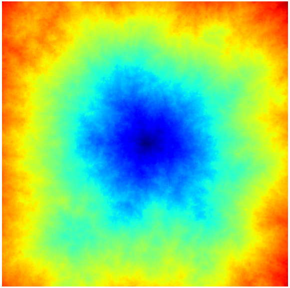
1-dependent first passage percolation

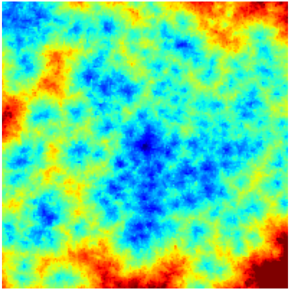
Júlia Komjáthy

joint w: John Lapinskas, Johannes Lengler, Ulysse Shaller, Zsolt Bartha, Rick Reubsaet.

Workshop on geometric random graph models and percolation

October 18, 2021

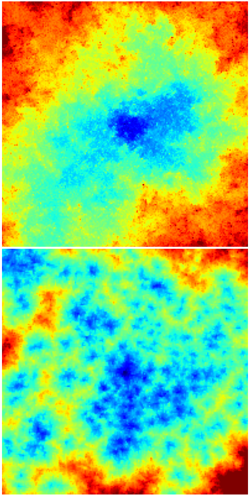




First passage percolation

FPP: (Hammersley and Welsh, 1965).

- At time $t = 0$ the source node is infected, all other nodes are susceptible.
- if, on an edge $\{u, v\}$, u is infected and v is not, then v becomes infected after a random transmission delay $\sigma_{(u,v)}$.



The epidemic curve⋆

The set of infected nodes before time t:

$$\mathcal{I}(t) = \{ \textit{ infected nodes before time } t\}$$

and

$$I(t) := |\mathcal{I}(t)|$$

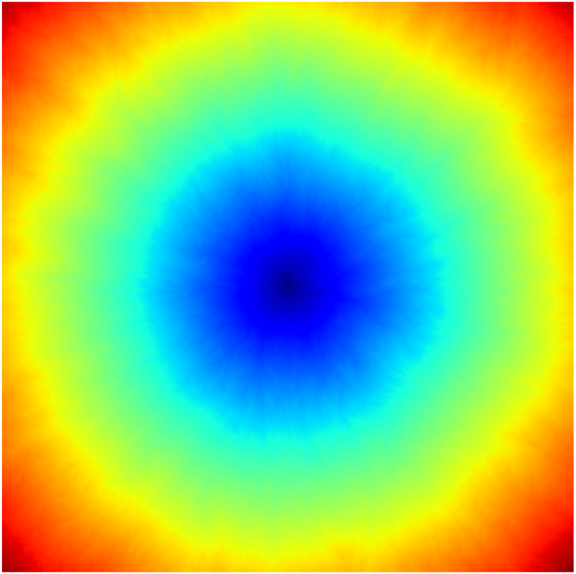
⋆: The first phase of the epidemic, before herd immunity/saturation is reached.

Question:

What shapes of the epidemic curves are possible?

On the lattice

FPP on lattice-like graphs:
$I(t) = \Theta(t^d)$



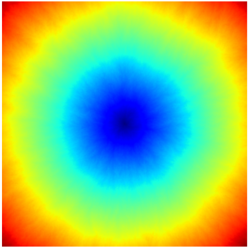
FPP on $\mathbb{Z}^d$

Shape theorem; Cox Durrett, 1981

When $\sigma_{(u,v)}$ is iid, $\mathbb{P}(\sigma = 0) < p_c(\mathbb{Z}^d)$ and σ has sufficiently high moments:

$$\frac{I(t)}{t} \to \mathcal{B}$$

for some compact set $\mathcal{B}$.
$\mathcal{B}$ depends on the distribution of τ.



Interesting results & questions

- the limiting shape (convex, differentiable boundary, etc)
- geodesics, their deviation from straight line
- 50 years of FPP (Auffinger, Damron, Hanson '16)

Long-range FPP

Model by Sh. Chatterjee and Dey

- edge set is $\mathbb{Z}^d \times \mathbb{Z}^d$,
- transmission time: $\sigma_{(u,v)} \overset{d}{=} \text{Exp}(1) \cdot \|u - v\|^{\alpha d}$, for some $\alpha > 0$.
- small α: quick transmission to far away

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Growth of $\mathcal{I}(t)$ (Chatterjee and Dey, '16)

