The Contact Process on a Graph Adapting to Infection Density

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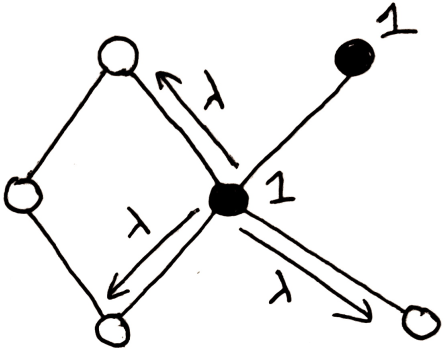
20th October 2021

Contact Process a.k.a. SIS infection

Contact process on a graph has infected vertices make each of their neighbours infected independently at rate $\lambda > 0$.

Simultaneously, they each recover independently at rate 1.



Context: Contact Process on Static Erdős-Rényi

Take $\beta > 0$. On the Erdős-Rényi graph on N vertices with every edge independently present with probability $\frac{\beta}{N}$:

Theorem (Bhamidi, Nam, Nguyen, Sly 2019)

For the contact process with initially every vertex infected $\exists \underline{\lambda}(\beta), \overline{\lambda}(\beta)$ *with* $0 < \underline{\lambda}(\beta) \leq \overline{\lambda}(\beta) < \infty$ *such that:*

- $\lambda < \underline{\lambda}(\beta) \implies$ *infection lives for time* $N^{O_{\mathbb{P}}(1)}$,
- $\lambda > \overline{\lambda}(\beta) \implies$ *infection lives for time* $e^{\Omega_{\mathbb{P}}(N)}$.

Theorem (Nam, Nguyen, Sly 2019)

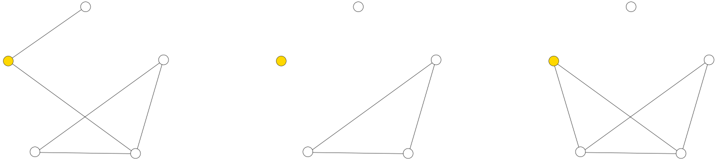
Further $\underline{\lambda}(\beta) \sim \overline{\lambda}(\beta) \sim \frac{1}{\beta}$ *as* $\beta \to \infty$.

Motivation

Stationary independent dynamics of e.g. [Mörters, Jacob 2017]:

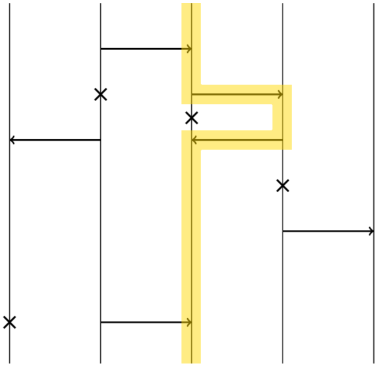
- *Vertex updates* occuring at each $i \in [N]$ at rate $\kappa > 0$, which re-randomise the neighbourhood of i.

$$\mathrm{d}(i) = d \mapsto \mathrm{d}(i) \sim \mathrm{Bin}\left(N-1, \frac{\beta}{N}\right)$$



Duality

As contact process self-duality is by time reversal, we can apply it on a dynamic graph by simply reading the graph history in reverse.



That's convenient when the dynamic is independent, stationary and reversible (w.r.t. the inhomogenous random graph measure).

Context: Contact Process on Dynamic Erdős-Rényi

Take $\beta, \kappa > 0$. On the dynamic Erdős-Rényi graph on N vertices where now every vertex *updates* at rate κ:

Theorem (Jacob, Mörters 2017)

Still from initially every vertex infected, $\exists \underline{\lambda}(\beta) > 0$ such that:

- $\lambda < \underline{\lambda}(\beta) \implies$ *infection lives for time* $N^{O_{\mathbb{P}}(1)}$.

Theorem (Mörters, Ortgiese, F.)

$\exists \overline{\lambda}(\beta)$ *with* $\overline{\lambda}(\beta) \geq \underline{\lambda}(\beta)$ *such that:*

- $\lambda > \overline{\lambda}(\beta) \implies$ *infection lives for time* $e^{\Omega_{\mathbb{P}}(N)}$.

Further $\underline{\lambda}(\beta) \sim \overline{\lambda}(\beta) \sim \frac{1}{\beta}$ *as* $\kappa \to \infty$.

Modification

A natural modification of this dynamic is to:

> only run the vertex update clock at each $i \in [N]$ when there is infection in the neighbourhood of i.

i doesn't check its own infection state at all – it just updates ("evacuates") at rate κ while adjacent to infection.

