Percolation phase transition in weight-dependent random connection models

Joint work with Peter Gracar and Peter Mörters

Lukas Lüchtrath, 20.10.2021

Percolation

Let $\mathscr{G}(\beta)$ be a random graph, defined on a Poisson process on $\mathbb{R}^d$ such that:

- each vertex has finite degree
- $\beta > 0$ controls the edge density, i.e. the larger β the more edges on average

Percolation is the event that $\mathscr{G}(\beta)$ contains an infinite connected component

Question: Is there a critical edge density $\beta_c \in (0,\infty)$ such that almost surely

- if $\beta < \beta_c$, the graph does not percolate but
- if $\beta > \beta_c$, the graph percolates

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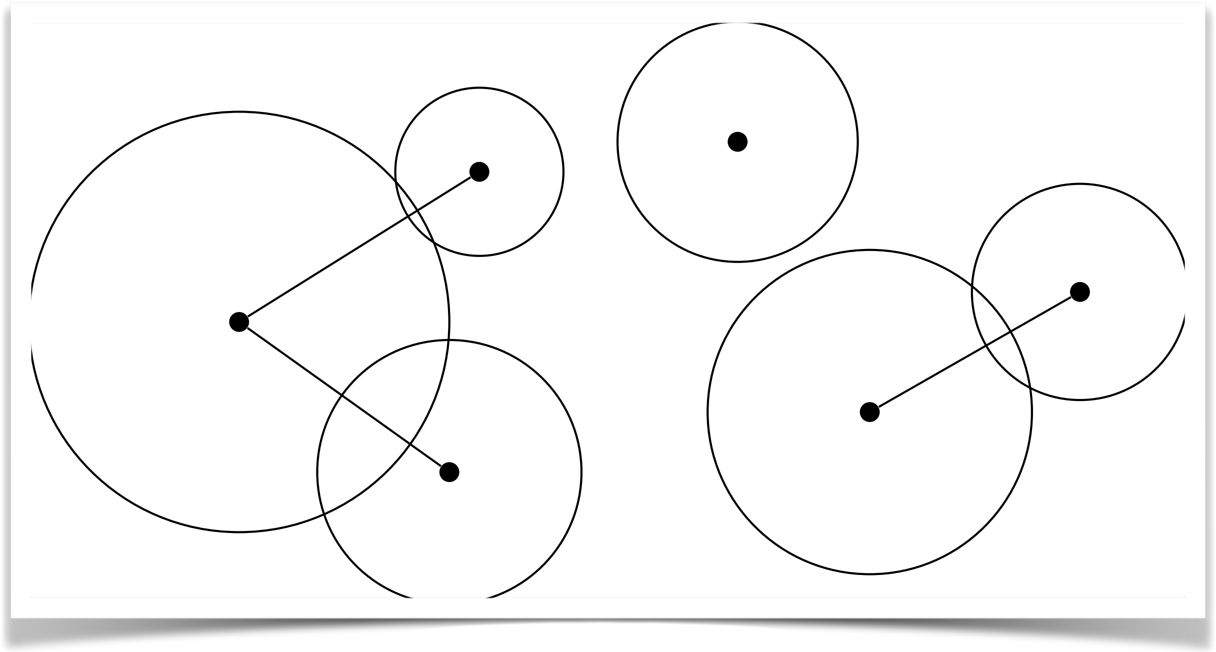

Each vertex x has assigned a radius βR_x and connect two vertices if their corresponding balls intersect

Heavy-tailed R_x lead to heavy-tailed degree distribution

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  Two vertices are connected with probability $1 - \exp(-\beta |x - y|^{-d\delta})$ for $\delta > 1$

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  Each vertex x is assigned a heavy-tailed weight W_x and two vertices are connected with probability $1 - \exp(-\beta W_x W_y |x - y|^{-d\delta})$ for $\delta > 1$. Heavy-tailed degree distribution with power-law exponent τ.

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Theorem: If $\tau > 3$, then $\beta_c > 0$, but if $\tau < 3$, then $\beta_c = 0$

Weight-dependent random connection model

- The vertex set is a Poisson point process on $\mathbb{R}^d \times (0,1)$
- Connect two vertices (x, t) and (y, t) (independently) with probability

$$\rho(\beta^{-1} g(s, t) \, |x - y|^d)$$

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- Assume $\int \rho(|x|^d)dx = 1$ since then the degree distribution only depends on the kernel g and β

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  - Describe the influence of the kernel g on the connection probability via a parameter $\gamma \in [0,1)$. All kernels lead to power-law degree distributions with exponent $\tau = 1 + 1/\gamma$.

