Cargèse Fall School on Random Graphs
Cargèse, Corsica, September 20-26, 2015

Random groups

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QUOTE

I feel, random groups altogether may grow up as healthy as random graphs, for example.

Misha Gromov *Spaces and questions* 1999

Why do we care?

Reason no. 3

Random (i.e. ‘typical’) graphs have got many ‘exotic’ properties.

WHY DO WE CARE?

REASON NO. 3

Random (i.e. 'typical') graphs have got many 'exotic' properties. We hope the same is true for random groups.

Plan of the talk

- Random graphs (and matrices)
- Random groups: the first few
  (natural, yet unsuccessful) approaches
- The cycle space and its generalizations
- Fundamental groups
- Finitely presented groups
- The evolution of the random triangular group

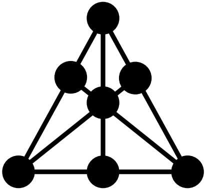
GRAPHS ARE MATRICES

REMARK

In the talk we shall often identify a graph $G = (V, E)$ with its **incidence** matrix of dimension $|V| \times |E|$.

GRAPHS ARE MATRICES

Example



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Graphs are matrices

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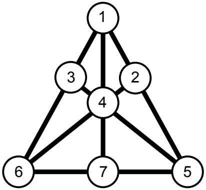

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Random graph models

The binomial random graph $G(n, p)$

$G(n, p)$ is the (random) graph on vertices $\{1, 2, \ldots, n\}$ in which each of $\binom{n}{2}$ possible pairs appears as an edge independently with probability p.

The uniform random graph $G(n, M)$

$G(n, M)$ is the (random) graph chosen uniformly at random from the family of all graphs on vertices $\{1, 2, \ldots, n\}$ and M edges.

The unlabelled random graph $U(n, M)$

$U(n, M)$ is the **unlabelled** (random) graph chosen uniformly at random from the family of all unlabelled graphs with n vertices and M edges.

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Asymptotics

In this talk we are interested only in the asymptotic behaviour of discrete random structures.
In particular, aas means 'tending to 1 as $n \to \infty$'.

Asymptotics

(Most of) asymptotic properties of $G(n, M)$ and $G(n, p)$ are very similar provided $p = M/\binom{n}{2}$.

On the other hand, the asymptotic properties of $G(n, M)$ and $U(n, M)$ are basically the same if M is large (say $M \gg n \ln n$) but when M is small (say $M \ll n \ln n$) they can be quite different.

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This is because aas as soon as one before last isolated vertex disappears $G(n, M)$ becomes asymmetric.

Random groups: the first attempt

Definition of ‘labelled’ random group

Given n and M choose uniformly at random a subgroup of the permutation group S_n with M elements.

Definition of ‘unlabelled’ random group

Given M choose uniformly at random a group from the family of all groups with M elements (classified up to isomorphism).

Random groups: the first attempt

Definition of 'labelled' random group

Given n and M choose uniformly at random a subgroup of the permutation group S_n with at most M elements.

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Unfortunately, we do not have a slightest idea how to deal with such random groups; counting subgroups of S_n is already a big challenge!

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Theorem Dixon'69

Aas two random elements of S_n generate either A_n or S_n.

Another idea

Generate random groups
based on random graphs!

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or, more specifically,

Generate random groups
based on random matrices!

Random groups: the third attempt

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Take the automorphism group of $G(n, M)$.

Metatheorem

Automorphism groups of **finite** random structures are not very exciting (e.g. dense random structures are very often asymmetric).

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The cycle space

Definition

The cycle space of a graph G is the linear space (over $\mathbb{F}_2$) which consists of all subgraphs of G with all degrees even with the symmetric difference as the addition.

Remark

The cycle space of G is spanned by cycles of G. Moreover, given a graph G, it is very easy to find a cycle basis of its cycle space.

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Moreover, given a graph G, it is very easy to find a cycle basis of its cycle space.

The cycle space is the right kernel of the incidence matrix

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Question 1

When the cycle space of $G(n, M)$ is trivial?

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When the cycle space of $G(n, M)$ is spanned by some special family of cycles?

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The threshold function for the property that the cycle space of $G(n, M)$ is spanned by its triangles is the same as the threshold that each edge of $G(n, M)$ is contained in a triangle.

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The incidence matrix of a 3-graph

Definition

A 3-graph G is a pair (V, E), where V is the set of vertices of G, and E is the set of 3-element subsets of V called edges.

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In the talk, by the incidence matrix of a 3-graph (V, E) we mean a zero-one matrix whose rows correspond to **pairs** of vertices, and columns to edges of G, and 1 appears only if a pair of vertices is contained in an edge.
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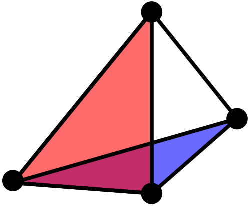

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The (homology) group $H_2(G)$

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The group $H_2(G)$ of a 3-graph G is the right kernel of its incidence matrix.

Equivalent combinatorial definition

For a 3-graph $G = (V, E)$ the group $H_2(G)$ is defined as the set of all subsets of E' of E such that each pair of vertices is contained in an even number of elements of E'. The addition in $H_2(G)$ is just the symmetric difference of sets.

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H_2 (for 3-graphs) is more interesting than the cycle space (for graphs)

Fact

There are no natural 'combinatorial' bases for $H_2(G)$.

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Collapsibility

Definition

If in every subgraph H of a 3-graph G one can find an edge e and a pair of vertices $\{v, w\} \subset e$ such that no other edge of H contains $\{v, w\}$, then we say that G is collapsible.

Observation

If G is collapsible, then $H_2(G) = 0$.

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Every subgraph of a graph G has a vertex of degree 1 (i.e. G is collapsible) if and only if G is a tree (i.e. its cycle space is trivial).

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There are non-collapsible 3-graphs G with $H_2(G) \neq 0$.

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Remark

This question is, in a way, analogous to the question on the k-colorability threshold and the threshold for the existence of k-core Mike Molloy mentioned in his talk.

$H_2(G_3(n, M))$

The following results have been proved by different subsets of the set {Aronstadt, Linial, Łuczak, Meshulam, Peled}.

Theorem

There are explicitly computable constants $a_h > a_c > 0$ such that the threshold function for collapsibility of $H_2(G_3(n, M))$ is $M_c = (a_c + o(1))n^2$, while the threshold function for the property that $H_2(G_3(n, M))$ is nontrivial is $M_h = (a_h + o(1))n^2$.

Theorem

For every constant $a > 0$ there exists a constant $b > 0$ such that aas each subgraph of $G_3(n, M)$ of at most bn^2 is collapsible.

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The theory of codes

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Given k and n find the largest family of vectors from $\{0, 1\}^n$ such that the Hamming distance between each pair of vectors is at least k.

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Remark

We are particularly interested in an efficient construction which, furthermore, allows efficient error-correcting procedure.

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Thus, we are after a graph G with large $H_2(G)$ such that each pair of vectors of $H_2(G)$ differ on more than k places. But this condition holds whenever $H_2(G)$ contains no vectors with fewer than k ones, e.g. for G in which each subgraph on k edges is collapsible.

Random groups: more geometric approach

Idea

Consider a 3-graph as a geometric realization of a 2-dimensional simplicial complex,

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Consider a 3-graph as a geometric realization of a 2-dimensional simplicial complex,
i.e. imagine the 1-dimensional skeleton of $(n-1)$-dimensional simplex and glue into some of its triangles 2-dimensional triangular cells.

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Consider a 3-graph as a geometric realization of a 2-dimensional simplicial complex,
i.e. imagine the 1-dimensional skeleton of $(n-1)$-dimensional simplex and glue into some of its triangles 2-dimensional triangular cells.
Then study its fundamental group.

