Isolated and Extreme Points in Hyperbolic Random Geometric Graphs

Joe Yukich

Lehigh University

Based on a joint paper with Nikolaos Fountoulakis (Birmingham)

I Introduction: Poincaré disk

We study the random geometric graph on the hyperbolic plane as introduced by Krioukov et al. (2010).

The standard Poincaré disk representation of hyperbolic plane is the open unit disk $\mathcal{D} := \{(u,v) \in \mathbb{R}^2 \ : \ u^2 + v^2 < 1\}$ equipped with the hyperbolic metric d_H given by $ds^2 = 4 \, \frac{du^2+dv^2}{(1-u^2-v^2)^2}$.

Given $\nu \in (0,\infty)$ a fixed constant and a natural number $n > \nu$, we let

$$R := 2\log(\frac{n}{\nu}), \quad \text{i.e.,} \quad n = \nu \exp(\frac{R}{2}).$$

We will consider a random geometric graph on Poisson nodes in $\mathcal{D}_R$, a disk of radius R.

I Introduction: Poisson point process on disk

For every $\alpha \in (0, \infty)$, consider the probability density function

$$\rho_{\alpha,n}(r) := \begin{cases} \alpha \frac{\sinh(\alpha r)}{\cosh(\alpha R) - 1} & 0 \le r \le R \\ 0 & \text{otherwise} \end{cases}.$$

We consider the probability density on $\mathcal{D}_R$: $\frac{1}{2\pi}\rho_{\alpha,n}(r)$.

$\mathcal{P}_{\alpha,n}$: the Poisson point process on disk $\mathcal{D}_R$ with intensity density $n\frac{1}{2\pi}\rho_{\alpha,n}$.

$\alpha = 1$: the uniform distribution on $\mathcal{D}_R$ under the hyperbolic metric.

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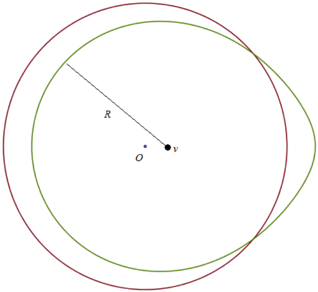
$\alpha = 1$: the uniform distribution on $\mathcal{D}_R$ under the hyperbolic metric.

General $\alpha \in (0, \infty)$: this is a *quasi-uniform* distribution on $\mathcal{D}_R$, since it arises as the projection onto $\mathcal{D}_R$ of the uniform distribution on a disc of hyperbolic radius R in the hyperbolic plane having curvature $-\alpha^2$ and equipped with the metric $\frac{4}{\alpha^2} \; \frac{du^2 + dv^2}{(1 - u^2 - v^2)^2}$.

I Introduction: Geometric Graph $\mathcal{G}_{\alpha,n}$ on disk

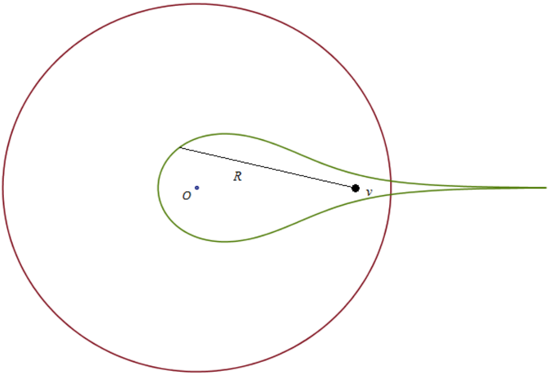
We join two points in $\mathcal{P}_{\alpha,n}$ with an edge iff they are within hyperbolic distance R of each other. The resulting graph is denoted by $\mathcal{G}_{\alpha,n}$ and it is called the *hyperbolic random geometric graph on the disk* $\mathcal{D}_R$.

What does the ball look like when its center is not the origin?



The ball $\mathcal{B}_R(v)$ around the point $v \in \mathcal{D}_R$. The closer v is to the origin, the more the ball resembles the disk $\mathcal{D}_R$.

I Introduction

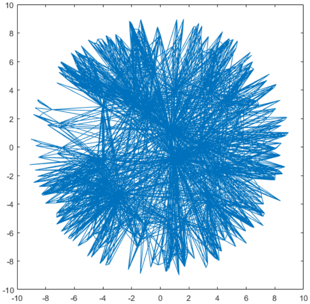


The disc $\mathcal{B}_R(v)$ around the point $v \in \mathcal{D}_R$.

What does the hyperbolic random geometric graph look like?

I Introduction

$R := 2\log(\frac{n}{\nu})$, α is inverse square root of curvature.



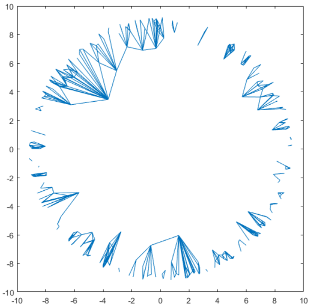
Samples of $\mathcal{G}_{\alpha,n}$ for $n = 300$, $\nu = 3$ and $\alpha = 0.7$

Average degree decreases as α increases.

Small α means more points near the origin. Now let α be equal to 2.

I Introduction

$R := 2\log(\frac{n}{\nu})$, α is inverse square root of curvature.