Difficulties

Such *adaptive* dynamics have been little studied as then:

- the graph is dependent on the infection so the complete graph graphical construction is not so useful;
- the contact process is not stationary (until it dies out) so nor is the graph, in fact it has the empty graph as an absorbing state;
- "more infection $\implies$ fewer edges" and thus the current infection state has a non-monotone effect on the future infection states.

Model

We consider this modified dynamic on the Erdős-Rényi IRG.

$\hookrightarrow$ Thus after every update at i we generate $\mathrm{d}(i) \sim \mathrm{Bin}\left(N-1, \frac{\beta}{N}\right)$.

$\hookrightarrow$ So we can start the graph with the Erdős-Rényi distribution, even if it's immediately lost.

Despite the previous difficulties, this model is a very natural and simplistic for selfish agents in an epidemic and so warrants investigation.
However we must modify the problem:

- We expect this model might never display metastability when $\kappa > 0$.
- No duality relations between different initial infection states anyway, so single vertex initially infected seems the practical case.

Problem

- ξ_t Contact process infection set.
- I_t Historical infection set: $\bigcup_{s=0}^t \xi_t$.
- ρ_ϵ Epidemic probability: $\mathbb{P}\left(|I_\infty| \geq \epsilon N \middle| |\xi_0| = 1\right)$.

Want to find (β, λ, κ) regions of infection subcriticality

$$\forall \epsilon > 0 : \quad \lim_{N\to\infty} \rho_\epsilon = 0$$

and supercriticality

$$\exists \epsilon > 0 : \quad \liminf_{N\to\infty} \rho_\epsilon > 0.$$

Results

Theorem (Mörters, Ortgiese, F.)

For the contact process on an adaptive dynamic Erdős-Rényi graph we find if $\beta, \lambda, \kappa \geq 0$ *satisfy*

$$\lambda\beta < 0.21 \quad \text{or} \quad \lambda(2\beta - 1) < \kappa$$

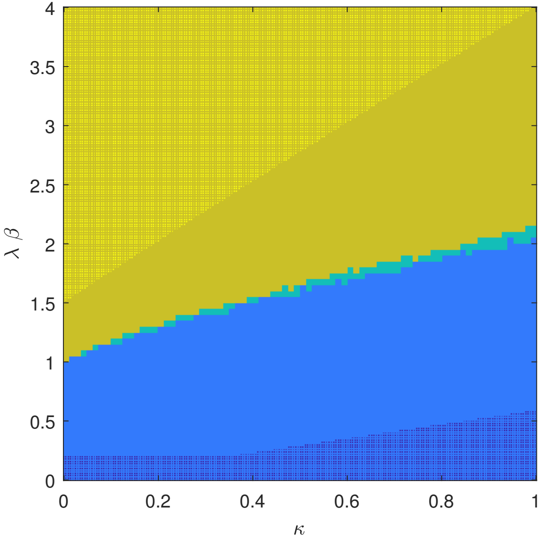
then the sampled infection sets $(|I_\infty|)_{N \in \mathbb{N}^+}$ *are tight in* $\mathbb{N}^+$ *and in particular the infection is subcritical.*

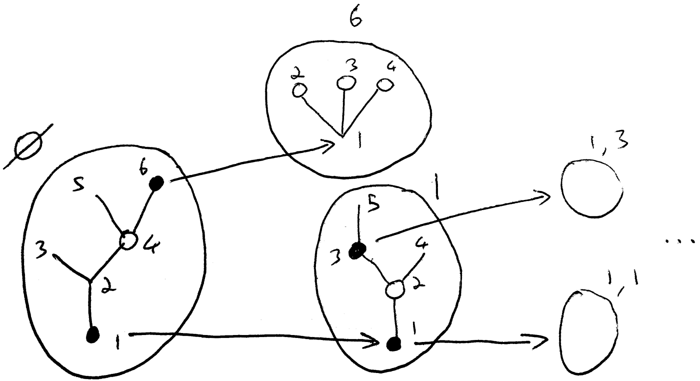
Theorem (Mörters, Ortgiese, F.)

For the contact process on an adaptive dynamic Erdős-Rényi graph we find the infection is supercritical if $\beta, \lambda, \kappa \geq 0$ *satisfy*

$$\frac{\lambda\beta}{1 + 2\kappa + \lambda} > \sqrt{1 - \frac{\kappa\lambda\left(1 - e^{-\beta}\right)}{(1+\kappa)(1 + 2\kappa + \lambda)}}.$$

Results ($\beta = 3$)





If $\beta > 1$ then the Pois(β)-GW-trees can be infinite, and if further λ is large enough for *tree survival* then the meta-GW-tree can be locally infinite.