Random groups: more geometric approach

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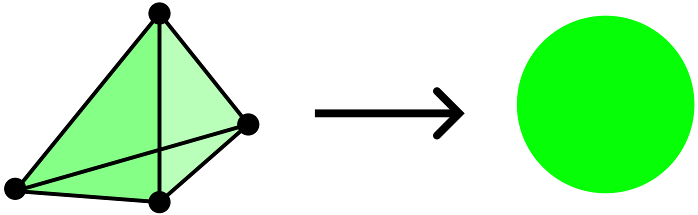

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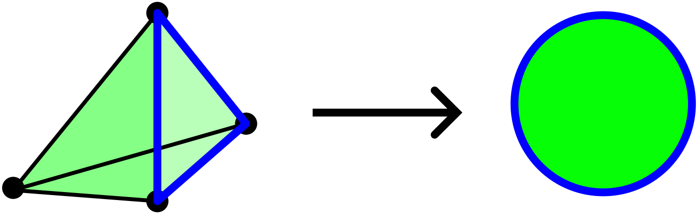

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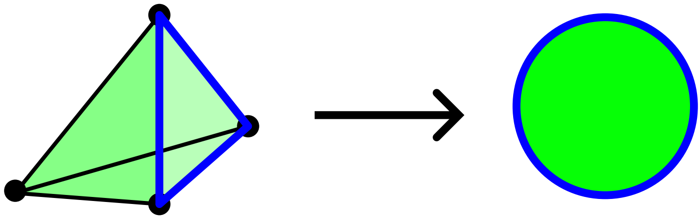

$\pi_1(G_3(4,3))$ is trivial

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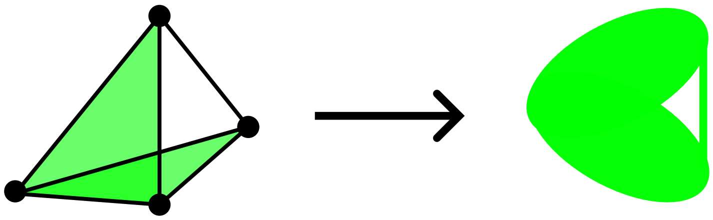

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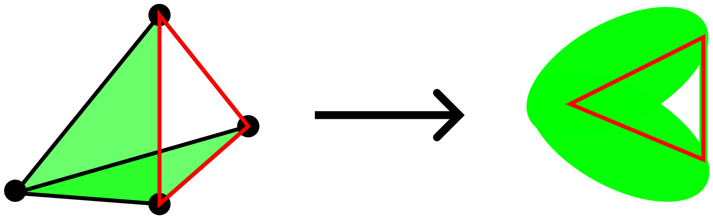

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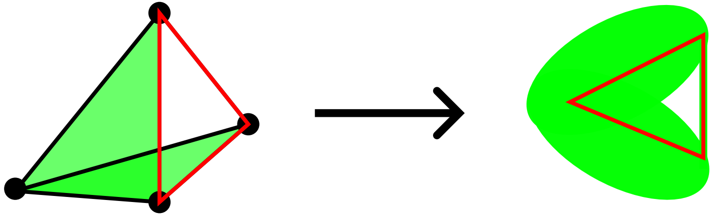

$$\pi_1(G_3(4,3)) \cong \mathbb{Z}$$

$\pi_1(G_3(n, M))$

Theorem Babson, Hoffman, Kahle'11

For every $\epsilon > 0$, if $M > n^{5/2+\epsilon}$ then aas $\pi_1(G_3(n, M)) = 0$, while for $M < n^{5/2-\epsilon}$ the group $\pi_1(G_3(n, M))$ is aas non-trivial (and hyperbolic).

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The proof is quite involved and applies some non-trivial topological tools.
We do not know how to generalize it for $G_k(n, M)$.

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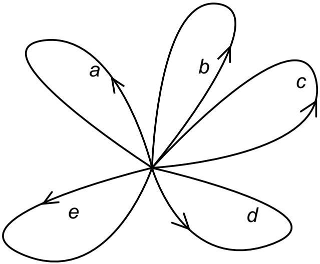

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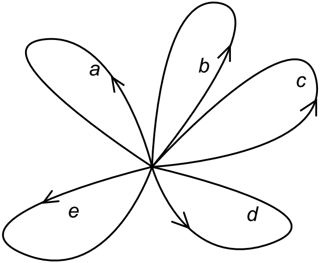

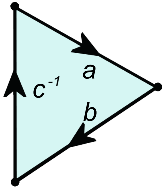

Group presentations

$$G = \langle S | R \rangle$$

is a group which consists of words with letters $a, b, \ldots$ (as well as its formal inverses $a^{-1}, b^{-1}, \ldots$) from an alphabet S in which we can cancel all words from set R.

Example

In the group

$$G = \langle \{a, b\} | aba^{-1}b^{-1} \rangle$$

we have $aba^{-1}b^{-1} = e$, i.e.

$$ab = aba^{-1}b^{-1}ba = aba^{-1}b^{-1}ba = ba\,,$$

so

$$G = \{a^n b^m : a, b \in \mathbb{Z}\} = \mathbb{Z}^2\,.$$

THE PRESENTATION COMPLEX

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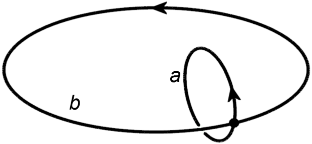

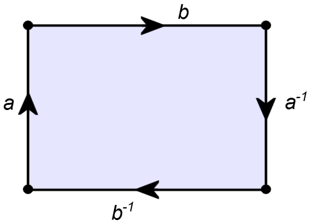

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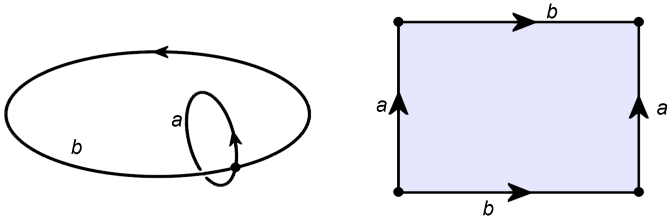

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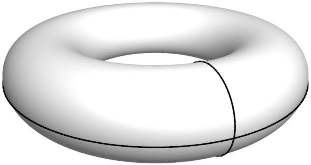

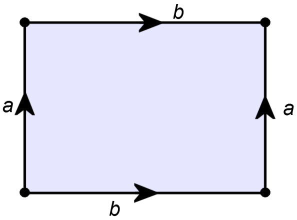

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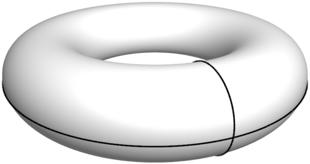

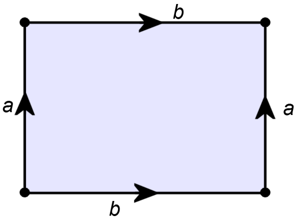

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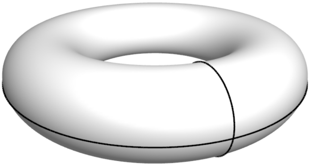

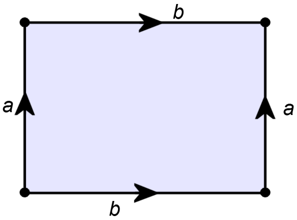

FINITELY PRESENTED GROUPS ARE '2-DIMENSIONAL'

Thus,

$$G = \langle \{a, b\} | aba^{-1}b^{-1} \rangle = \pi_1(S^1 \times S^1) = \mathbb{Z}^2,$$

and, in general, each finitely presented groups can be viewed as the fundamental group of its (2-dimensional) presentation complex.

van Kempen diagrams

Group presentation have a strong combinatorial flavour

If $abc^{-1} = e$ and $aba^{-1}d = e$, then

$$a^{-1}dc = b^{-1}a^{-1}aba^{-1}dabc^{-1}c = b^{-1}a^{-1}aba^{-1}dab = e.$$

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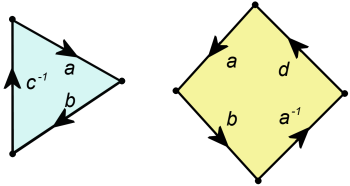

