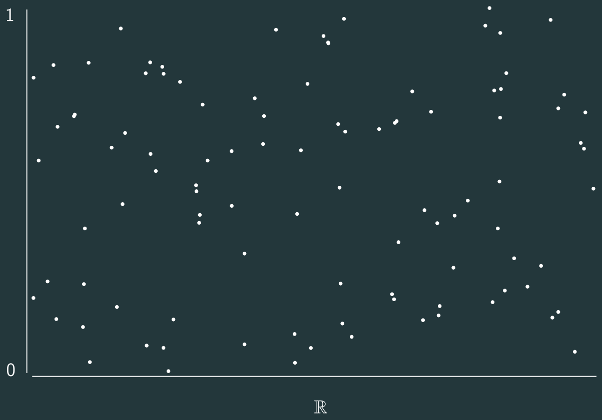
Chemical distance in geometric random graphs with long edges and scale-free degree distribution

joint work with Peter Gracar and Peter Mörters

Arne Grauer

October 19th to October 20th 2021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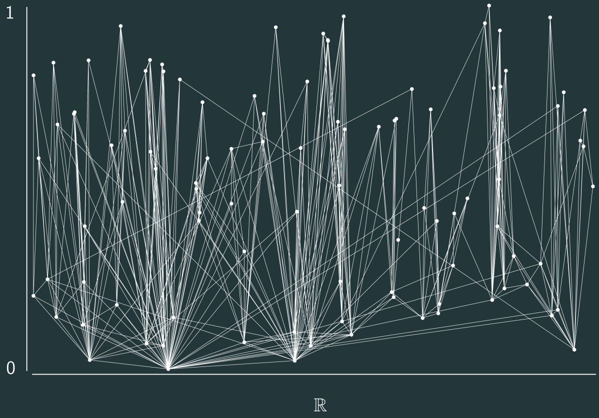

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Ultrasmallness in scale-free networks

Consider a sequence (G_N) of random graphs.

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Ultrasmallness in scale-free networks

Consider a sequence (G_N) of random graphs.

- The graph G_N has N vertices which carry independent uniform marks.
- Given the marks, vertices are connected by an edge independently with probability

$$\frac{1}{Ng(s,t)}.$$

- The kernel g depends on the marks s and t of the endvertices of the potential edge.

Ultrasmallness in scale-free networks

$g^{prod}(s,t) = s^\gamma t^\gamma$	$g^{pa}(s,t) = (s \wedge t)^\gamma (s \vee t)^{1-\gamma}$
• scale-free degree distribution with power-law exponent $\tau = 1 + \frac{1}{\gamma}$ • not ultrasmall for $\gamma < 1/2$ • ultrasmall for $\gamma > 1/2$ and $\frac{d(x,y)}{\log\log N} \to \frac{c}{\log \frac{\gamma}{1-\gamma}}$ with high probability as $N \to \infty$ for randomly chosen vertices x, y by *Dereich, Mönch, Mörters ('12)*	
$c = 2$	$c = 4$

The weight-dependent random connection model

- Vertex set: Poisson process of unit intensity on $\mathbb{R}^d \times (0,1)$.
- A vertex $\mathbf{x} = (x,t)$ has *location* $x \in \mathbb{R}^d$ and *mark* $t \in (0,1)$.

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- The *profile function* ρ non-increasing with $\rho(x) \sim cx^{-\delta}$ for chosen $\delta > 1$.
- The *kernel* g is symmetric and non-decreasing in both arguments.

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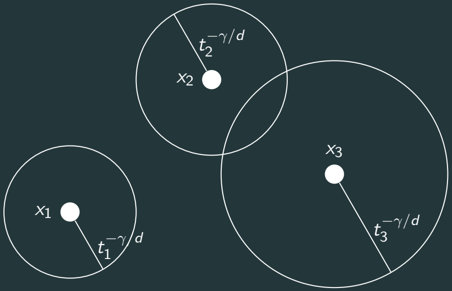

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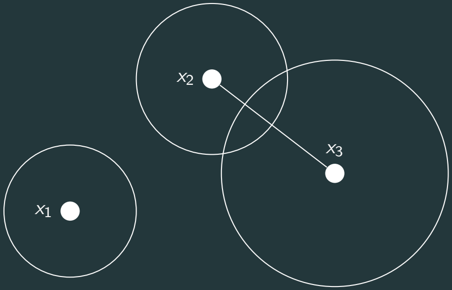