$\alpha < 1$	$\alpha = 1$	$\alpha \in (1, 2)$	$\alpha \in (2, 2 + \frac{1}{d})$	$\alpha > 2 + \frac{1}{d}$
$\mathcal{I}(t) = \mathbb{Z}^d \ \forall t > 0$	$I(t) = e^{\Theta(t)}$	$I(t) = e^{\Theta(t^{\Delta})}$	$I(t) = t^{\zeta + o(1)}$	$I(t) = \Theta(t^d)$
instantaneous	exponential*	stretched exp.	polynomial	lattice-like

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$\mathcal{I}(t) = \mathbb{Z}^d \ \forall t > 0$ instantaneous	$I(t) = e^{\Theta(t)}$ exponential*	$I(t) = e^{\Theta(t^{\Delta})}$ stretched exp.	$I(t) = t^{\zeta + o(1)}$ polynomial	$I(t) = \Theta(t^d)$ lattice-like

Comments:

*: slowly varying correction terms are added/needed in the transmission delay.

$\Delta = \log 2 / \log(2d/\alpha) \in (0,1)$, similar to long range percolation (Biskup '04)

$\zeta = (\alpha - 2)d$

Long-range FPP

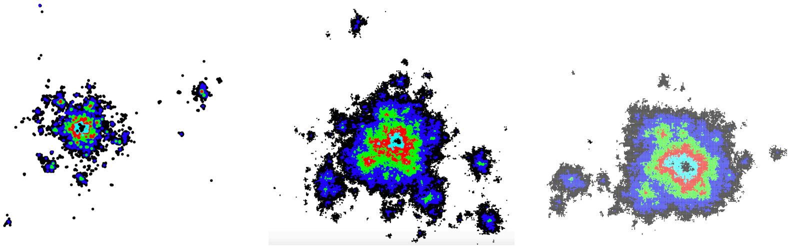

Figure: Long-range FPP, in $d = 2$, $\alpha = 1.75$ (left), 2 (middle) and 2.5 (right) by Chatterjee and Dey.

Interpolation between lattice and complete graph

original FPP

nearest neighbor graph of $\mathbb{Z}^d$

Long-range FPP

Complete graph on the vertex set $\mathbb{Z}^d$

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Geometric inhomogeneous random graph

Changing the vertex set to a Poisson PP on $\mathbb{R}^d$
Trimming edges in 'complete graph' inhomogeneously

Infinite Geometric Inhomogeneous Random Graphs

Ingredient 1:
Poisson point process
for the location
of vertices

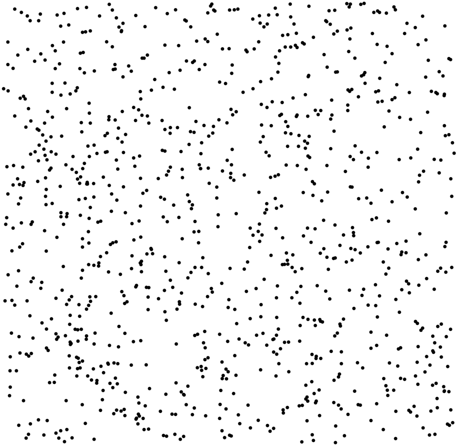
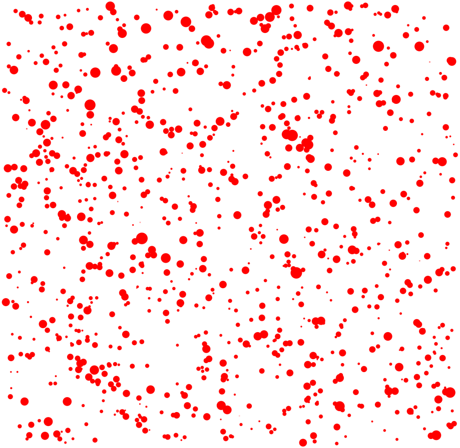


Figure: GIRG simulation by Joost Jorritsma

Infinite Geometric Inhomogeneous Random Graphs

Ingredient 2:
i.i.d. fitnesses for vertices.