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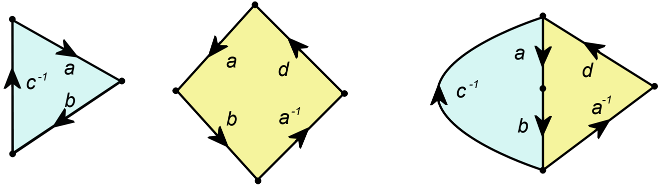

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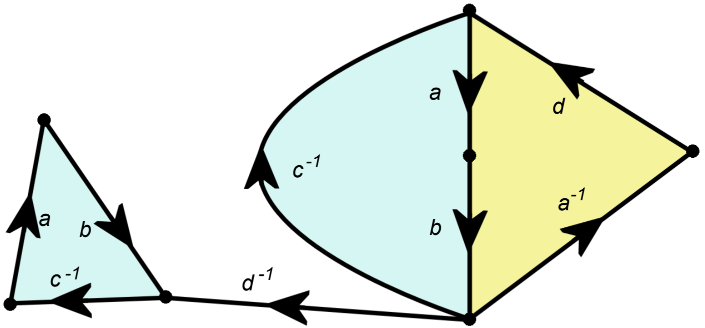

Finitely presented groups are often hard to study

Presentations are sometimes hard to deal with, both in theory

Theorem

Given presentation $\langle S|R\rangle$ of a group Γ it is undecidable if a given word is equivalent to 0 in Γ.

Many properties of groups with natural short finite presentations are unkown (e.g. it is not known if Thompson group F is amenable).

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and in practice

Many properties of groups with natural short finite presentations are unkown (e.g. it is not known if Thompson group F is amenable).

Gromov's idea:

Choose a random presentation!

RANDOM GROUP $\Gamma(n, k; p)$

DEFINITION

$$\Gamma(n, k; p) = \langle \{g_1, g_2, \dots, g_n\} | \mathcal{R}_p \rangle$$

where each relation of length k belongs to $\mathcal{R}_p$ independently with probability p.

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Gromov'83: $\Gamma(2, k; p)$, where $p = p(k)$ and $k \to \infty$,
Żuk'06: $\Gamma(n, p) = \Gamma(n, 3; p)$, where $p = p(n)$ and $n \to \infty$.

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$\Gamma(n, k; p)$ was formally introduced by
Antoniuk, Łuczak, Świątkowski'14.

THE EVOLUTION OF $\Gamma(n, p)$

THEOREM Żuk'03

For every constant $\epsilon > 0$ the following holds.

- If $p \leq n^{-2-\epsilon}$ then aas $\Gamma(n, p)$ is free.
- If $n^{-2+\epsilon} \leq p \leq n^{-3/2-\epsilon}$, then aas $\Gamma(n, p)$ is infinite, hyperbolic, and has Kazdhan's property (T).
- If $p \geq n^{-3/2+\epsilon}$, then aas $\Gamma(n, p)$ is trivial.

Collapsing of the random group

Theorem Żuk'03

Let $\epsilon > 0$. Then

- If $p \geq n^{-3/2+\epsilon}$, then aas $\Gamma(n, p)$ is trivial.
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$$\mathcal{R}(n, p) \supseteq \mathcal{R}_1(n, p/3) \cup \mathcal{R}_2(n, p/3) \cup \mathcal{R}_3(n, p/3) .$$

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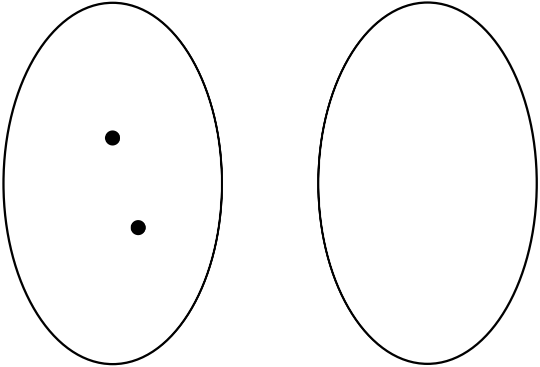
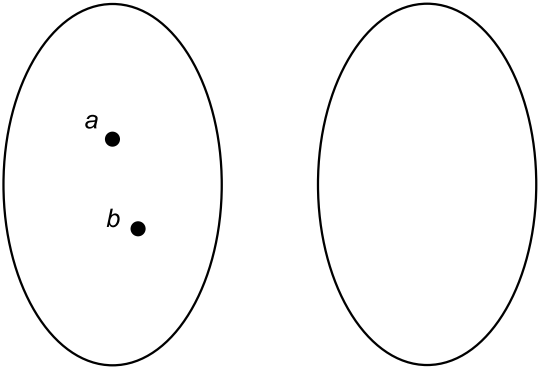
Now let us split the set S of n generators into two roughly equal parts $S_1 \cup S_2$ and define an auxiliary graph on the vertex set $S_1 \cup S_1^{-1}$.

The auxiliary graph



$S = S_1 \cup S_2$, the vertices in the left set are labelled by $S_1 \cup S_1^{-1}$ the vertices in the right set by $S_2 \cup S_2^{-1}$.

The auxiliary graph



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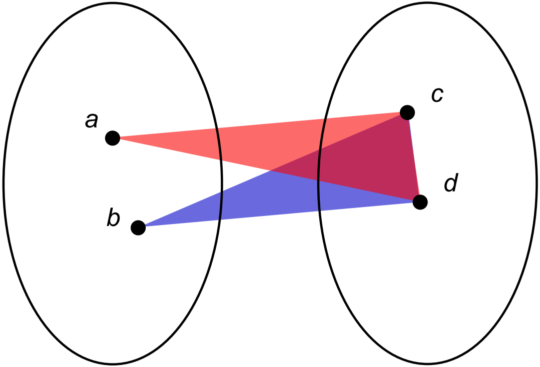

$$acd = e \ \& \ bcd = e \Longrightarrow a = d^{-1}c^{-1} = b$$

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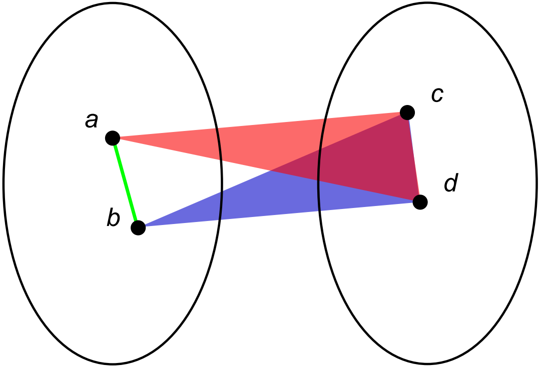

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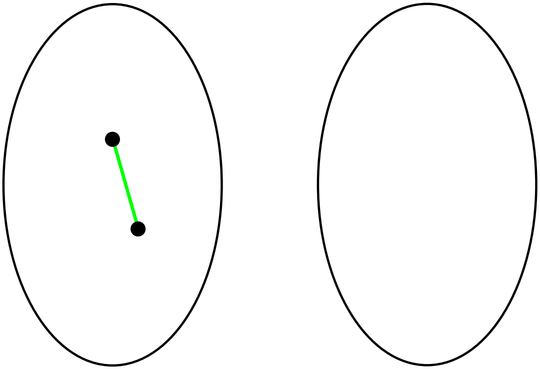
The first stage



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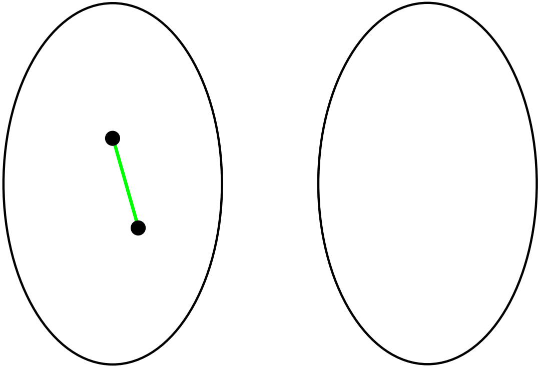
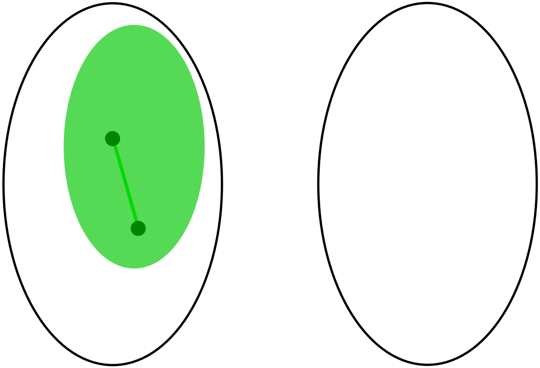


