The giant component after percolation of product graphs

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Probabilistic preliminaries

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Often observed: for $(G_n)_{n\geq 1}$ there is a critical function p_c such that:

- if $p \geq (1+\varepsilon)p_c$, $(G_n)_p$ contains a unique component of size $\Theta(|G_n|)$ whp.
- if $p \leq (1-\varepsilon)p_c$, the largest component of $(G_n)_p$ is of size $o(|G_n|)$ whp.

Combinatorial preliminaries

Cartesian product of G_1 and G_2: graph with vertex set $(u, v)_{u \in V_1, v \in V_2}$ and edge set $(u, v)(u', v')$ where $u = u', vv' \in E_2$ or $uu' \in E_1, v = v'$.

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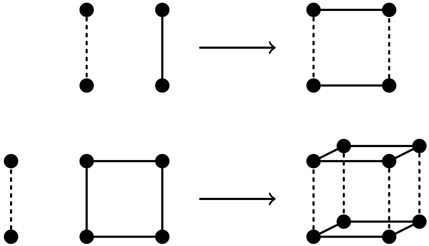

$$\mathbb{Z}\square\mathbb{Z} \longrightarrow \mathbb{Z}^2$$

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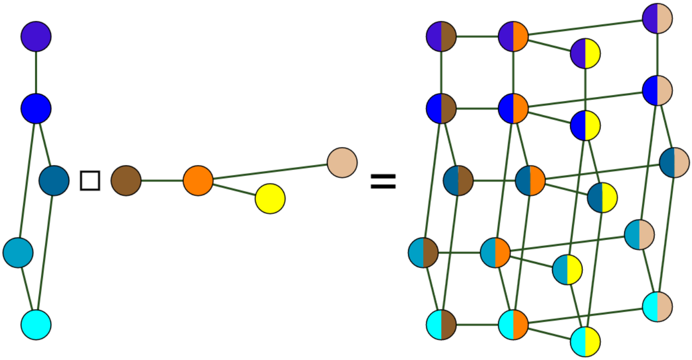

Figure: Another cool example from Wikipedia

Combinatorial preliminaries

The *isoperimetric constant* of G is given by

$$\iota(G) = \min_{S \subseteq V, |S| \le |V|/2} \frac{|E_G(S, V \setminus S)|}{|S|}.$$

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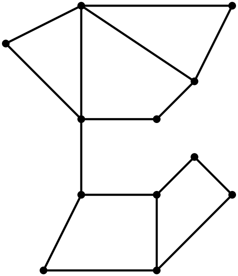

G

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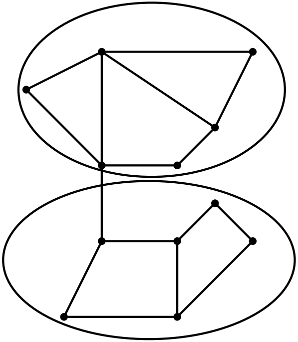

$\iota(G) = 1/6$

The giant component after percolation of product graphs

The main theorem

Theorem (L., '21)

Let $G_1, G_2, \dots, G_n$ be connected graphs with

- *at least two vertices each,*
- *maximum degree at most $C \in \mathbb{N}$,*
- *isoperimetric constants at least $n^{-\gamma}$.*

Then $G = G_1 \square \dots \square G_n$ with average degree $\overline{d} = \overline{d}(n)$ whp satisfies:

1. *if $p \le (1-\varepsilon)/\overline{d}$, then the largest component in G_p is of size $o(|G|)$.*
2. *if $p \ge (1+\varepsilon)/\overline{d}$ then G_p contains a component of size $\Theta(|G|)$.*

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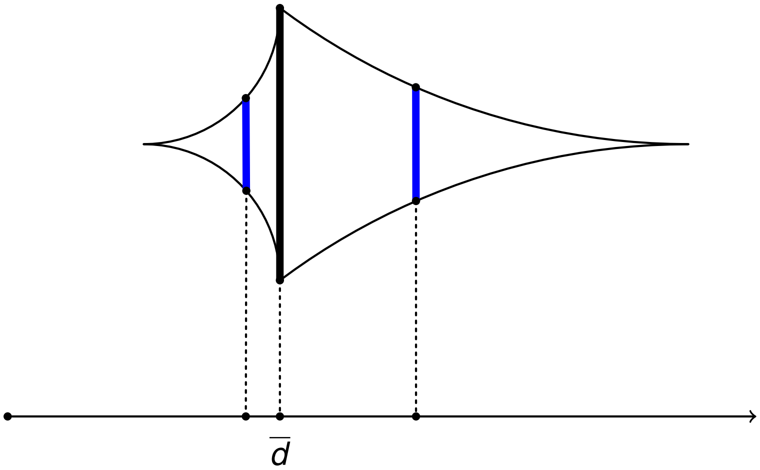
The subcritical regime: first step

Standard concentration inequalities give the following degree profile of G:



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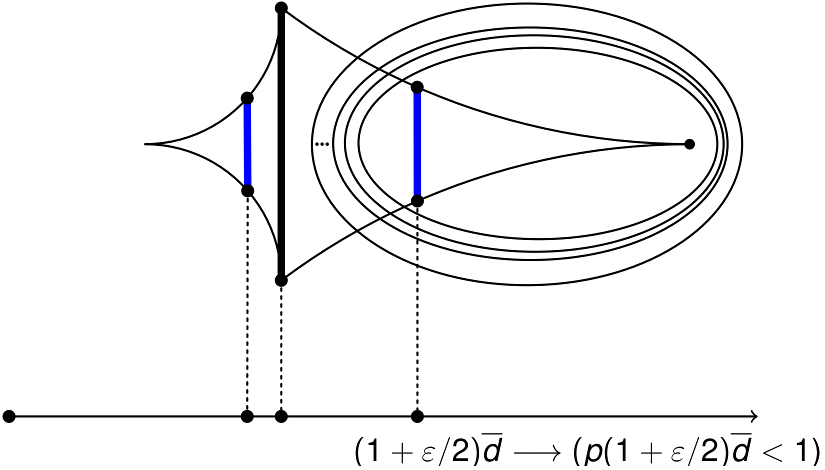

The subcritical regime: second step

Conclusion: the union of all connected components of G_p, containing a vertex of degree at least $(1+\varepsilon/2)\overline{d}$, contains $o(|G|)$ vertices.