(e.g.) fat tailed,
$\mathbb{P}(W > x) \asymp 1/x^{\tau-1}$



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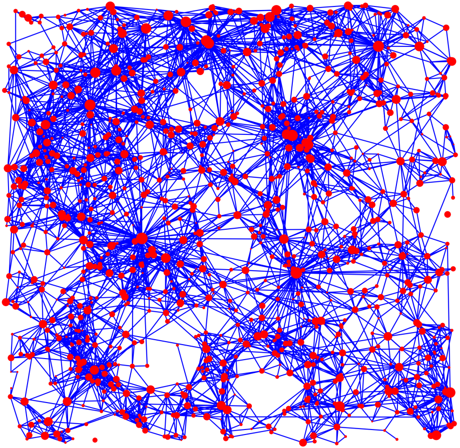
Ingredient 3:
random edges

probability
increasing with fitness,
decaying with distance.



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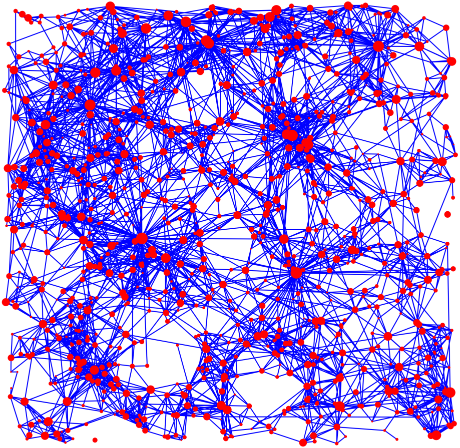
$$p(u,v) = \Theta\Big(\min\Big\{1, \Big(\frac{W_u W_v}{\|u-v\|^d}\Big)^{\alpha}\Big\}\Big),$$



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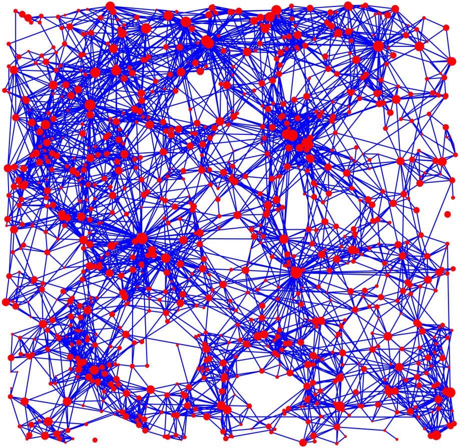
Threshold case:

$$p(u,v) = \mathbb{1}\{\|u-v\|^d \le \Theta(W_u W_v)\}.$$



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Some properties of Infinite GIRGs

Theorem (DHH'13)

If $\alpha \leq 1$ or $\tau < 2$, each vertex has infinite degree.
(NOT locally finite)

Theorem (BKL'17, BKL'16)

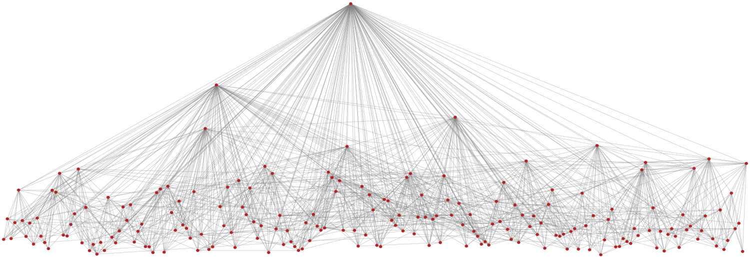
Let $\alpha > 1$, $\tau > 2$: model locally finite and:

Fitness distribution W power law with $\tau > 2 \Rightarrow$
degree distribution power law with $\tau > 2$.

iid FPP on GIRGs

Transmission delays $\sigma_{(u,v)} \overset{d}{=} \text{Exp}(1)$ on existing edges

α \ Fitnesses	**fat-tailed**	**light-tailed**
weak decay		
strong decay		
	K-Lodewijks '20	open* (Chatterjee-Dey)



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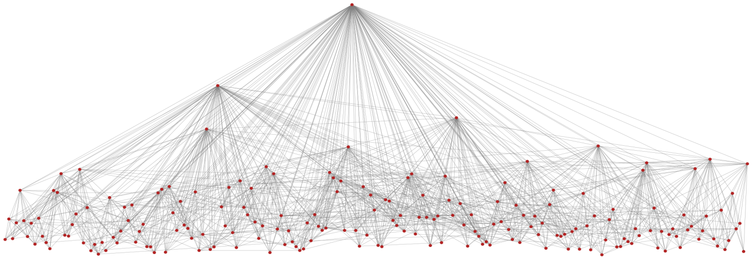