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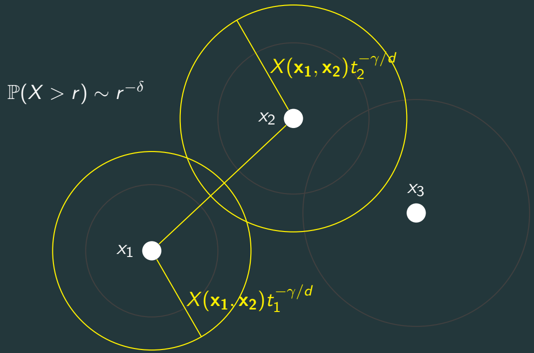

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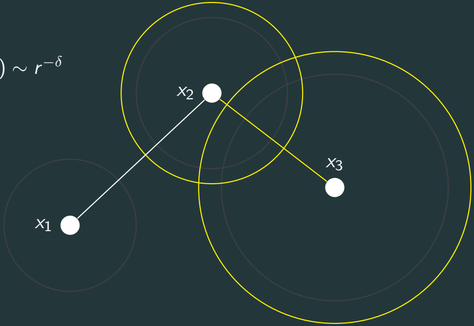

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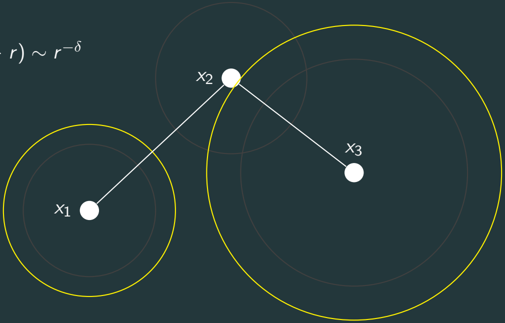

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Have scale-free degree distribution with power-law exponent $\tau = 1 + \frac{1}{\gamma}$.
For g^{prod} analogous behaviour to the non-spatial case is identified.
See *Deijfen, van der Hofstad, Hooghiemstra ('13)*, *Deprez, Wüthrich ('19)* and *Bringmann, Keusch, Lengler ('18)*.

Main Result

Theorem (Gracar, G., Mörters ('21))

Let $\mathscr{G}$ be the weight-dependent random connection model with kernel g^{pa}, g^{sum} or g^{min}.

- If $\gamma < \frac{\delta}{\delta+1}$, then $\mathscr{G}$ is **not ultrasmall**.
- If $\gamma > \frac{\delta}{\delta+1}$, then $\mathscr{G}$ is **ultrasmall** and, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d \times (0,1)$, under $\mathbb{P}_{\mathbf{x},\mathbf{y}}(\ \cdot \ \mid \mathbf{x} \leftrightarrow \mathbf{y})$ we have

$$\frac{\mathrm{d}(\mathbf{x},\mathbf{y})}{\log\log|x-y|} \to \frac{4}{\log \frac{\gamma}{\delta(1-\gamma)}}$$

  with high probability as $|x-y| \to \infty$.

Proof ideas for the lower bounds for chemical distance

- We want to establish an upper bound for $\mathbb{P}_{\mathbf{x},\mathbf{y}}\{d(\mathbf{x},\mathbf{y}) \leq 2\Delta\}$ for $\Delta \in \mathbb{N}$.

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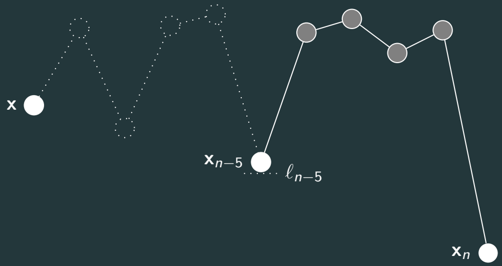
- We want to establish an upper bound for $\mathbb{P}_{\mathbf{x},\mathbf{y}}\left\{d(\mathbf{x},\mathbf{y}) \leq 2\Delta\right\}$ for $\Delta \in \mathbb{N}$.
- Let $(\ell_k)_{k=\mathbb{N}_0}$ be a decreasing sequence. A path of length n is *good* if the k-th (resp. $n-k$-th) vertex of the path has a mark larger than ℓ_k for all $k = 0, \dots, n$.