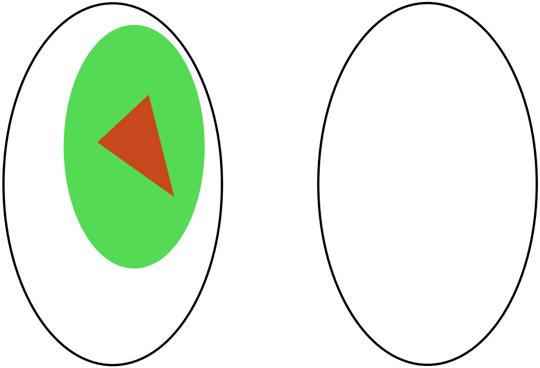
There exists a large green component which contains more than half of all vertices of $S_1 \cup S_1^{-1}$

THE FIRST STAGE



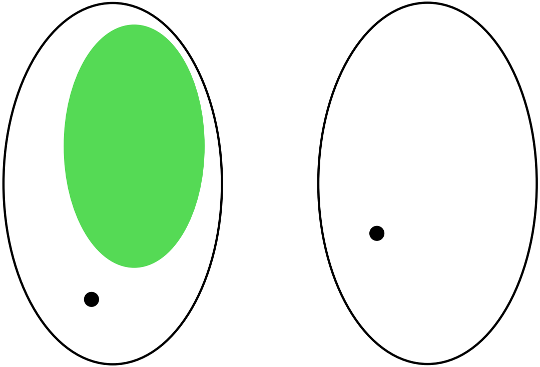
All generators c in the green component are the same and, since $c = c^{-1}$, all of them are of rank 2.

The second stage



For each generators c in the green component we have
$c^2 = e$ & $c^3 = e \Longrightarrow c = e$.

The third stage



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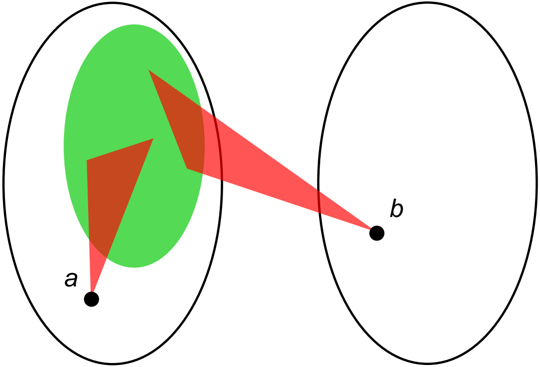
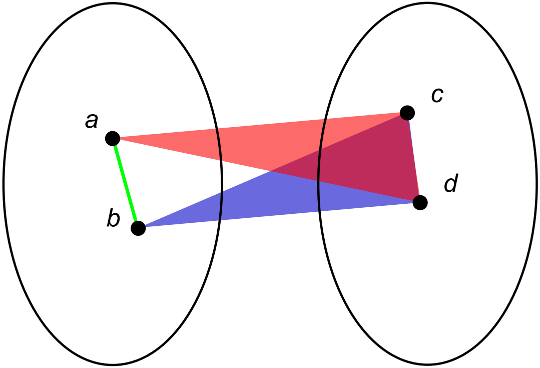


$aee = e \implies a = e,$

$bee = e \implies b = e.$ QED

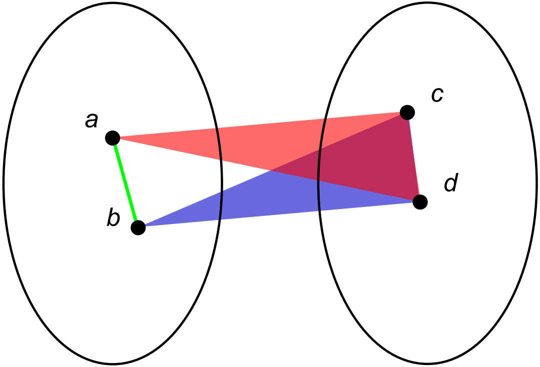
Large green component



The probability that a and b are adjacent is roughly

$$\binom{n}{2}p^2 \sim \frac{1}{3}n^2(cn^{-3/2})^2 \geq c'/n.$$

LARGE GREEN COMPONENT



Unfortunately, the events that the edges $\{a, b\}$ and $\{b, c\}$ appear in the green graph are positively correlated.

Large green component

The green graph is a random graph but the existence of its edges are positively correlated.

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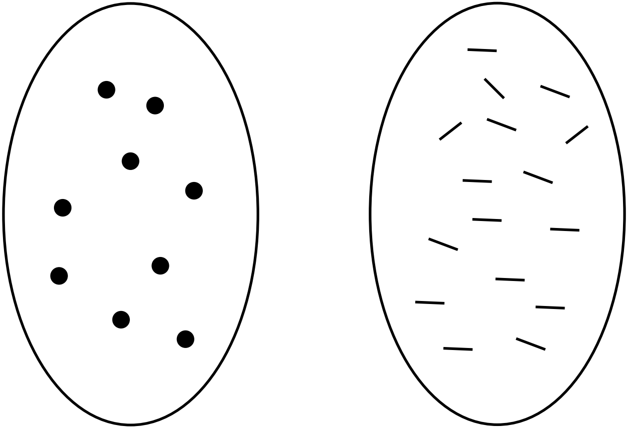
Fortunately, we can apply an analogous BEHRISCH'07 result for the random intersection graph

(which, by the way, is one of a very few models of 'small world graphs' which have both the power law degree distribution and large clustering coefficient; that is precisely why Paweł Prałat mentioned it in his talk).

Random intersection graphs



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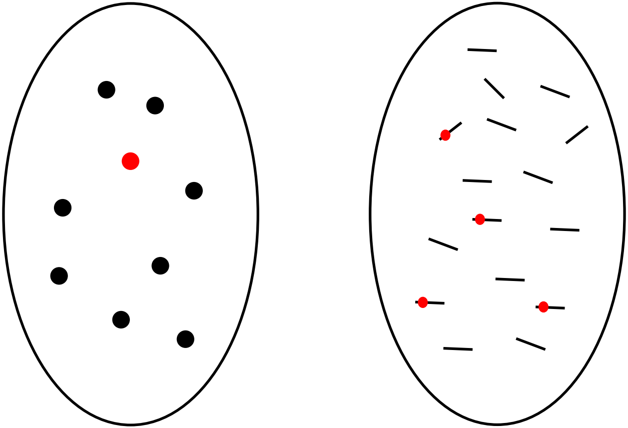

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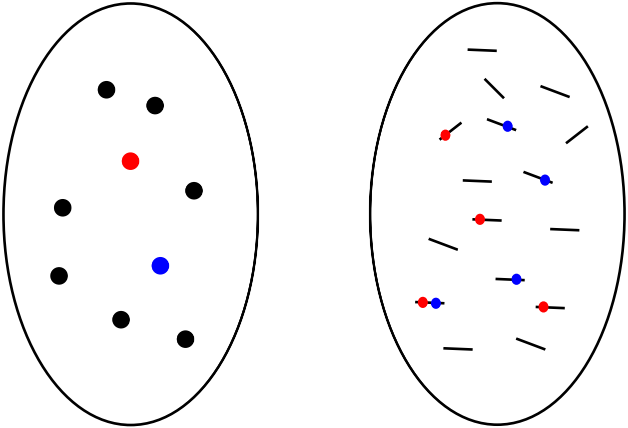

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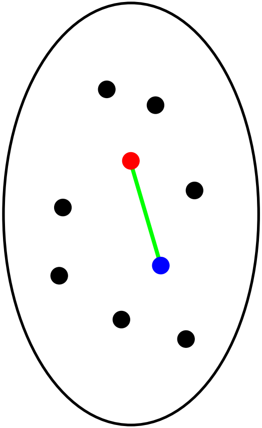

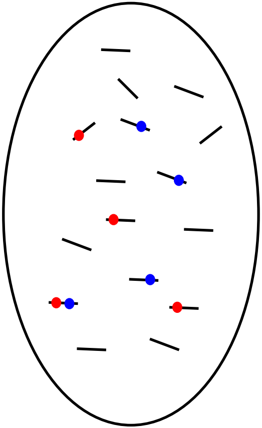

Collapsing of the random group revisited

Theorem Żuk'03

Let $\epsilon > 0$. Then

- If $p \geq n^{-3/2+\epsilon}$, then aas $\Gamma(n,p)$ is trivial.
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CONJECTURE ANTONIUK, ŁUCZAK, ŚWIĄTKOWSKI'14

There exists a constant $c' > 0$ such that if $p \leq c' n^{-3/2}$, then aas $\Gamma(n, p)$ is infinite (and hyperbolic).