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strong decay $\alpha \in [2,\infty]$		linear
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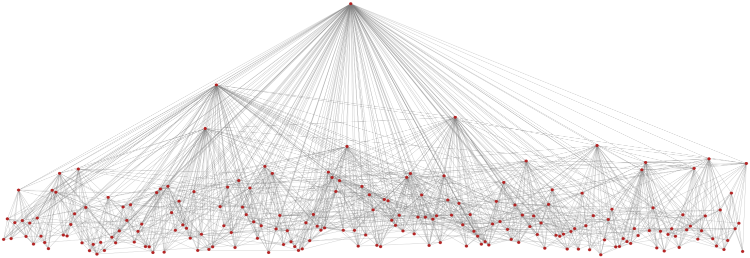

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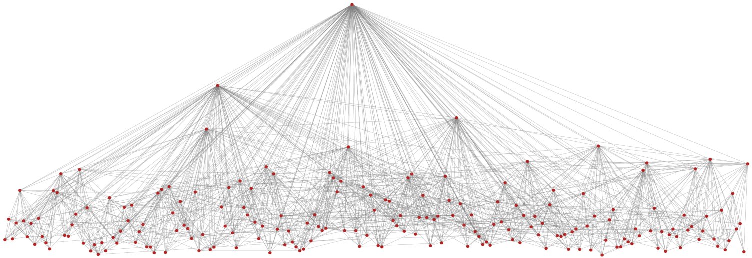
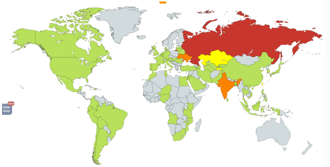


What is explosion here?

NOT instantaneous.

On infinite networks

A spreading process is explosive on an infinite, locally finite network if $I(t) = \infty$ for some $t < \infty$.



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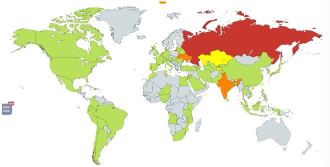

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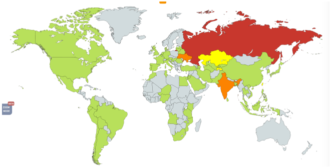

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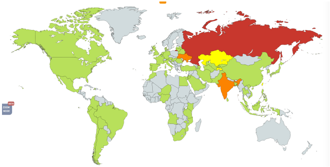

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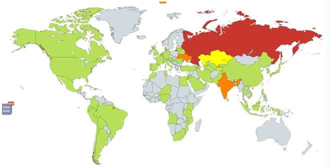

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- 2017+: Me: explosion on networks

1-dependent FPP

Observation

Disease spreading, real-world communication: Large-degree nodes have a limited “time-budget” to meet and infect.

Miritello *et. al.* ‘13, Feldman Janssen ‘17, Giuraniuc *et al.* ‘16, Karsai *et. al.* ‘11

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1-FPP:

- Transmission delay through an edge:

$$\sigma_{(u,v)} \stackrel{d}{=} \mathrm{Exp}(1) \cdot f(W_u, W_v, \|u - v\|)$$

- Rate: $f(W_u, W_v, \|u - v\|)$ depends on the spatial distance and fitnesses
- (Our result is more general, $\mathrm{Exp}(1)$ can be replaced).

Result: Explosion with degree-penalties

Is explosion still possible with these penalty factors?

Theorem (K-Lapinskas-Lengler (2021), Bartha-K-Reubsaet)

Take 1*-FPP on infinite GIRG, with*

$$\sigma_{(u,v)} := Exp(1) \cdot \text{poly}(W_u, W_v, \|u - v\|).$$

Explosive *if and only if* $\deg_d(f) < (3 - \tau)$

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- *Explosion not possible when* $\tau > 3$*.*

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- *Explosion not possible when $\tau > 3$.*
- $\tau \in (2,3)$*:*

define for monomials $g = W_u^{\mu} \cdot W_v^{\nu} \cdot \|u-v\|^{\zeta}$,

$$\deg_d(g) = \mu + \nu + \zeta \cdot \frac{2}{d}.$$

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$$\mathrm{deg}_d(g) = \mu + \nu + \zeta \cdot \frac{2}{d}.$$

Explosive *if and only if* $\mathrm{deg}_d(f) < (3 - \tau)/\beta$

Generally, $Exp(1) \to L$ arbitrary nonnegative distribution: $3 - \tau$ is replaced by $(3 - \tau)/\beta$ when $\mathbb{P}(L \le t) \asymp t^{\beta}$ close to 0.