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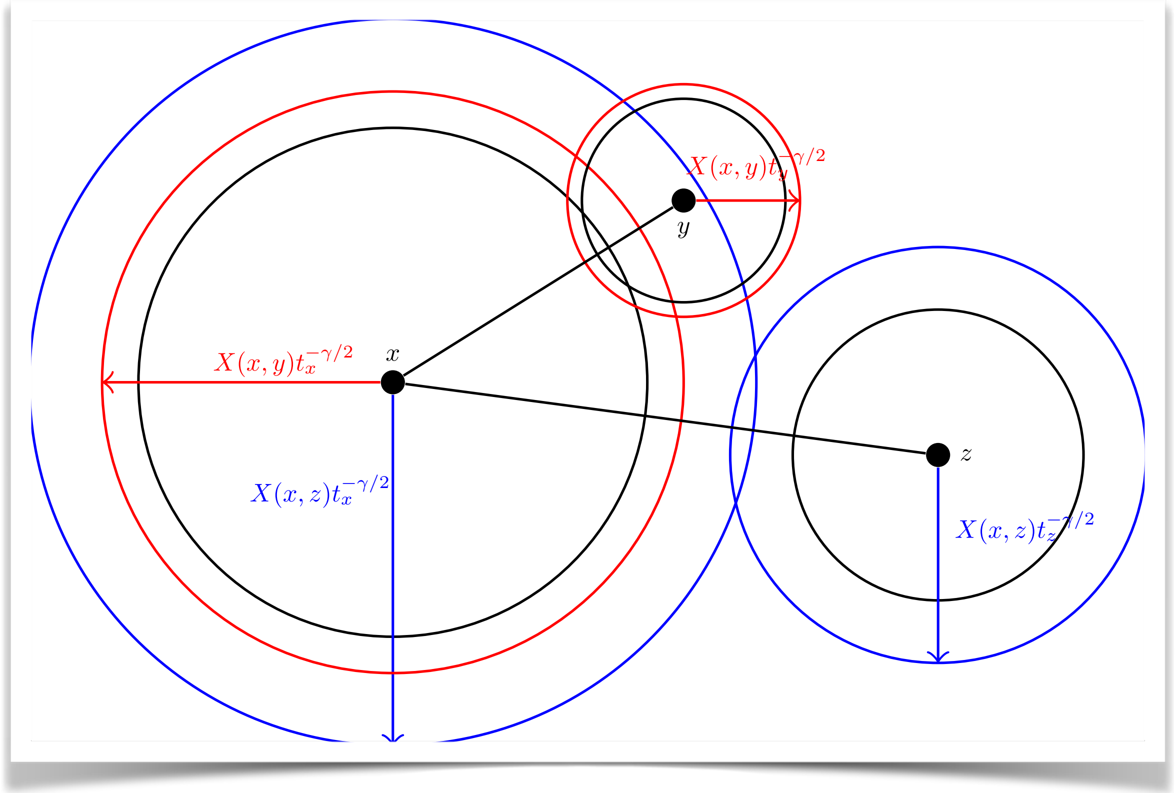

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- **Scale-free percolation:** product kernel $g^{\text{prod}}(s,t) = s^{\gamma}t^{\gamma}$

Main result

Theorem (Gracar, L, Mörters, 2020): If the kernel g satisfies $c_1(s \wedge t)^\gamma \geq g(s,t) \geq c_2(s \wedge t)^\gamma (s \vee t)^{1-\gamma}$ then

1. if $\gamma < \delta/(\delta+1)$ or equivalently $\tau > 2 + 1/\delta$, then $\beta_c > 0$
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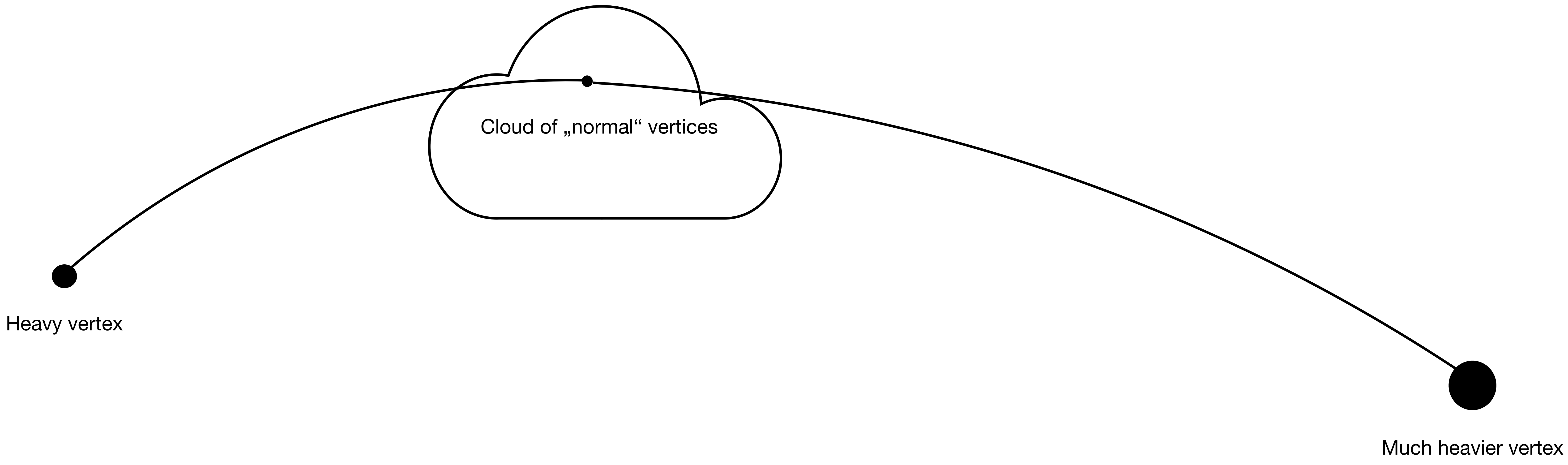