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Theorem Antoniuk, Friedgut, Łuczak'15+

There exists a function $c(n)$ such that for every $\epsilon > 0$ the following holds.

- If $p \geq (1+\epsilon)c(n)n^{-3/2}$, then aas $\Gamma(n, p)$ is trivial.
- If $p \leq (1-\epsilon)c(n)n^{-3/2}$, then aas $\Gamma(n, p)$ is not trivial.

Two types of thresholds

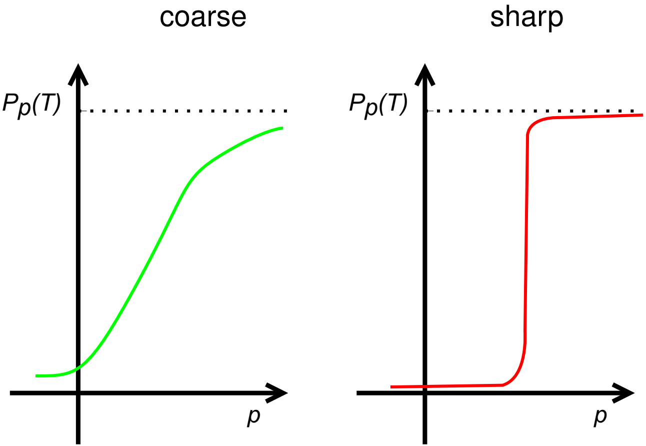
The (coarse) threshold for '$G(n,p) \supseteq K_3$'

If $np \to 0$, then aas $G(n,p) \not\supseteq K_3$, while
if $np \to \infty$, then aas $G(n,p) \supseteq K_3$.

The (sharp) threshold for connectivity

Let $\omega(n) \to \infty$.
If $p = \frac{1}{n}(\log n - \omega(n))$, then aas $G(n,p)$ is not connected,
while if $p = \frac{1}{n}(\log n + \omega(n))$, then aas $G(n,p)$ is connected.

Two types of thresholds



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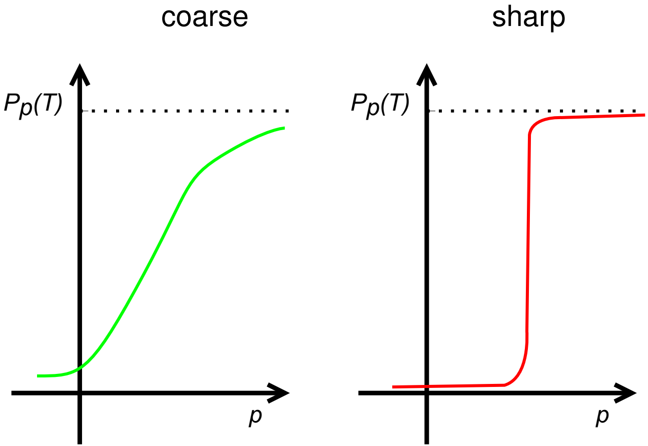

We claim that the threshold for collapsing is of the latter kind.

General theory of (sharp) thresholds

Suppose a random subset $\mathcal{R}_p$ of a set Ω is obtained choosing elements of Ω independently at random with probability p.
Let A be an increasing property of subsets of Ω.

Theorem Friedgut+Bourgain'99

A property A has a coarse threshold if and only if it is 'local'.

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Example

Consider the following properties of $\Gamma(n,p) = \langle S | \mathcal{R}(n,p) \rangle$
A_1: five generators of $\Gamma(n,p)$ are equivalent to the identity,
A_2: all generators of $\Gamma(n,p)$ are equivalent to the identity.

Then, A_1 has a coarse threshold, while, as we see shortly, the threshold for A_2 is sharp.

General theory of (sharp) thresholds

Kahn, Kalai, Linial'88

$\Downarrow$

Bourgain, Kahn, Kalai, Katznelson, Linial'92

$\Downarrow$

Friedgut+Bourgain'99

Sharp threshold for collapsing

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The threshold for collapsing $\Gamma(n,p)$ which occurs for $p \sim n^{-3/2+o(1)}$ is sharp.

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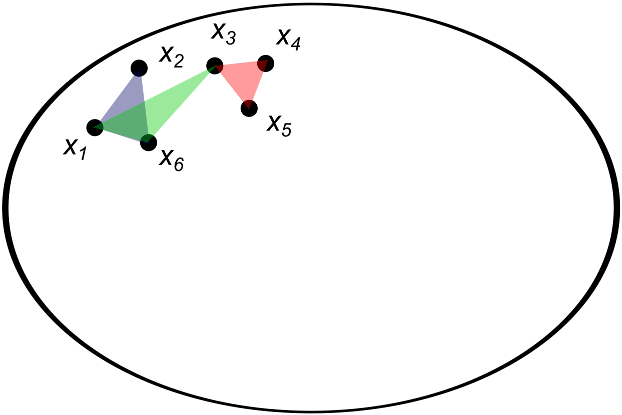
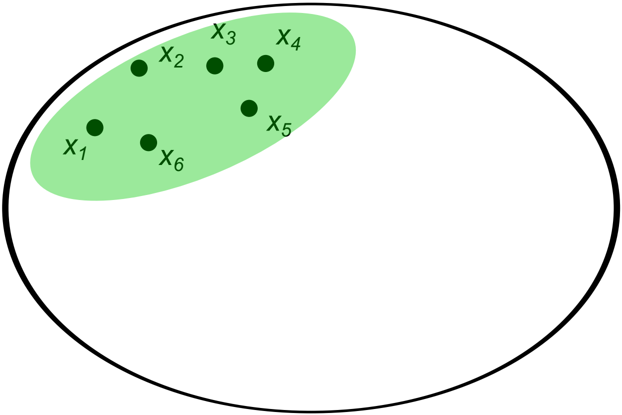
Proof We have to show that collapsing is not 'local', i.e. adding a few relations to $\Gamma(n, p)$ does not change probability of collapsing more than changing probability p to $(1 + \epsilon)p$, for some $\epsilon > 0$.

