(x)$ to model the behaviour of the scaled random variable $\frac{N(0,xn)}{n}$. If we presume that the changes in the function correspond to the expected changes of random variable, we obtain the following differential equation

$$f'(x) = 1 - f(x)\frac{pA_2}{x}$$

with the initial condition $f(0) = 0$.

A little taste of DEs method

The general solution of this equation can be put in the form

$$f(x)x^{pA_2} - \frac{x^{1+pA_2}}{1+pA_2} = C.$$

Consider the following real-valued function

$$H(x,y) = yx^{pA_2} - \frac{x^{1+pA_2}}{1+pA_2}.$$

(We expect $H(\mathbf{w}_t) = H(t, N(0,t))$ to be close to zero.)

$$\begin{aligned} \mathbb{E}(H(\mathbf{w}_{t+1}) - H(\mathbf{w}_t) \mid G_t) &= O(t^{pA_2-1}) \\ |H(\mathbf{w}_{t+1}) - H(\mathbf{w}_t)| &= O(t^{pA_2}\log^2 n). \end{aligned}$$

Use generalized Azuma-Hoeffding inequality: a.a.s.

$$|H(\mathbf{w}_t) - H(\mathbf{w}_{t_0})| = O(n^{1/2+pA_2}\log^3 n).$$

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Degree distribution

Out-degree: An important difference between the SPA model and many other models is that the out-degree is not a parameter of the model, but is the result of a stochastic process.

Theorem (Aiello, Bonato, Cooper, Janssen, Prałat)

A.a.s.

$$\max_{0\le i\le t} \deg^+(v_i,t) \ge (1+o(1))p\frac{\log t}{\log\log t}.$$

However, a.a.s. all nodes have out-degree $O(\log^2 t)$.

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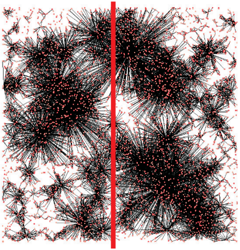
Sparse cuts



Let us partition the vertex set V_t as follows:

$$V'_t = \left\{ x = (x_1, x_2, \ldots, x_m) \in V_t : x_1 < \frac{1}{2} \right\}$$

and $V''_t = V_t \setminus V'_t$.

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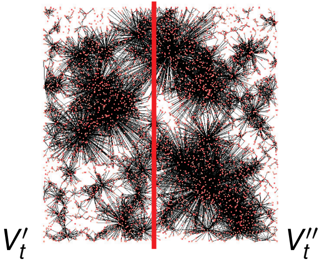

Theorem (Cooper, Frieze, Prałat)

A.a.s. the following holds $|V'_t| = (1+o(1))t/2$, $|V''_t| = (1+o(1))t/2$, *and*

$$|E(V'_t, V''_t)| = O(t^{\max\{1-1/m, pA_1\}} \log^5 t) = o(t).$$

Diameter

Let $l(v_i, v_j)$ denote the length of the shortest directed path from v_j to v_i if such a path exists, and let $l(v_i, v_j) = 0$ otherwise.

The directed diameter of a graph G_t is defined as

$$D(G_t) = \max_{1 \le i < j \le t} l(v_i, v_j).$$

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Theorem (Cooper, Frieze, Prałat)

There exists absolute constant c_2 such that a.a.s.

$$D(G_t) \geq \frac{c_2 \log t}{\log \log t}.$$

(The lower bound requires the additional assumption that $A_1 < 3A_2$, and it is showed for dimension 2 only. However, it can be easily generalized.)

Estimating distnaces

The distance between u and v can be estimated from the graph properties ($cn(u, v, n)$, $\deg^-(u)$ and $\deg^-(v)$).

Theorem (Janssen, Prałat, Wilson)

Theorem 3.1. *Let $\omega = \omega(n)$ be any function tending to infinity together with n. The following holds a.a.s. Let v_k and v_ℓ be vertices such that*

$$k = \deg(v_k, n) \geq \deg(v_\ell, n) = \ell \geq \omega^2 \log n$$

in a graph generated by the SPA model. Let $d = d(v_k, v_\ell)$ be the distance between v_k and v_ℓ in the metric space. Finally, let $T = f^{-1}(\ell/(\omega \log n))$. Then,

Case 1. *If $d \geq \varepsilon(\omega \log n/T)^{1/m}$ for some $\varepsilon > 0$, then*

$$cn(v_\ell, v_k, n) = O(\omega \log n).$$

Case 2. *If $k \geq (1+\varepsilon)\ell$ for some $\varepsilon > 0$ and*

$$d \leq \left(\frac{A_1 k + A_2}{c_m n}\right)^{1/m} - \left(\frac{A_1 \ell + A_2}{c_m n}\right)^{1/m} = \Theta\left(\left(\frac{k}{n}\right)^{1/m}\right), \tag{5}$$

then

$$cn(v_\ell, v_k, n) = (1 + o(1))p\ell.$$

If $k = (1+o(1))\ell$ and $d \ll (k/n)^{1/m} = (1+o(1))(\ell/n)^{1/m}$, then $cn(v_\ell, v_k, n) = (1+o(1))p\ell$ as well.