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Remainder: vertices of degree at most $(1+\varepsilon/2)\overline{d}$.

Comparison with a subcritical Bienaymé-Galton-Watson process $\Rightarrow$ whp all other components have size $O(\log|G|)$.

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So $1 - p = (1 - p_1)(1 - p_2)$.

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- Almost all vertices of G are adjacent to $\Omega(n)$ disjoint cells of G_{p_2} of size $\Omega(n)$ (property P_{p_2}) whp.

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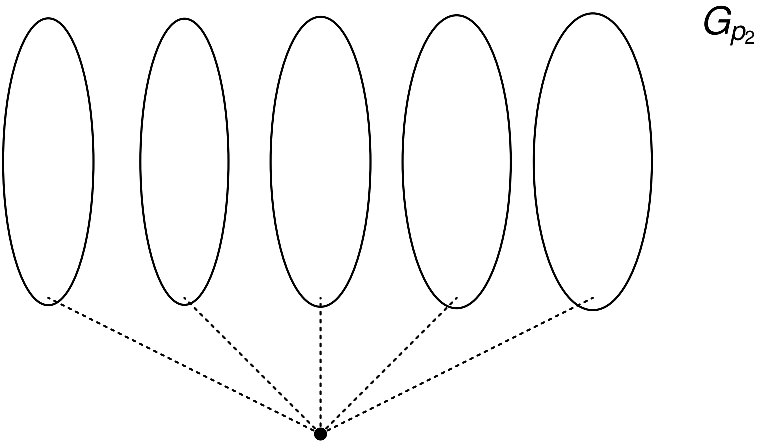

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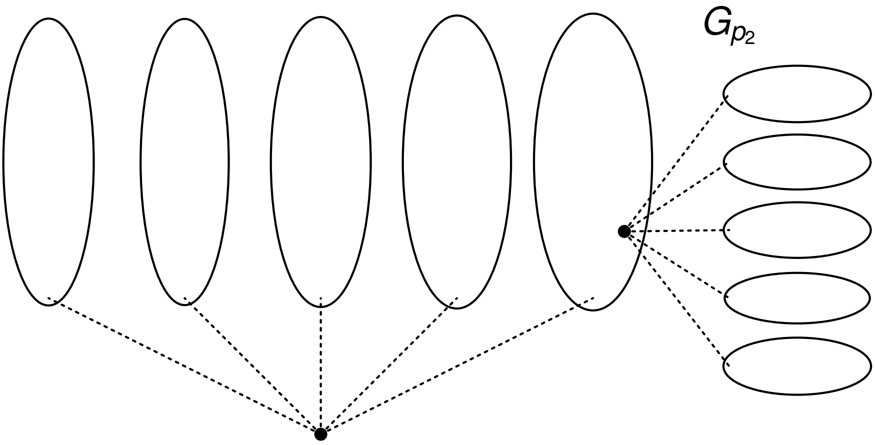

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- Almost all vertices of G are adjacent to $\Omega(n)$ disjoint cells of G_{p_2}, containing at least $\Omega(n)$ vertices with the property P_{p_2} whp.
- Almost all vertices of G are adjacent to $\Omega(n)$ vertices in connected components of G_{p_1} (not necessarily distinct) of size $\Omega(n^2)$ whp.

The supercritical regime: second step

An iteration procedure increases the sizes of the cells, adjacent in G to a typical vertex, to n^k for any $k \in \mathbb{N}$.

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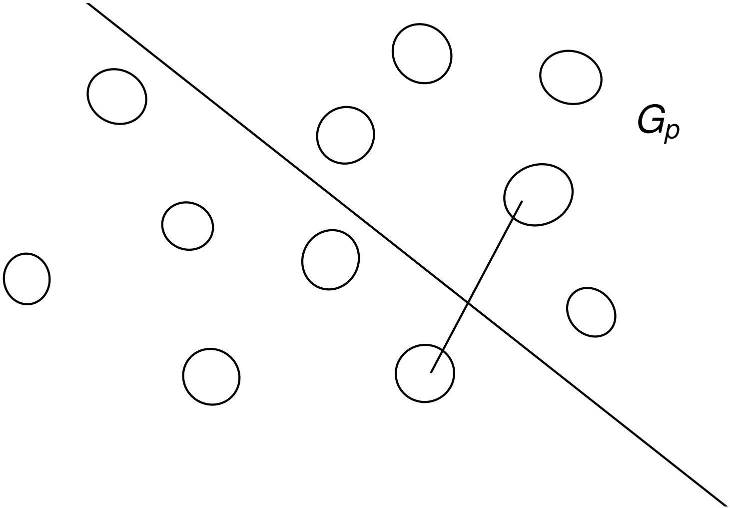
Result: a bunch of connected components of size $\Omega(n^k)$ in G_{p_0} ($p_0 = (1 + \varepsilon/2)/\overline{d}$).

One final picture ($p = (1+\varepsilon)/\overline{d}$)



Thank you for your attention!

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