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Theorem (Deijfen et al, Deprez and Wüthrich): If the kernel is $g(s,t) = s^\gamma t^\gamma$, then

1. if $\gamma \leq 1/2$ or equivalently $\tau \geq 3$, then $\beta_c > 0$
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Proof of 1.:

What one hopes to get:

$\mathbb{P}_{\mathbf{0}}\{\mathbf{0}$ starts a self-avoiding, shortcut free path of length $n\} \leq \mathbb{E}\#\{$such paths$\} \leq (\beta C)^n$

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Solution:

➡ Understand the structure of the paths (identify key vertices and how they are connected within the path)

➡ Only bound the expectation of paths with such idealized structure

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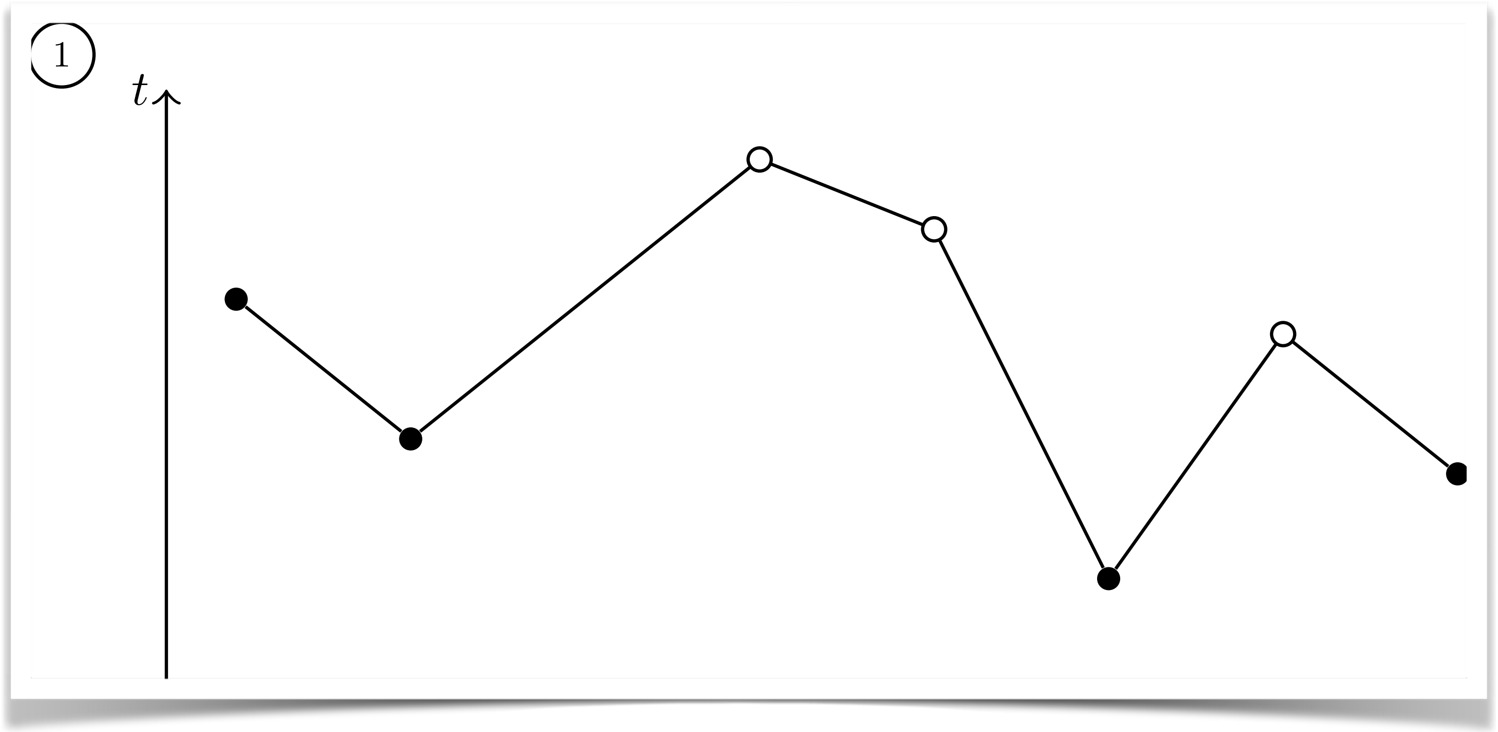