$$\begin{aligned}
&\mathbb{P}_{\mathbf{x},\mathbf{y}}\left\{d(\mathbf{x},\mathbf{y}) \leq 2\Delta\right\} \\
&\quad \leq \sum_{n=1}^{2\Delta} \mathbb{P}_{\mathbf{x},\mathbf{y}}\left\{\exists \text{ good path of length } n \text{ between } \mathbf{x} \text{ and } \mathbf{y}\right\} \\
&\qquad + \mathbb{P}_{\mathbf{x},\mathbf{y}}\left\{\exists \text{ bad path starting in } \mathbf{x}, \text{ resp. } \mathbf{y}\right\}
\end{aligned}$$

Connection between powerful vertices

- Consider paths which connect as quickly as possible to vertices with small marks, i.e powerful vertices.

Connection between powerful vertices



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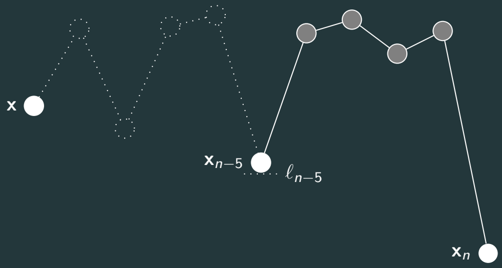

$\rightarrow$ Observe connections between two powerful vertices.

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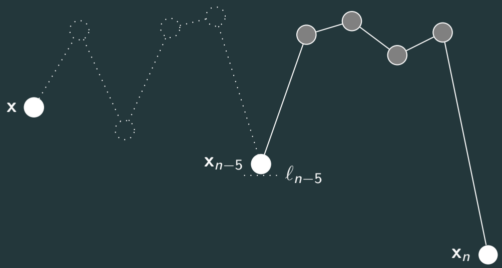

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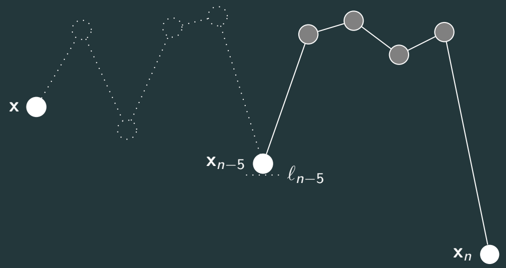

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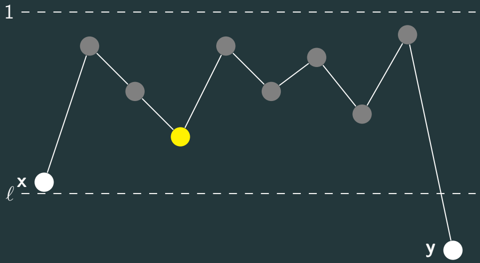
- If $\gamma > \frac{\delta}{\delta+1}$, connection of two powerful vertices via some *connector*, i.e. a vertex with large mark, is more 'probable' than direct connection.
- If $\gamma < \frac{\delta}{\delta+1}$, connection of two powerful vertices via a connector is not beneficial in comparison to a direct connection.

Are connections via multiple connectors better than via one?

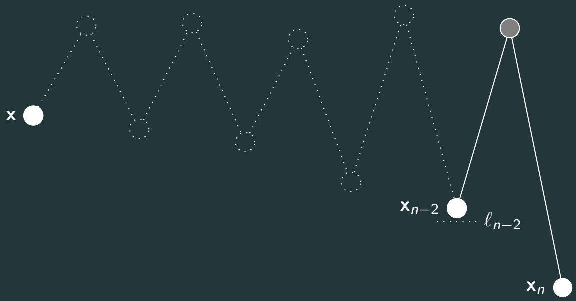


Lemma

Let **x** *and* **y** *be two given vertices and* $\left\{\mathbf{x} \overset{k}{\leftrightarrow} \mathbf{y}\right\}$ *be the event that* **x** *and* **y** *are connected via* $k-1$ *connectors. Then, there exists* $C(k,l)>0$*, depending on* k *and the truncation, such that*

$$\mathbb{P}_{\mathbf{x},\mathbf{y}}\left\{\mathbf{x} \overset{k}{\leftrightarrow} \mathbf{y}\right\} \le C(k,\ell)\rho(\frac{1}{\beta}(t\wedge s)^{\gamma}(t\vee s)^{\gamma/\delta}\left|x-y\right|^{d}).$$