Current and future work

Growth of $I(t)$ with degree penalties

Penalty & α \ Fitnesses	**fat-tailed** $\tau \in (2, 3)$
small	explosive
medium	
high	
very high	

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Current and future work

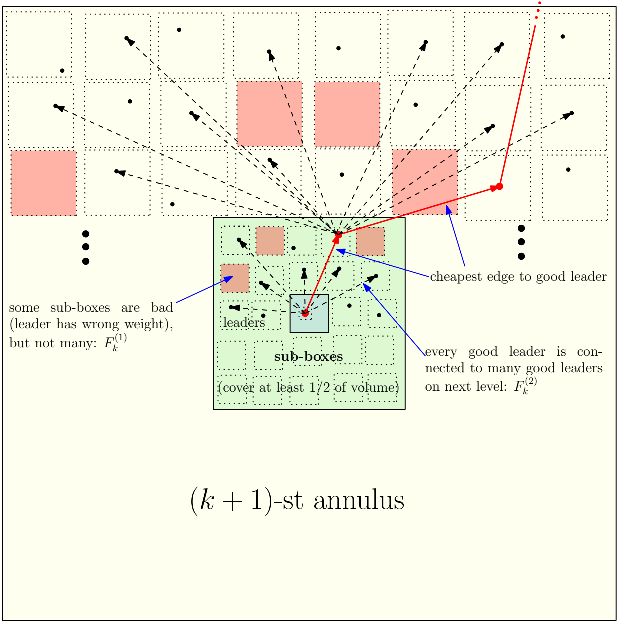
Growth of $I(t)$ with degree penalties

Penalty & α \ Fitnesses	**fat-tailed** $\tau \in (2,3)$
small $\deg_d(f) < (3-\tau)$	explosive
medium $\deg_d(f) < 2(3-\tau)$ or $\alpha \in (1,2)$	stretched exponential
high $\deg_d(f) < \frac{2}{d} + 2(3-\tau) \vee 2\frac{\alpha-\tau+1}{d(\alpha-2)}$ and $\alpha > 2$	polynomial (faster than grid-like)
very high $\deg_d(f) > \frac{2}{d} + 2(3-\tau) \vee 2\frac{\alpha-\tau+1}{d(\alpha-2)}$ and $\alpha > 2$	linear (grid-like)

Proof ideas

Proof of explosion when $\deg_d(\mathrm{f}) < (3 - \tau)$

Construction of a greedy path with finite total length



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- Let $M, A, B > 1$, $\text{Annulus}(\mathrm{k})_{k \geq 1}$ be consecutive annuli of volume

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- 'Leader' of a subbox := maximal weight vertex inside it

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- $\#\{\text{leader neighbors in Annulus}(k+1) \text{ of a leader}(k)\}$

$$\text{LeaderDeg}(k) = cM^{(A-1)B^{k+1}(1-\varepsilon)}$$

with summable error probability as long as $\frac{1-\delta}{\tau-1}(1+B) \geq AB$.

Construction of a greedy path with finite total length

Greedy path

- Assume $0 \in \mathcal{C}_\infty$
- From 0, follow a path to $\text{leader}(0)$ (its length is some finite random variable)
- Take the edge *with minimal* Exp_e between $\text{leader}(0)$ and its $\text{leader}(1)$ neighbors.
- continue with this rule

Cost of the greedy path

$$\text{Cost of } \pi_{\text{greedy}} \le \text{Cost to go to leader of Annulus(0)}$$
$$+ \sum_{k=0}^{\infty} W^{\mu}_{\text{leader}(k)} W^{\nu}_{\text{leader}(k+1)} M^{AB^{k+1}\zeta/d} \cdot \min_{j \le LeaderDeg(k)} \text{Exp}_{kj}$$

Estimate the minimum, and plug everything in, we need that the sum is finite:

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Estimate the minimum, and plug everything in, we need that the sum is finite:

$$\sum_{k=0}^{\infty} M^{B^k\left((\mu+\nu B)\frac{1+\delta}{\tau-1}+\zeta B/d-(A-1)B(1-\varepsilon)\right)} < \infty$$

Cost of the greedy path

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Path is present:

$$\frac{1-\delta}{\tau-1}(1+B) \geq AB$$

Finite-cost:

$$(\mu+\nu B)\frac{1+\delta}{\tau-1} + B\zeta/d - (A-1)B(1-\varepsilon) < 0$$

This system of inequalities have a solution for $A, B > 1$ and $\varepsilon, \delta > 0$ if $\tau \in (1,3)$ and

$$\mu + \nu + 2\zeta/d < 3 - \tau.$$

Greedy path has finite cost. $\square$

Proof of non-explosion when $\deg_d(f) > 3 - \tau$

Understanding explosion to show non-explosion

Explosion time: $Y(v) = \inf_t\{I(t) = \infty\}$.