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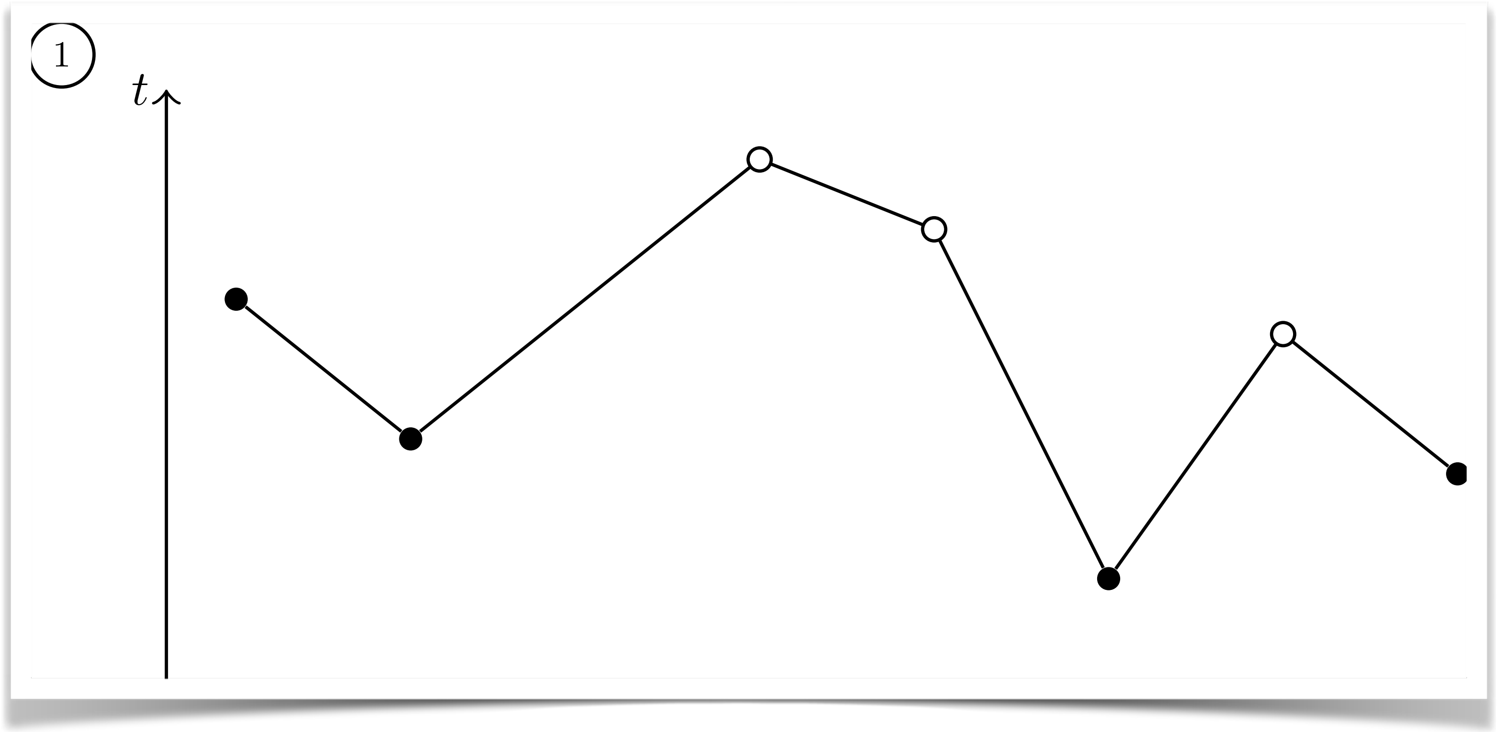