A Galton-Watson Infection

For this local approximation model we are interested in

$$\mu := \mathbb{E}\left(\#\left\{v \in I_\infty^\emptyset : v \text{ updates while infected}\right\}\right),$$

or precisely in where $\mu < 1$ or $\mu > 1$.

Because, locally, updating is permanent recovery, we can *lower bound* μ using the SIR which accepts every update attempt.

Small κ Upper Bound

Large $\kappa \implies$ percolated local tree.
Small $\kappa \implies$ neglect the local effect of updates entirely and use the unmodified contact process.

Note $\mu \leq \mathbb{E}\left(\left|I_\infty^\emptyset\right|\right) - 1$.

Target: $\mathbb{E}\left(\left|I_\infty^\emptyset\right|\right) < 2$ for the contact process.

$\mathbb{E}\left(\left|I_\infty^\emptyset\right|\right) < 2$ for the Contact Process

Loosely inspired by [Bhamidi, Nam, Nguyen, Sly 2019] we upper bound by the subtree contact process, which adds an extra permanently infected vertex adjacent to the root and forbids recoveries that would disconnect the infection set.

i.e. after full recovery of the tree we return to the initial state at rate λ.

We do not include the extra vertex in the state and so this process is on *rooted subtrees* of the host tree, and is also reversible w.r.t. $\pi_\lambda(T) \propto \lambda^{|T|}$.

$$T \underset{1}{\overset{\lambda}{\rightleftarrows}} T \cup \{v\}$$

$\mathbb{E}\left(\left|I_\infty^\emptyset\right|\right) < 2$ for the Subtree Contact Process

For a fixed tree we introduce a partition function $\pi_\lambda(T) = \frac{\lambda^{|T|}}{Z_\lambda}$.
Kac's formula for the SCP:

$$\mathbb{E}_\emptyset\left(H_\emptyset\right) = \frac{\lambda}{\pi_\lambda(\emptyset)} = \lambda Z_\lambda$$

Consider instead the set r containing only the root:

$$\mathbb{E}_\emptyset\left(H_\emptyset\right) = \frac{1}{\lambda} + \mathbb{E}_r\left(H_\emptyset\right)$$

Thus to obtain the (annealed) expected recovery time we need $\mathbb{E}\left(Z_\lambda\right)$.

$\mathbb{E}(Z_\lambda)$

By linearity of the expectation:

$$\mathbb{E}(Z_\lambda) = 1 + \lambda + \sum_{k=2}^{\infty} \lambda^k \mathbb{E}(\#\text{ rooted subtrees of size } k).$$

Using local weak convergence, we relate the number of rooted subtrees in the GW tree to the $N \to \infty$ limit of that in the ER graph:

$$\left(\frac{\beta}{N}\right)^{k-1} \binom{N}{k-1} k^{k-2}$$

It turns out $\mathbb{E}(Z_\lambda) < \infty$ when $\lambda\beta < e^{-1}$.

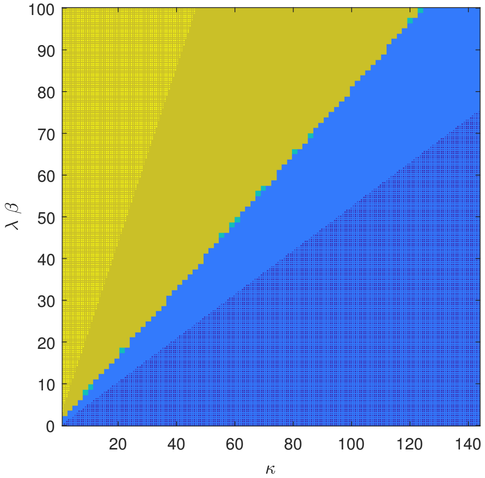
Recovery Time $\leftrightarrow$ Infection Size

For $\rho > 1$, introduce the *slowed* SCP which leaves state T a factor $\rho^{-|T|}$ of the SCP speed.

This has stationary distribution $\pi_{\lambda\rho}$ and so we understand the recovery time by Kac's formula in the same way as for the SCP with infection rate $\lambda\rho$. Hence, we now require $\lambda\beta\rho < e^{-1}$ for positive recurrence.

- At any possible state of a $\rho = 4/3$ slowed SCP, the tree has total recovery rate bounded by $\frac{243}{256} \approx 0.95$.
- Deduce expected infection size is bounded by 2 when $\boxed{\lambda\beta < 0.21}$.

Results ($\beta = 10$)



Further, simulations suggest criticality for $\kappa, \beta \to \infty$ is $\lambda\beta \sim c\kappa$ with $c \approx 0.7005$.