Lemma (1: Excluding sideways explosion)

Sideways explosion cannot happen when for all $T > 0$,

$$\mathrm{N}(v, \leq T) = \#\{u : (u, v) \in E(G), \sigma_{(u,v)} \leq T\} < \infty \ a.s.$$

Lemma (2: Explosion can happen arbitrarily fast)

For some $t > 0$, $I(t) = \infty \Rightarrow$ for all $t > 0$, $\mathbb{P}(Y(v) < t) > c_t > 0$.

Statement is known for Branching Processes, but nontrivial for spatial random graphs

Corollary (Corollary to Lemmas 1 & 2)

Explosion happens $\Rightarrow \forall t > 0; \mathbb{P}(\exists$ infinite path $\pi : \|\pi\|_\sigma < t) > 0$

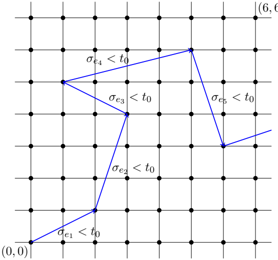
Restricted path counting to show non-explosion

Explosion $\Rightarrow \forall t > 0; \mathbb{P}(\exists \text{ infinite path } \pi : \|\pi\|_\sigma < t) > 0$.
For conservativeness, the opposite statement:

$$\exists t_0 > 0; \mathbb{P}(\exists \text{ infinite path } \pi : \|\pi\|_\sigma < t_0) = 0.$$

$$\Leftarrow \mathbb{P}(\exists \text{ infinite path } \pi, \forall e \in \pi : \sigma_e < t_0) = 0.$$

Idea to show this:



Restricted path counting to show non-explosion

Explosion $\Rightarrow \forall t > 0; \mathbb{P}(\exists \text{ infinite path } \pi : \|\pi\|_\sigma < t) > 0$.
For conservativeness, the opposite statement:

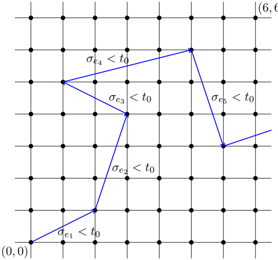
$$\exists t_0 > 0; \mathbb{P}(\exists \text{ infinite path } \pi : \|\pi\|_\sigma < t_0) = 0.$$

$$\Leftarrow \mathbb{P}(\exists \text{ infinite path } \pi, \forall e \in \pi : \sigma_e < t_0) = 0.$$

Idea to show this:

Lemma (Restricted Path counting)

$\mathbb{E}[\#\{$ *self-avoiding paths on k edges, with all* $\sigma_e < t_0\}]$*,*
is exponentially decaying in k.



Restricted path counting to show non-explosion

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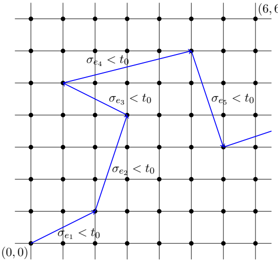
Idea to show this:

Lemma (Restricted Path counting)

$\mathbb{E}[\#\{$ self-avoiding paths on k edges, with all $\sigma_e < t_0\}]$, is exponentially decaying in k.

$A_k := \{\exists$ a length-k self-avoiding path with all $\sigma_e < t_0\}$.
Markov's inequality + Borel-Cantelli lemma:
a.s. only finitely many A_ks occur.
i.e., no such infinite path, hence no explosion.



Non-explosive regimes

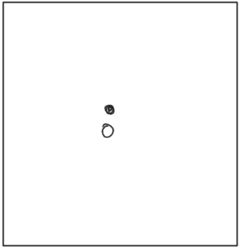
Stretched exponential and polynomial growth

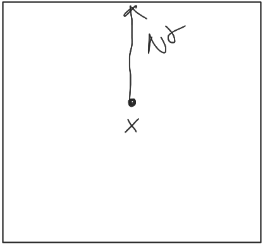
See jamboard.