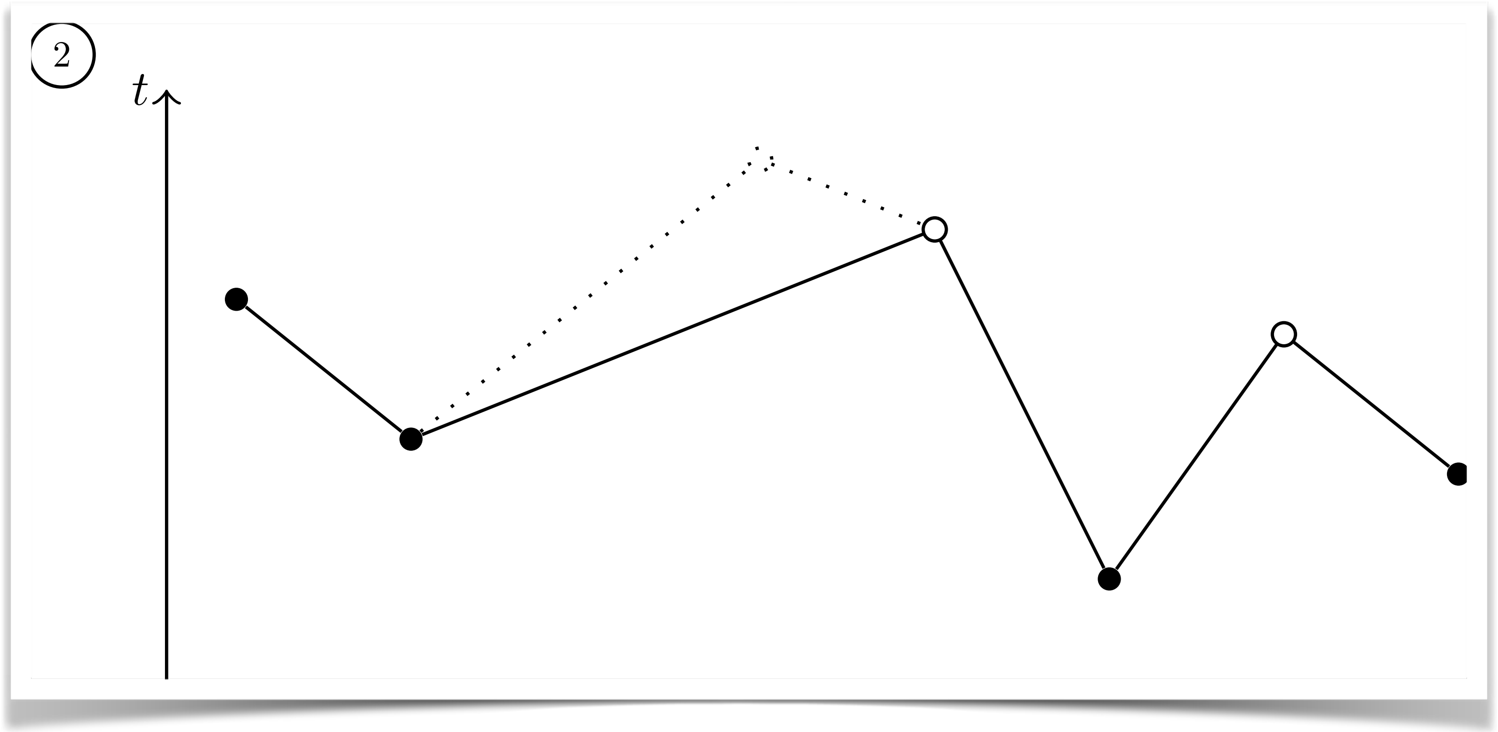

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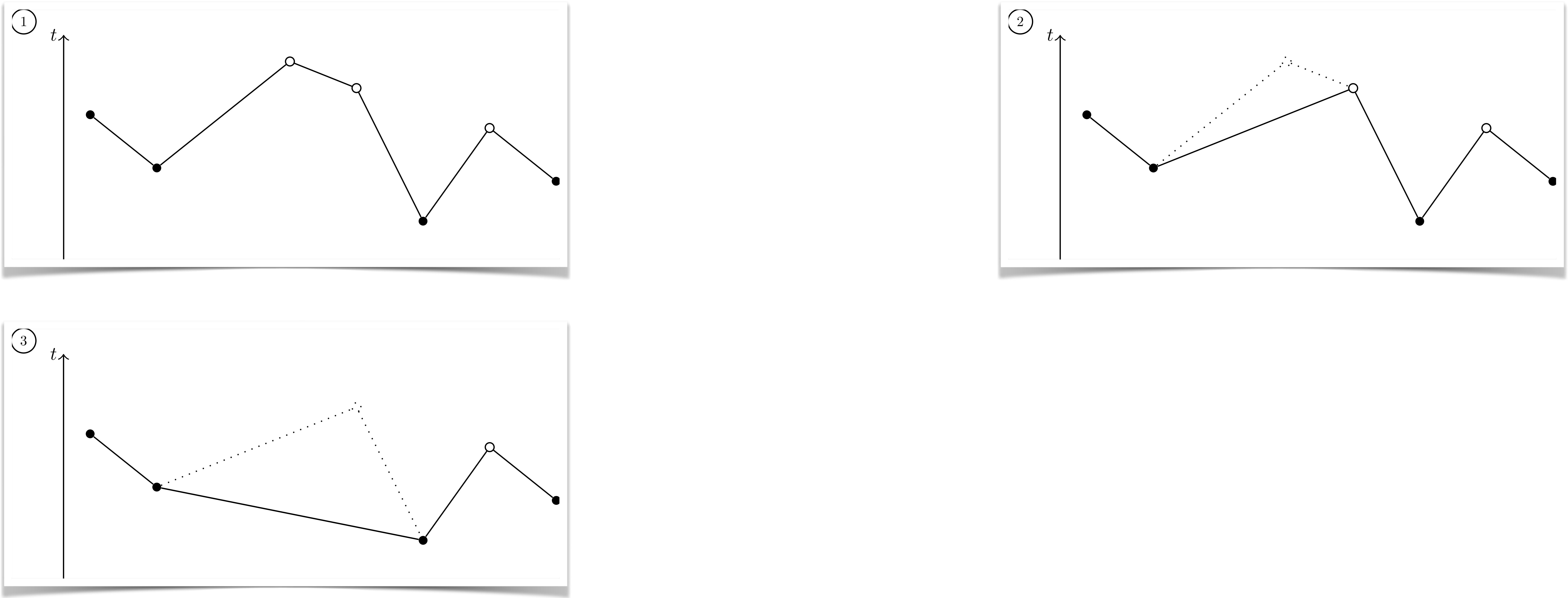