Case 3. *If $k \geq (1+\varepsilon)\ell$ for some $\varepsilon > 0$ and*

$$\left(\frac{A_1 k + A_2}{c_m n}\right)^{1/m} - \left(\frac{A_1 \ell + A_2}{c_m n}\right)^{1/m} < d \ll (\omega \log n/T)^{1/m}, \tag{6}$$

then

$$cn(v_\ell, v_k, n) = C i_k^{-\frac{(pA_1)^2}{1-pA_1}} i_\ell^{-pA_1} d^{-\frac{mpA_1}{1-pA_1}} \left(1 + O\left(\left(\frac{i_k}{i_\ell}\right)^{pA_1/m}\right)\right), \tag{7}$$

where $i_k = f^{-1}(k)$ and $i_\ell = f^{-1}(\ell)$ and $C = pA_1^{-1} A_2^{\frac{1}{1-pA_1}} c_m^{-\frac{pA_1}{1-pA_1}}$.

If $k = (1+o(1))\ell$ and $\varepsilon(k/n)^{1/m} < d \ll (\omega \log n/T)^{1/m}$ for some $\varepsilon > 0$, then

$$cn(v_\ell, v_k, n) = \Theta\left(i_k^{-\frac{(pA_1)^2}{1-pA_1}} i_\ell^{-pA_1} d^{-\frac{mpA_1}{1-pA_1}}\right).$$

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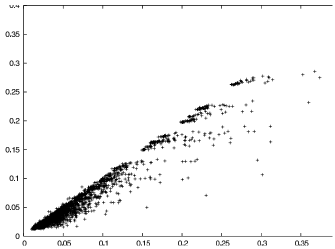


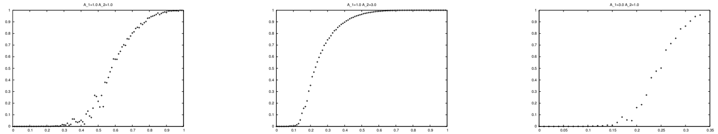
Actual distance vs. estimated distance from simulated data

Outline

Giant component

Conjecture (Cooper, Frieze, Prałat)

$p_3 := (2A_1 + 2A_2)^{-1}$ is the threshold for the giant component.



(a) $A_1 = 1, A_2 = 1$ (b) $A_1 = 1, A_2 = 3$ (c) $A_1 = 3, A_2 = 1$

Conjecture

The clustering coefficient of a vertex of degree k is of order $1/k$.

Common directions

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- Find the right parameters for power law exponent etc.
- Validate the model by comparing graph properties
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Spatial Preferred Attachment (SPA) Model

- Generalize the model:
  - Node and edge deletion
  - Adding edges to existing nodes
  - Updating the out-links of a node
  - Shifting coordinates ("learning process")
- Undirected graphs
- Non-uniform distribution of points

- Use the model to estimate the underlying geometry of the nodes.

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- Use the model to estimate the underlying geometry of the nodes.

Spatial Preferred Attachment (SPA) Model

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  - Node and edge deletion
  - Adding edges to existing nodes
  - Updating the out-links of a node
  - Shifting coordinates ("learning process")
- Undirected graphs
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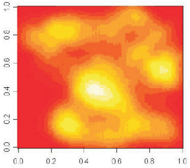
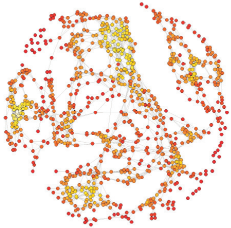


$\Rightarrow$



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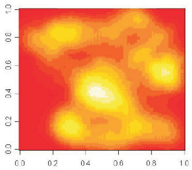

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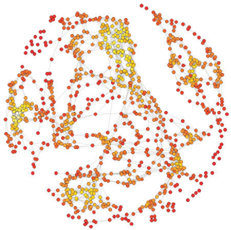

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Story 1: Social Learning (BlackBerry)

Consider two homophily hypotheses:

- the likelihood of tie formation between two actors increases with greater similarities in the actors' tastes
- the likelihood of tie deletion between two actors increases with greater differences in the actors' tastes

The role of social influence—third main hypothesis:

- actors tend to adopt the tastes of others they share direct connections with

Story 2: GEO-P model and domination number