- Upper bounds: Constructing bridges (ala Kleinberg or ala Biskup)
- Lower bounds: Robust renormalisation techniques (ala Berger)

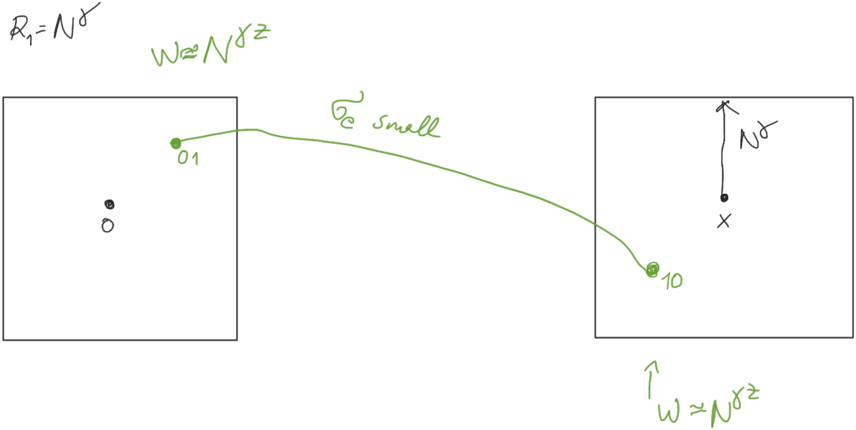
Stretched exponential and polynomial regime

$R_1 = N^\gamma$

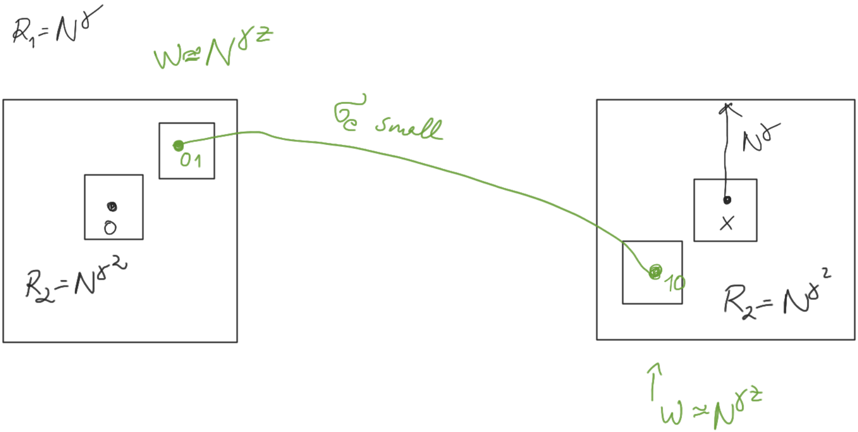




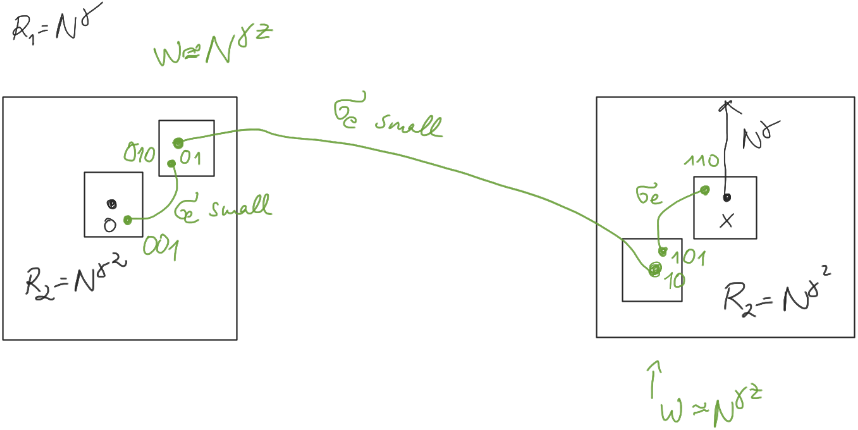
Stretched exponential and polynomial regime



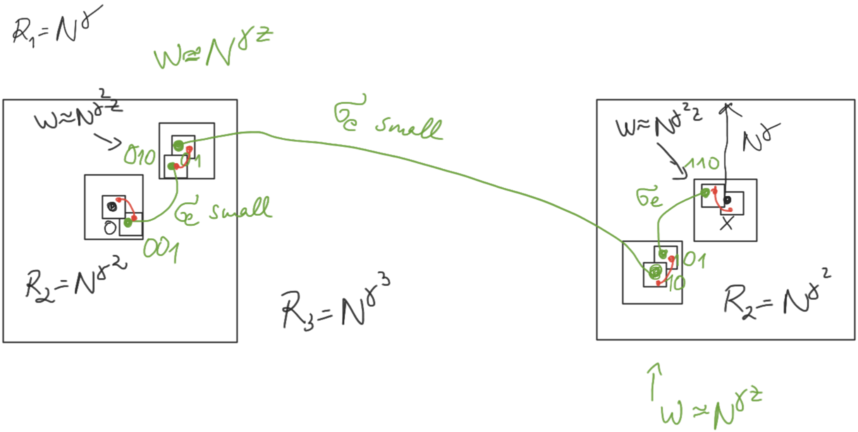
Stretched exponential and polynomial regime