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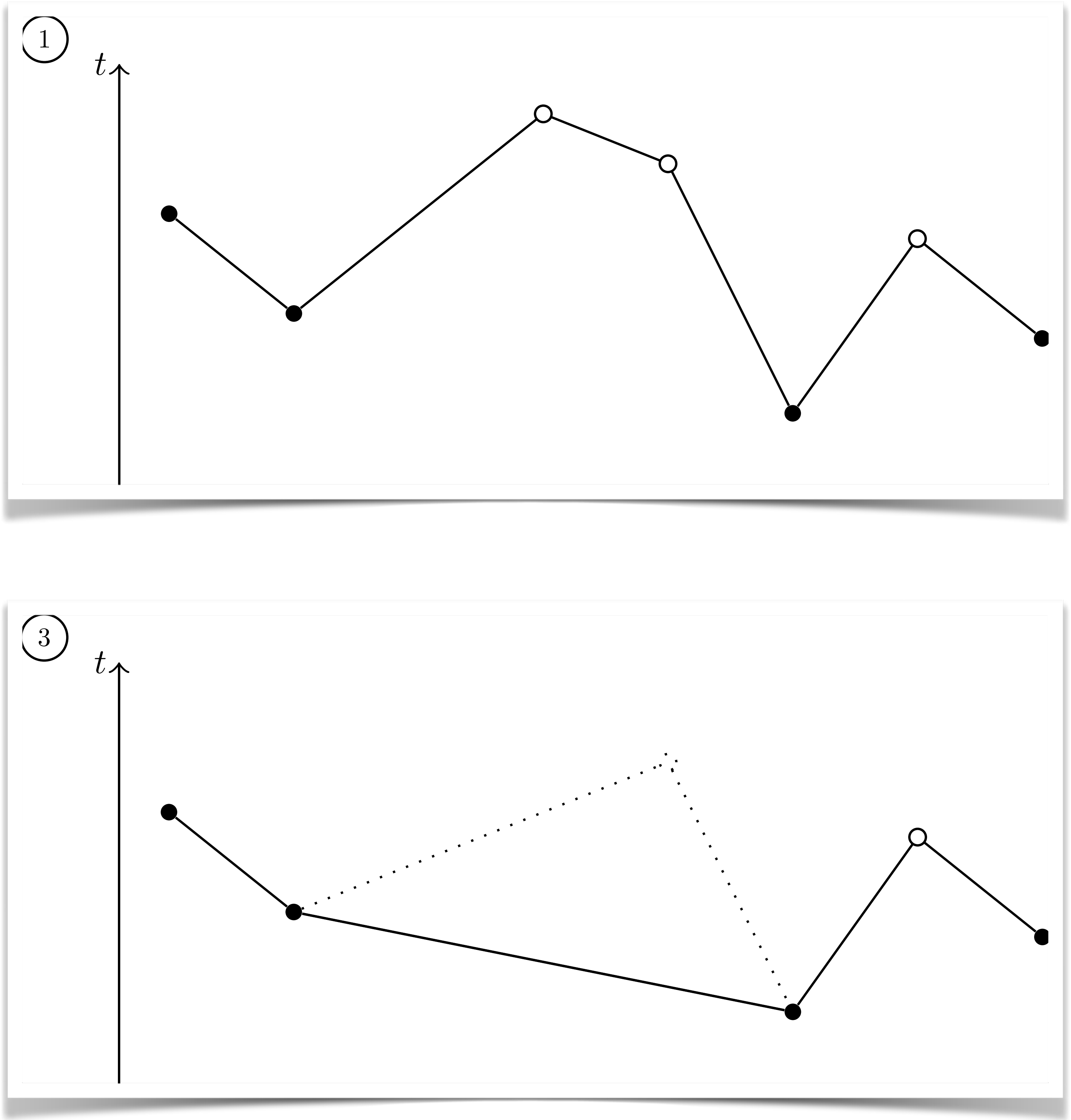

