CONTACT ON HYP. RANDOM GRAPH

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Geometric Random Graphs and percolation

joint work with Amitai Linker, Dieter Mitsche and Daniel Valesin.

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Edge set :

$$(x, h) \sim (x', h') \iff |x - x'| \leq e^{(h+h')/2}.$$

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One has a.s. $\lambda_c = 0$, and as $\lambda \to 0$,

$$\gamma(\lambda) \asymp \begin{cases} \lambda^{\frac{1}{2-2\alpha}}, & \alpha \in (\frac{1}{2}, \frac{3}{4}]; \\ \frac{\lambda^{4\alpha-1}}{\log(1/\lambda)^{2\alpha-1}}, & \alpha \in (\frac{3}{4}, 1). \end{cases}$$

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• Dynamical random graphs : Jacob–Linker–Mörters.

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Furthermore, for any $(t_n)_{n\geq 1}$, with $t_n \to \infty$, and $t_n < e^{cn}$, for any $\varepsilon > 0$,

$$\mathbb{P}_\lambda\left(\left|\frac{|\xi_{t_n}^{\mathbb{G}_n}|}{n} - \gamma(\lambda)\right| > \varepsilon\right) \underset{n\to\infty}{\longrightarrow} 0.$$

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$$B_{j,k} := [2^{j-1}k, 2^{j-1}(k+1)] \times [jL, (j+1)L], \quad j, k \geq 0,$$

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$B_{j+1,[k/2]}$ is called the **parent** of $B_{j,k}$, $\rightarrow$ this defines a **canopy tree**.

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Similar ideas show exponential survival time on $\mathbb{G}_n$.

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Upper bound (hardest part). One needs to control all possible infection paths...
Requires a precise understanding of geometric properties of the graph, and good control on probabilities of infection paths.

Thank you for your attention !