Percolation on hyperbolic Poisson-Voronoi tessellations

Tobias Müller
Groningen

(based on joint work with Benjamin Hansen)

workshop on geometric random graph models and percolation,
20 October 2021

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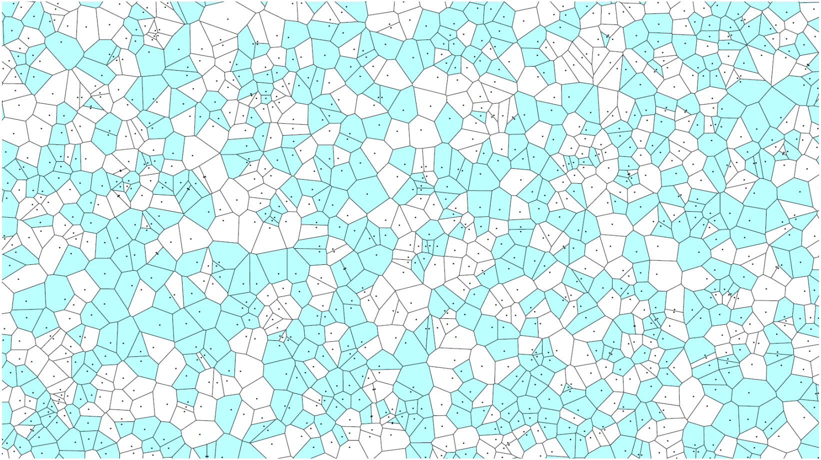
- (Independently) colour cells black with probability p, white with probability $1 - p$.

A computer simulation



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Does not depend on λ (follows from standard properties of PPPs).

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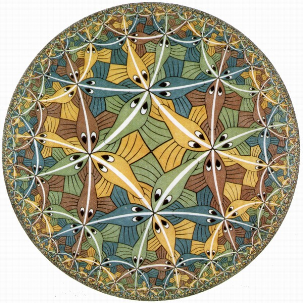
Work towards more detailed picture "at criticality" by Tassion'16, Ahlberg et al. '16, Ahlberg-Baldasso'18, Vanneuville'19, ...

Poincaré disk model

In this talk, all depictions of the hyperbolic plane, and all math, will take place in the Poincaré disk representation of $\mathbb{H}^2$.



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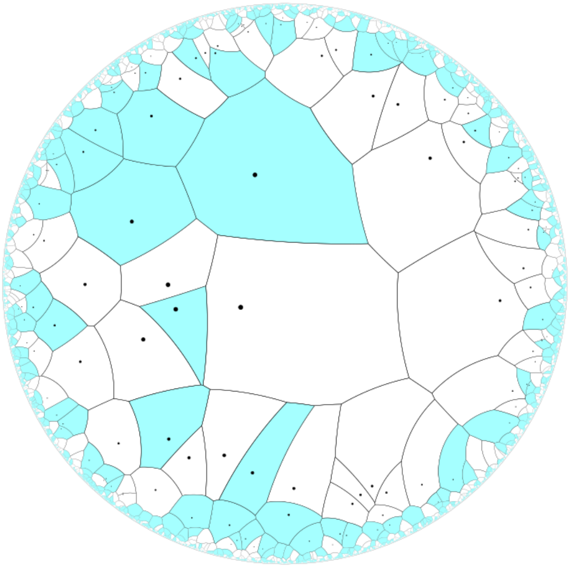

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(A priori we have no reason to assume p_c does not depend on λ.)

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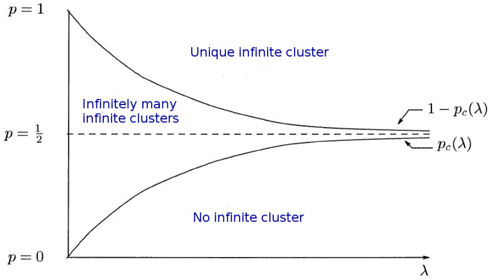
(iii) If $p_c(\lambda) < p < 1 - p_c(\lambda)$ then there are infinitely many unbounded black clusters (a.s.).

A diagram from Benjamini+Schramm'00

The BS paper contains the following diagram, several aspects of which are conjectures/open questions.



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Theorem. [Hansen+M '21+] The conjecture holds.