The optimal path structure



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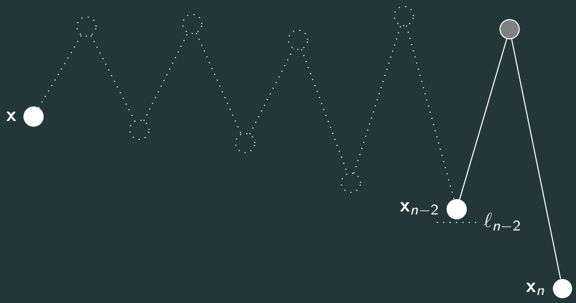

$$\mathbb{P}_{\mathbf{x}}\left\{\exists \text{ path starting in } \mathbf{x} \text{ with optimal structure fails to be good after exactly } n \text{ steps}\right\}$$

$$\leq \int_{\mathbb{R}^d\times(\ell_2,t_0]} \mathrm{d}\mathbf{z}_1 \cdots \int_{\mathbb{R}^d\times(\ell_{n-2},t_0]} \mathrm{d}\mathbf{z}_{n/2-1} \int_{\mathbb{R}^d\times(0,\ell_n]} \mathrm{d}\mathbf{z}_{n/2} \prod_{i=1}^{n/2} C\rho\big(\kappa^{-1/\delta} u_i^{\gamma} u_{i-1}^{\gamma/\delta} |z_i - z_{i-1}|^d\big),$$

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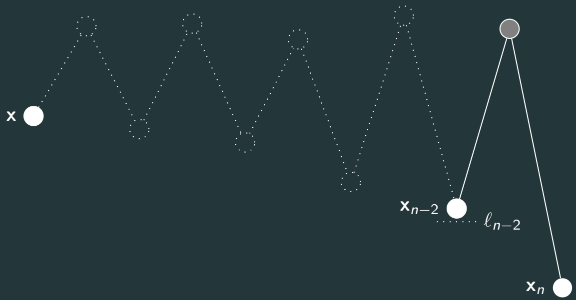

$$\Rightarrow \ell_n \approx b \exp\left(-B\left(\frac{\gamma}{\delta(1-\gamma)}\right)^{n/2}\right)$$

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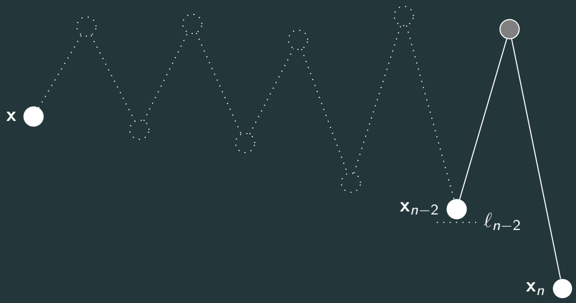

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$$\Rightarrow \mathbb{P}_{\mathbf{x},\mathbf{y}}\left\{d(\mathbf{x},\mathbf{y}) \leq 2\Delta\right\} \leq \epsilon + o(1) \text{ for } \Delta \leq \frac{2\log\log|x-y|}{\log\left(\frac{\gamma}{\delta(1-\gamma)}\right)} - c$$

General assumption for the results on lower bounds of the chemical distance

Assumption

There exists $\kappa > 0$ such that, for every set of pairs of vertices $I \subset \mathcal{X}^2$, we have

$$\mathbb{P}_{\mathcal{X}}\Big(\bigcap_{(\mathbf{x}_i, \mathbf{y}_i) \in I} \{\mathbf{x}_i \sim \mathbf{y}_i\} \Big) \le \prod_{(\mathbf{x}_i, \mathbf{y}_i) \in I} \kappa \, (t_i \wedge s_i)^{-\delta\gamma} (t_i \vee s_i)^{\delta(\gamma-1)} \, |x_i - y_i|^{-\delta d}$$

where $\mathbf{x}_i = (x_i, t_i)$, $\mathbf{y}_i = (y_i, s_i)$.

Thank you!