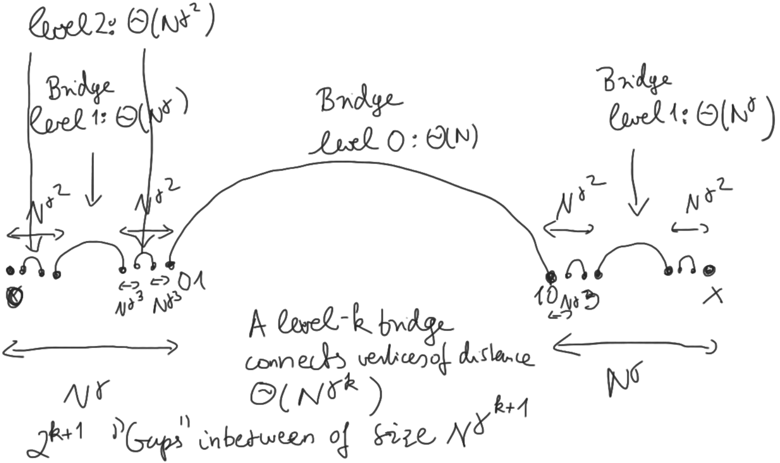
Stretched exponential and polynomial regime



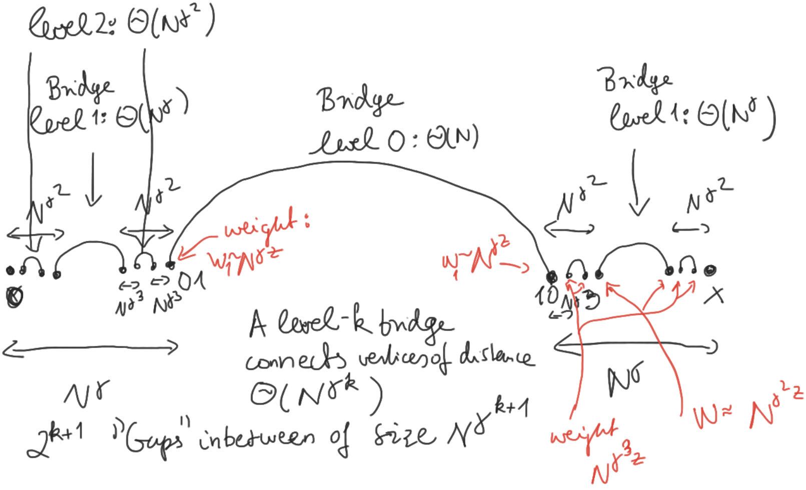
Stretched exponential and polynomial regime



Stretched exponential and polynomial regime



Stretched exponential and polynomial regime



Thank you for the attention!

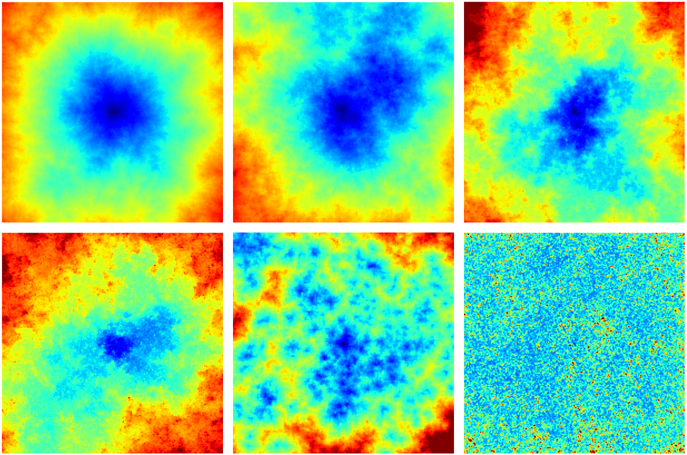


Figure: Six instances of an infection spreading on a two-dimensional SSNM with different parameters τ and α.