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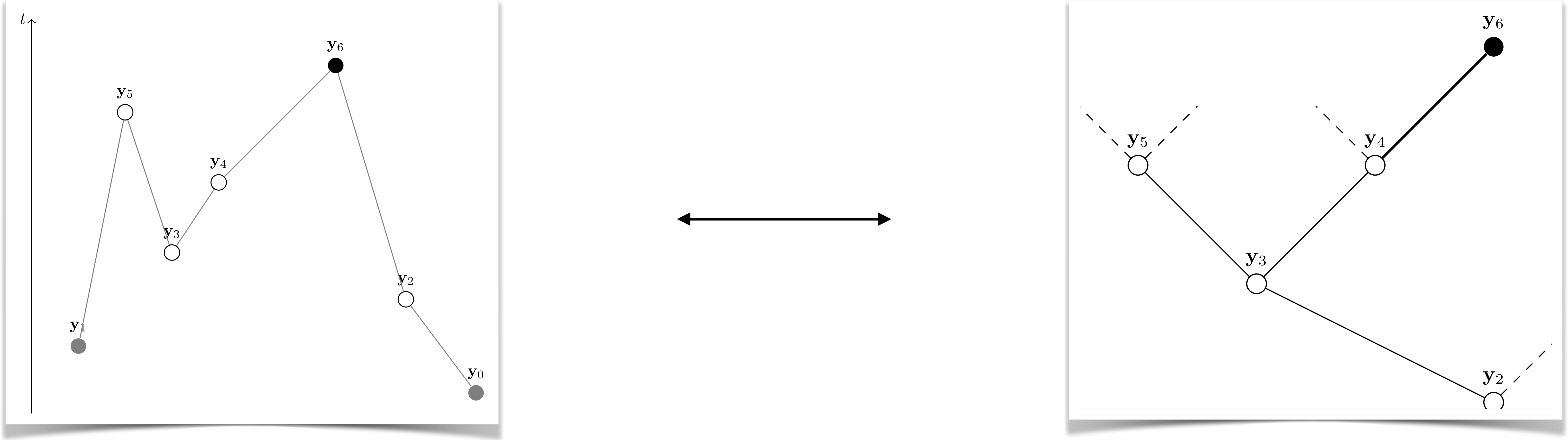