The 'local' graph



$x_1 x_2 x_6 = e$ & $x_3 x_5 x_4 = e$ & $x_1 x_3 x_6 = e$

THE 'LOCAL' GRAPH



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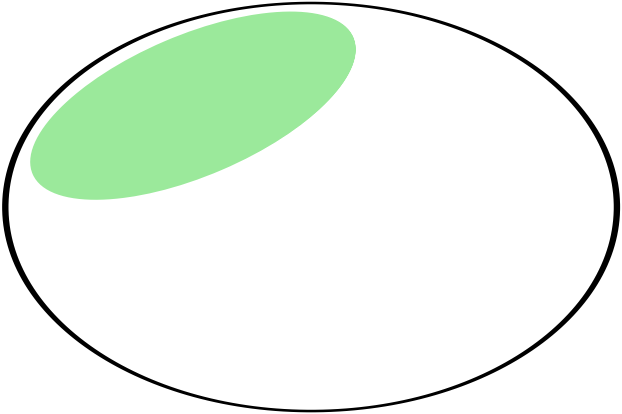

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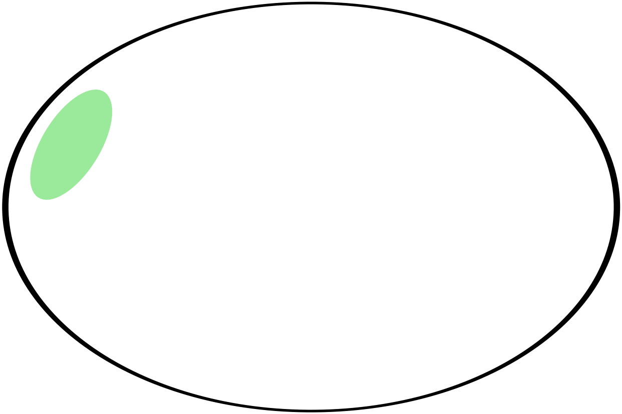

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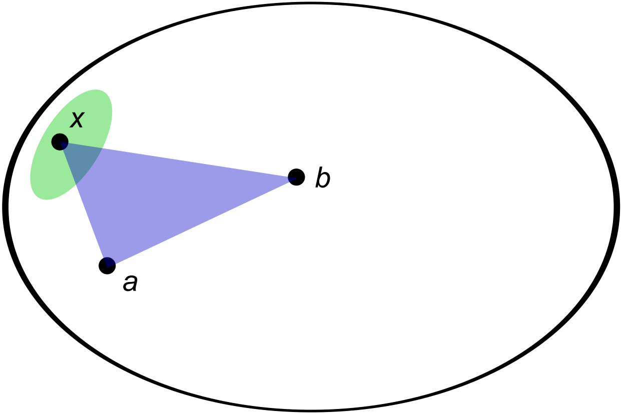
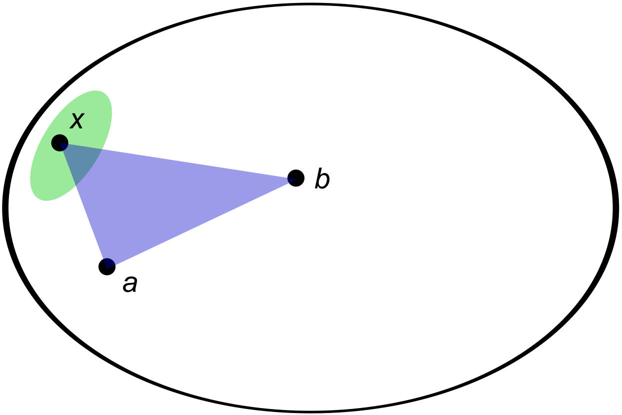


The 'local' graph



$$xab = e \Longrightarrow ab = e \Longrightarrow a = b^{-1}$$

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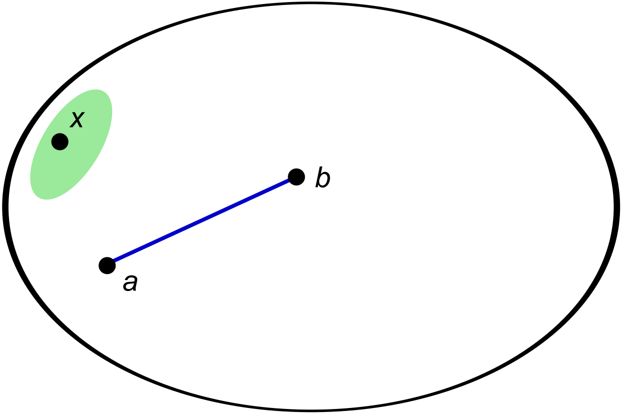

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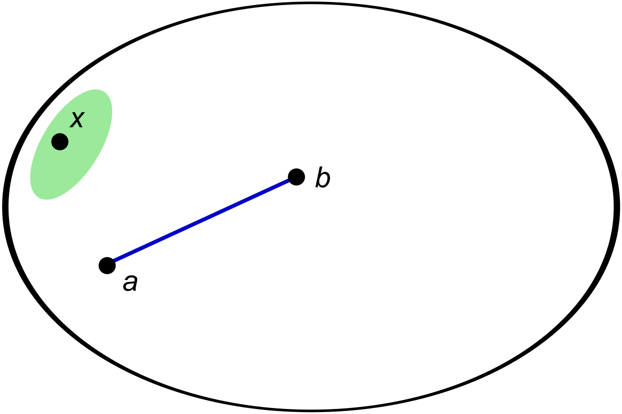
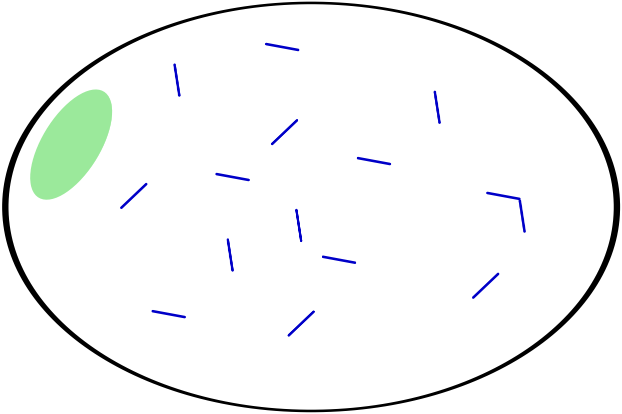


$a = b^{-1}$

$\rho_1 = \Theta(p)$

The blue 'local' graph



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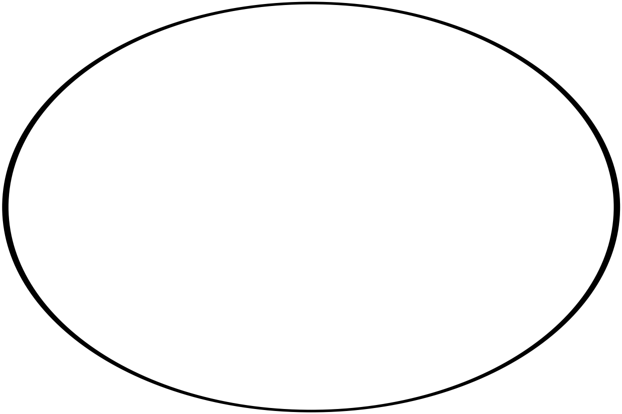
The 'global' graph



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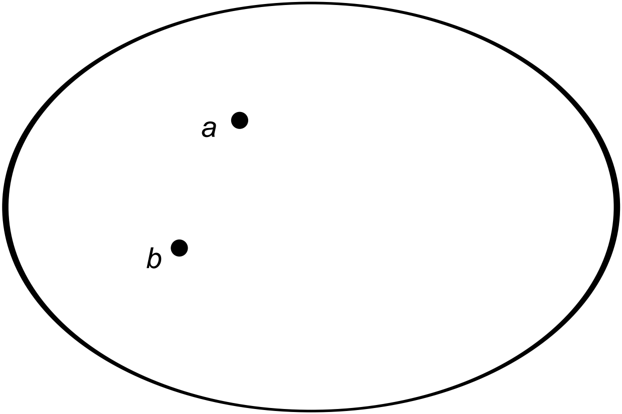

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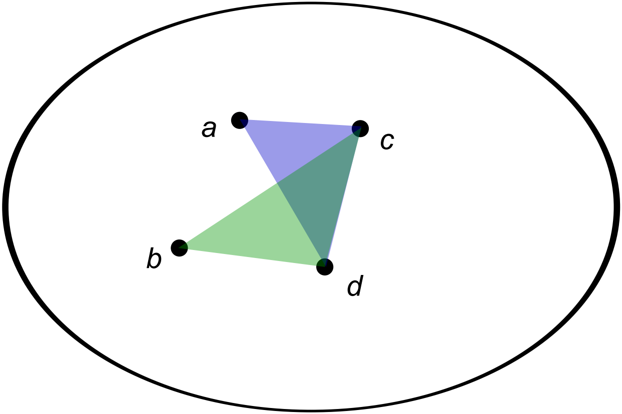