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Theorem. [Hansen+M '21+] $p_c(\lambda) = \frac{\pi}{3}\lambda + o(\lambda)$ as $\lambda \searrow 0$.

Some words on the $\lambda \to \infty$ result.

We leverage the results on Euclidean Poisson-Voronoi percolation. Intuition:

- If we "zoom in" the geometry of $\mathbb{H}^2$ looks more and more Euclidean.
- As $\lambda \to \infty$ the points get packed closer and closer together.

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Of course more ideas are needed. For details:

- Ben's talk in the online "Percolation Today" seminar (28 April 2020).
- Arxiv : 2004.01464

Strategy for the $\lambda \to 0$ result.

We add the origin o to $\mathcal{Z}$ and consider percolation on the Voronoi tesselation for $\mathcal{Z} \cup \{o\}$.

- For $p = (1+\varepsilon)\frac{\pi}{3}\lambda$ and λ small: show the cluster of $C(o)$ stochastically dominates a super-critical Galton-Watson branching process.
- For $p = (1-\varepsilon)\frac{\pi}{3}\lambda$ and λ small: show there is no infinite path starting from $C(o)$ (a.s.).

Where did $\frac{\pi}{3}\lambda$ come from?

The typical cell: let D denote the number of sides of $C(o)$ in the Voronoi tessellation for $\mathcal{Z} \cup \{o\}$.

Theorem. [Isokawa '01] $\mathbb{E}D = 6 + \frac{3}{\pi\lambda}$.

So, for small λ, the critical probability p_c is such that the average number of black neighbours is roughly one.

Almost all adjacent points have distance $2\log(1/\lambda) \pm \text{const}$.

Using a variation on Isokawa's computations, we can show that for almost all z such that $C(o)$ and $C(z)$ are adjacent,

$$\mathrm{dist}_{\mathbb{H}^2}(o, z) = 2\log(1/\lambda) \pm \text{const}.$$

Constructing a Galton-Watson tree inside the cluster of $C(o)$

Exploration of a tree inside the cluster of o:

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- When processing a point z, we find a collection of black "children" $z_1, \dots, z_k$ in the annulus

$$B(z, 2\log(1/\lambda) + K) \setminus B(z, 2\log(1/\lambda) - K),$$

  such that all angles $\angle z_i z z_j$ are at least ϑ, as well as the angles with the parent of z.

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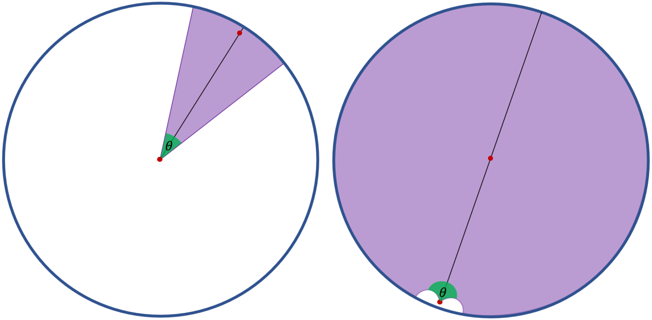
In each step, the previously explored region does not bother us too much. (Next slide.)

Why the past exploration does not bother us too much.

Left: perspective of parent, Right: perspective of child.



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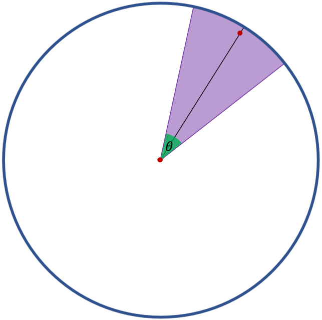

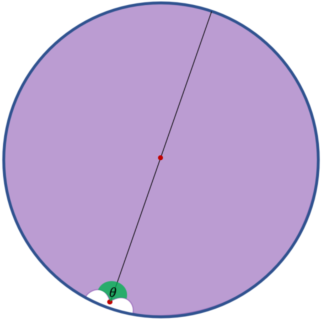

Computations for "typical point" show in each step of the exploration the expected # children is > 1.
(When $p = (1+\varepsilon)\frac{\pi}{3}\lambda$ and λ suff. small.)

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- Differentiable?
- **Opening for doing a PhD with me.**
  (not necessarily on these or related questions)

Thank you for your attention!