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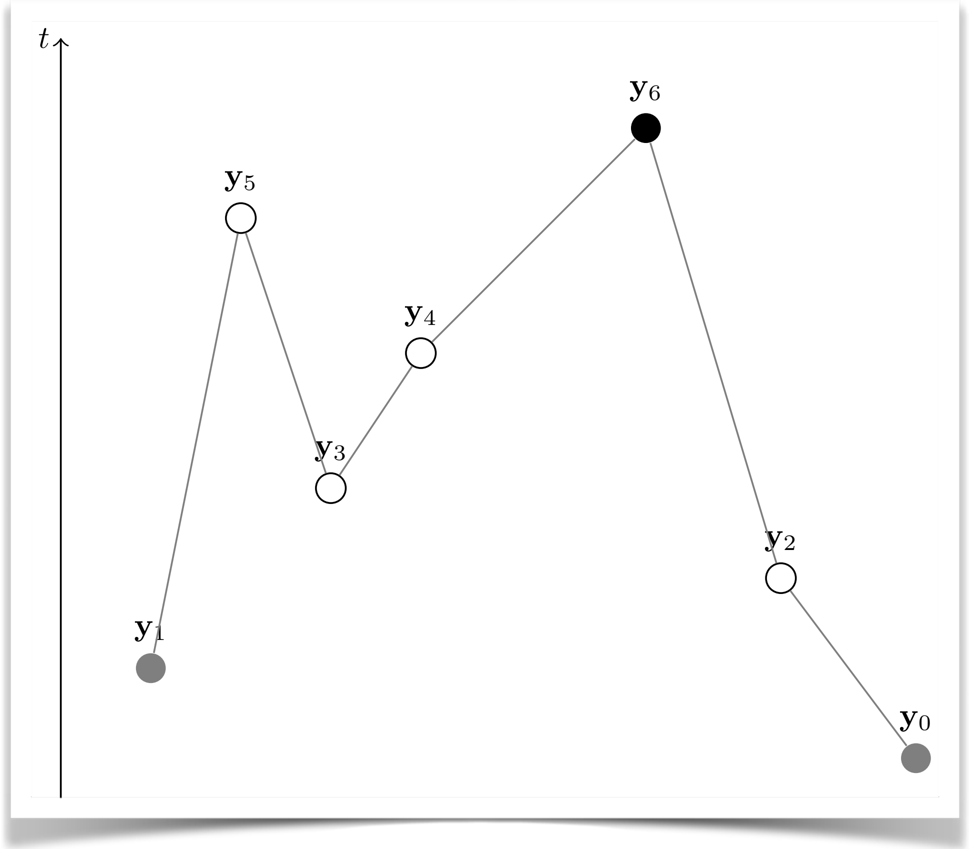
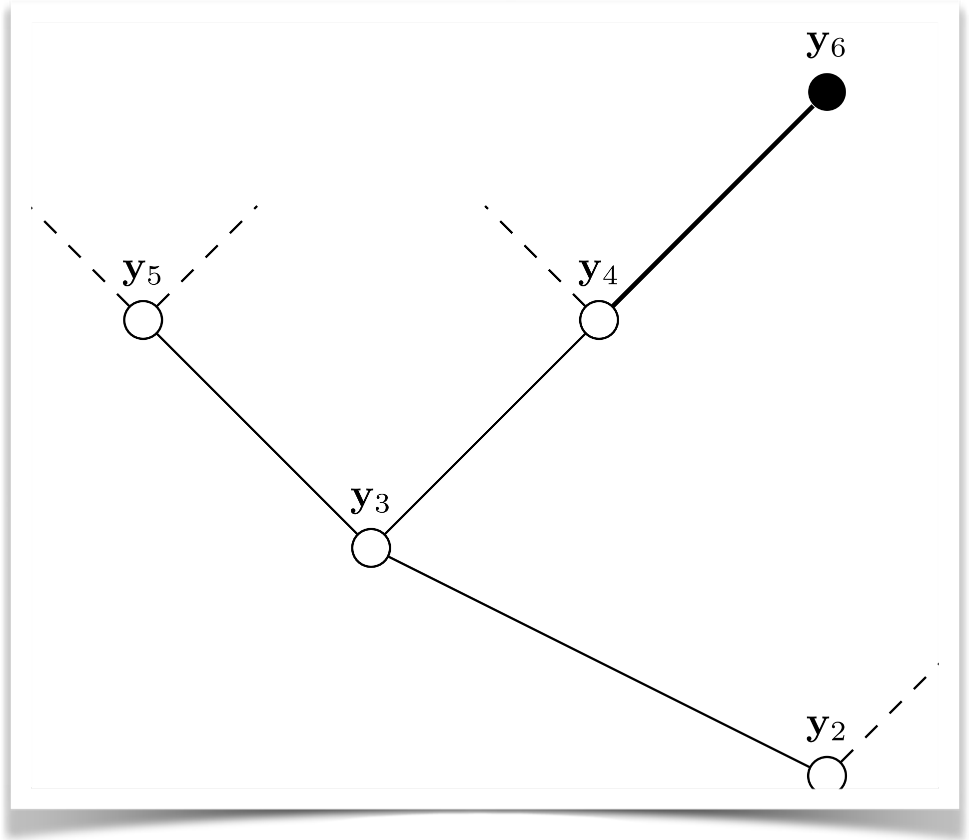


$\longleftrightarrow$



Key Lemma: If $\gamma < \delta/(\delta+1)$, then

$$\mathbb{P}_{\mathbf{y}_0,\mathbf{y}_1}\{\,\exists \text{a conector } \mathbf{z} : \mathbf{y}_0 \sim \mathbf{z} \sim \mathbf{y}_1\} \;\leq\; \int_{\mathbb{R}^d}\int_{t_1}^{1} \mathbb{P}_{\mathbf{y}_0,\mathbf{z}}\{\mathbf{y}_0 \sim \mathbf{z}\}\mathbb{P}_{\mathbf{z},\mathbf{y}_1}\{\mathbf{z} \sim \mathbf{y}_1\}d\mathbf{z} \;\leq\; (\beta C)\mathbb{P}_{\mathbf{y}_0,\mathbf{y}_1}\{\mathbf{y}_0 \sim \mathbf{y}_1\}$$

Hier is where the spatial embedding is dealt with

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Combining everything:

$$\mathbb{P}_{\mathbf{0}}\{\mathbf{0} \text{ starts a self-avoiding, shortcut free path of length } n\} \leq \sum_{m=1}^{n} (\beta C)^{n-m}\mathbb{E}[\#\{\text{skeleton paths of length } m\}] \leq (\beta C)^n$$

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