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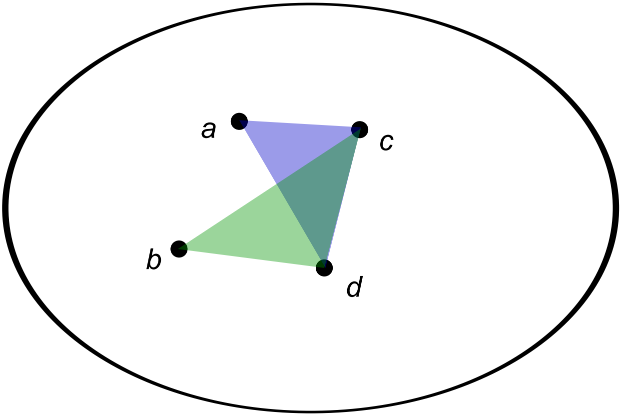
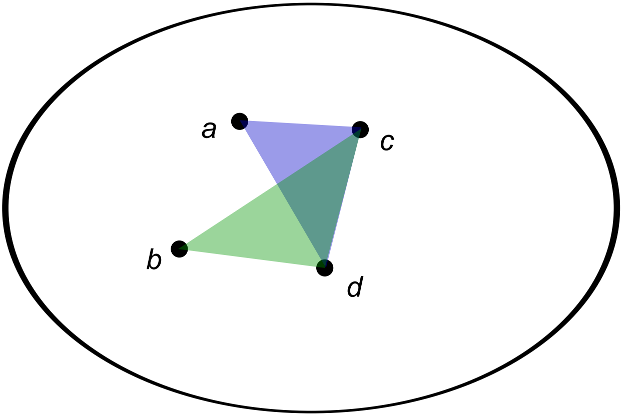


$acd = e$ & $b^{-1}cd = e$

The 'global' graph



$acd = e$ & $b^{-1}cd = e \implies a = b^{-1}$

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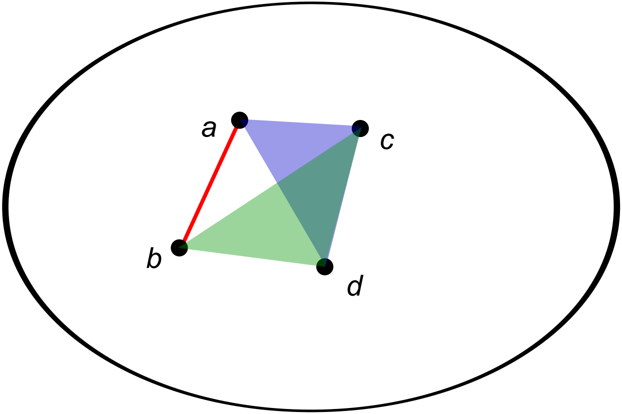

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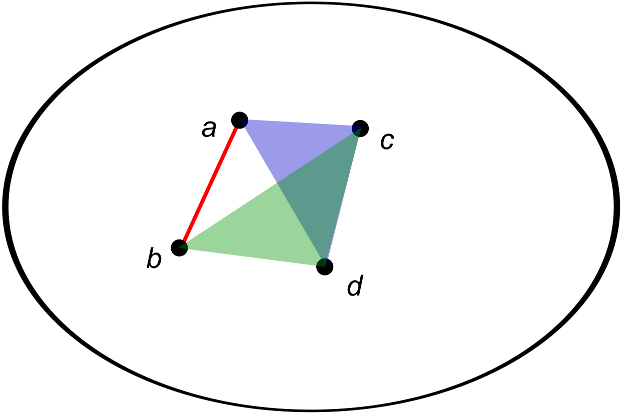

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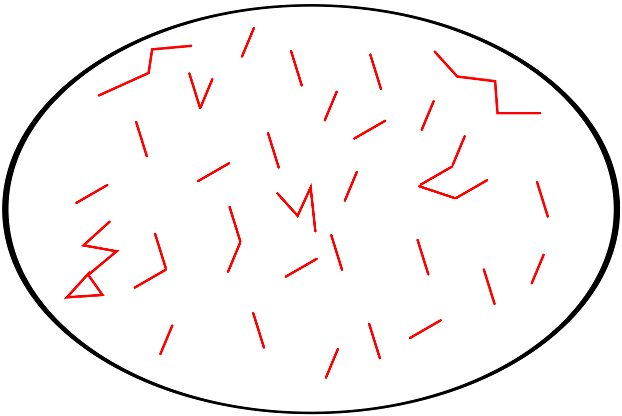
$\rho_2 = \Theta(n^2(\epsilon p)^2)$

THE RED 'GLOBAL' GRAPH



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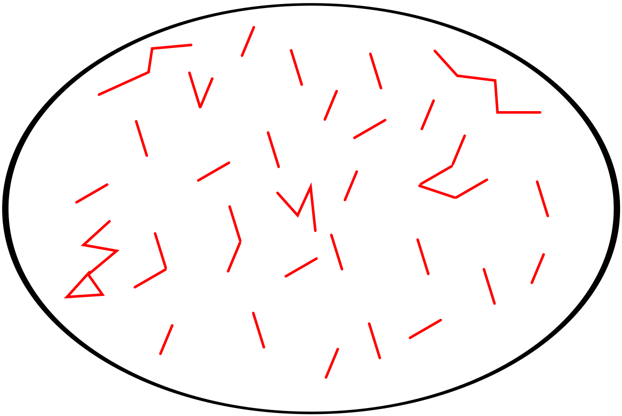
The red 'global' graph



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$\rho_2 = \Theta(n^2(\epsilon p)^2) \gg \rho_1 = \Theta(p)$ QED

The evolution of the random group

Theorem Żuk'03

For every constant $\epsilon > 0$ the following holds.

- If $p \leq n^{-2-\epsilon}$ then aas $\Gamma(n,p)$ is free.
- If $n^{-2+\epsilon} \leq p \leq n^{-3/2-\epsilon}$, then aas $\Gamma(n,p)$ is infinite, hyperbolic, and has Kazdhan's property (T).
- If $p \geq n^{-3/2+\epsilon}$, then aas $\Gamma(n,p)$ is trivial.

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THEOREM ANTONIUK, ŁUCZAK, ŚWIĄTKOWSKI'14

Let $\epsilon > 0$. Then there exists constants $c_2 \geq c_1 > 0$ and $c_4 \geq c_3 > 0$ so that the following holds.

- If $p \leq (c_1 - \epsilon)n^{-2}$ then aas $\Gamma(n,p)$ is free.
- If $(c_2 + \epsilon)n^{-2} \leq p \leq (c_3 - \epsilon)n^{-2} \log n$, then aas $\Gamma(n,p)$ is **not** free but **does not** have Kazdhan's property (T).
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When is $\Gamma(n,p)$ free?

Theorem Antoniuk, Łuczak, Prytuła, Przytycki, Zaleski'15+

There exist a constant $c > 0$ so that for every $\epsilon > 0$

- If $p \leq (c-\epsilon)n^{-2}$ then aas $\Gamma(n,p)$ is free.
- If $p \geq (c+\epsilon)n^{-2}$, then aas $\Gamma(n,p)$ is not free.

Theorem Antoniuk, Łuczak, Prytuła, Przytycki, Zaleski'15+

Aas $\Gamma(n,p) = \langle S|\mathcal{R}(n,p)\rangle$ becomes not free roughly at the moment when for some $S' \subseteq S$ and $R' \subseteq \mathcal{R}(n,p)$ for each $s \in S'$ there exists at least two relators $r_1, r_2 \in R'$ which contains either s or s^{-1}.

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OR IN MORE ACCESSIBLE LANGUAGE...

DEFINITION OF AN AUXILIARY RANDOM 3-GRAPH

Let $H(n, p)$ be a 3-graph whose vertex set consists of generators and the edge $\{a, b, d\}$ appears only if the random presentation of $\Gamma(n, p)$ contains a relation of type $ad^{-1}b$.

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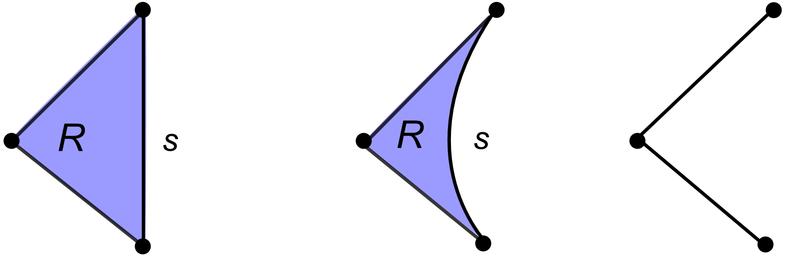
Observation

If for $\langle S' | R' \rangle$ there exists $s \in S'$ and $r \in R'$ such that

- r contains s,
- no $r' \in R' \setminus \{r\}$ contains either s or s^{-1},

then $\langle S' | R' \rangle = \langle S' \setminus \{s\} | R' \setminus \{r\} \rangle$.

Tietze moves



THE OTHER WAY IS NOT SO STRAIGHTFORWARD ...

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PROBLEM

How to show that a group is not a free group?

The other way is not so straightforward . . .

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Problem

How to show that a group is not a free group?

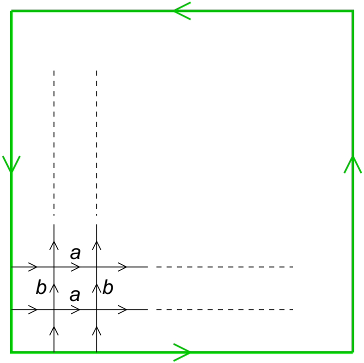
It is high time to define hyperbolicity

Definition

A finitely generated group is hyperbolic if there exists a constant c, such that for every word of length k which is equal to e, there exists a van Kampen diagram with at most ck cells which proves it.

Hyperbolic group: example

$\mathbb{Z}^2 = \langle \{a, b\} | aba^{-1}b^{-1} \rangle$ is **not** hyperbolic: van Kampen diagram showing that $a^k b^k a^{-k} b^{-k} = e$ has k^2 squares.



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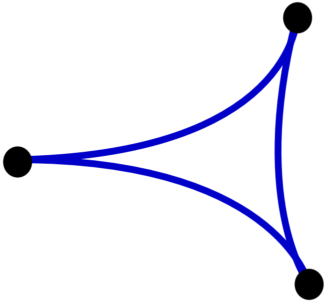

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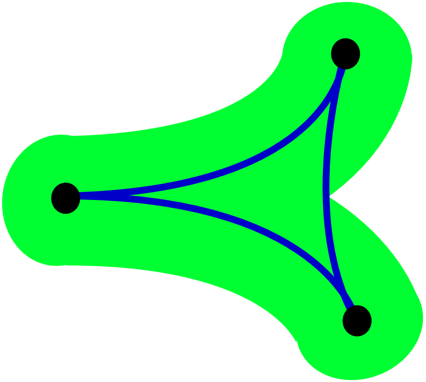

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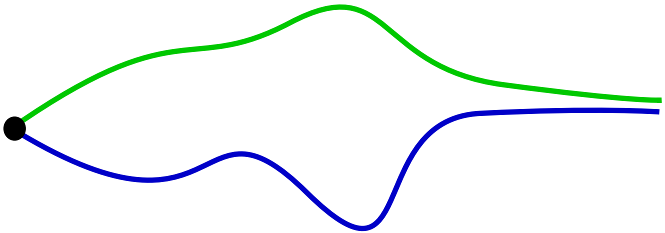

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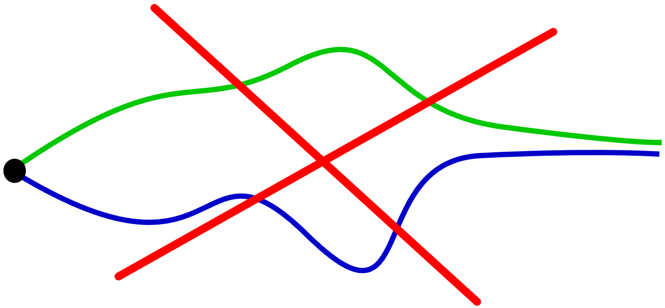

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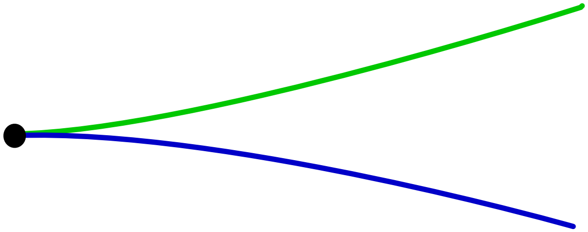

Gromov's boundary of a group

(Crude) definition

The boundary of the infinite group is, roughly speaking, the set of all infinite rays of its Cayley graph with a 'natural' topology.

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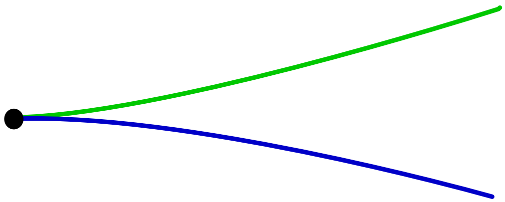

Gromov's boundary of a group

Definition

The boundary of the infinite group is, roughly speaking, the set of all infinite rays of its Cayley graph with a 'natural' topology.

Important example

The boundary of a free group has the same topology as the Cantor set.

When is a group free?

Problem

How to show that a group is not a free group?

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It is enough to show that each subgraph induced by the k-neighbourhood of a vertex in the Cayley graph is connected.

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It is enough to show that each subgraph induced by the k-neighbourhood of a vertex in the Cayley graph is connected.

Indeed, then Gromov's boundary (which is inverse limit of these graphs) is connected (and compact).
Thus the group is not free.

Kazdhan's property (T)

Theorem Hoffman, Kahle, Paquette'15+

There exists a constant C such that for every $\epsilon > 0$

- If $p \leq (c - \epsilon)n^{-2} \log n$, then aas $\Gamma(n, p)$ has not got property (T).
- If $p \geq (c + \epsilon)n^{-2} \log n$, then aas $\Gamma(n, p)$ has got property (T).

A few final remarks

In the first part of the talk I have only presented results on the right kernel of the incidence matrix whose elements belonged to $\mathbb{F}_2$.

A FEW FINAL REMARKS

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The left kernel of the incidence matrix is also interesting, and $\mathbb{F}_2$ can be replaced by any ring (but incidence matrix needs to be redefine in this case).

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The left kernel of the incidence matrix is also interesting, and $\mathbb{F}_2$ can be replaced by any ring (but incidence matrix needs to be redefine in this case).

Furthermore, there are also other models of random groups I have not mentioned.

Pseudorandom groups

There are naturally stated conditions
which makes the graph behavior ‘random-like’.
It would be nice to describe similar
conditions for some models of random groups.

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conditions for some models of random groups.

Unfortunately, I do not know
any result in this direction.

Thank you!

Never run overtime.
Running overtime is the one fatal mistake a lecturer can make.

Ten lessons I wish I had been taught
Gian-Carlo Rota

Further readings

If anyone would like to know more about random groups, there are three articles give a gentle introduction to the subject and a glimpse on some recent (and a bit older) results in this area.

T.Łuczak, Randomly generated groups. In *Survey in Combinatorics 2015*, London Math. Soc. Lecture Note Series 424, Cambridge University Press, 2015, 175-194.

M.Kahle, *Topology of random simplicial complexes: a survey.* AMS Contemporary Volumes in Mathematics 620 (2014) 201-222.

Y.Ollivier, *A January 2005 invitation to random groups*, Ensaios Matematicos [Mathematical Surveys] 10, Sociedade Brasileira de Matematica, Rio de Janeiro, 2005.
with
Y.Ollivier, *Random group update*,
http://www.yann-ollivier.org/rech/publs/rgupdates.pdf.