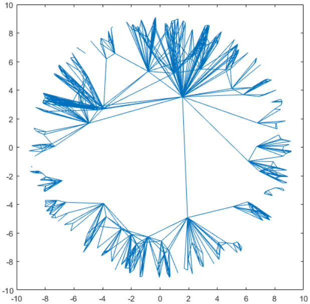


Samples of $\mathcal{G}_{\alpha,n}$ for $n = 300$, $\nu = 3$ and $\alpha = 2$.

Average degree decreases as α increases.

I Introduction

$R := 2\log(\frac{n}{\nu})$, α is inverse square root of curvature.

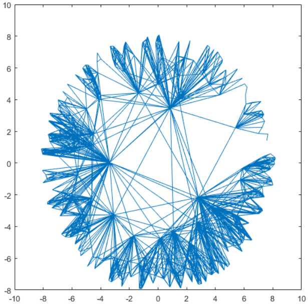


Samples of $\mathcal{G}_{\alpha,n}$ for $n = 300$, $\alpha = 1$ and $\nu = 3$.

Average degree increases as ν increases. Increase ν to 5.

I Introduction

$R := 2\log(\frac{n}{\nu})$, α is inverse square root of curvature.



Samples of $\mathcal{G}_{\alpha,n}$ for $n = 300$, $\alpha = 1$ and $\nu = 5$.

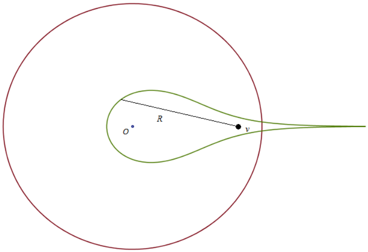
Average degree increases as ν increases.

I Introduction

Given $p \in \mathcal{D}_R$ we let $y(p)$ denote its distance to the boundary of $\mathcal{D}_R$.

Given a point process $\mathcal{P}$ on $\mathcal{D}_R$ and $p \in \mathcal{P} \cap \mathcal{D}_R$, we say that p is **isolated** with respect to $\mathcal{P}$ iff there is no $p' \in \mathcal{P}$, such that $d_H(p, p') \leq R$.

We say that p is **extreme** with respect to $\mathcal{P}$ iff there is no $p' \in \mathcal{P}$, such that $d_H(p, p') \leq R$ *and* $y(p') \in [0, y(p))$.



I Introduction

Given $p \in \mathcal{P}$, define the score $\xi^{iso}(p, \mathcal{P})$ to be 1 if p is isolated and zero otherwise. Likewise, define $\xi^{ext}(p, \mathcal{P})$ to be 1 if p is extreme and zero otherwise.

Goal: seek the limit theory for the number of isolated and extreme points in hyperbolic random geometric graph $\mathcal{G}_{\alpha,n}$ given respectively by

$$S^{iso}(\mathcal{P}_{\alpha,n}) := \sum_{p \in \mathcal{P}_{\alpha,n}} \xi^{iso}(p, \mathcal{P}_{\alpha,n})$$

and

$$S^{ext}(\mathcal{P}_{\alpha,n}) := \sum_{p \in \mathcal{P}_{\alpha,n}} \xi^{ext}(p, \mathcal{P}_{\alpha,n}).$$

I Introduction

Isolated points can not be 'reached' by other points. This has an interesting interpretation in cosmology.

In the cosmological set-up (see Krioukov et al.), in the large time limit, isolated points are precisely those whose past and future light cones are empty, i.e., the set of points neither accessible by the past nor having access to the future. Extreme points are those whose future light cones are empty, i.e., points which do not causally influence other points.

Extreme points are the analog of maximal points of a sample

Given $\mathcal{X} \subset \mathbb{R}^d$ locally finite, $x \in \mathcal{X}$ is **maximal** if no other point in $\mathcal{X}$ exceeds it in all coordinates.

If $\mathcal{X}$ is an i.i.d. sample uniformly distributed on a smooth convex body B in $\mathbb{R}^d$, $n := \text{card}(\mathcal{X})$, then both the expectation and variance of the number of maximal points in $\mathcal{X}$ are asymptotically $\Theta(n^{(d-1)/d})$, the order of the expected number of points close to the boundary of B.

We will see that the expectation and variance of the number of extreme points in the random geometric hyperbolic graph grow *linearly* with input size, which likewise is of the order of the expected number of points close to the boundary of $\mathcal{D}_R$.

II First and second order results for isolated pts

Theorem

We have for all $\alpha \in (\frac{1}{2}, \infty)$

$$\lim_{n\to\infty} \frac{\mathbb{E}[S^{iso}(\mathcal{P}_{\alpha,n})]}{n} = 2\alpha \int_0^\infty \exp\left(-\gamma e^{\frac{y}{2}}\right) \exp(-\alpha y) dy,$$

where $\gamma := \frac{8\nu\alpha}{\pi(2\alpha-1)}$. *Also*

$$\mathrm{Var}[S^{iso}(\mathcal{P}_{\alpha,n})] = \begin{cases} \Theta(n^{3-2\alpha}) & \alpha \in (\frac{1}{2}, 1) \\ \Theta(nR) = \Theta(n \log n) & \alpha = 1 \\ \Theta(n) & \alpha \in (1, \infty) \end{cases}.$$

II First and second order results for extreme pts

On the other hand, for all $\alpha \in (\frac{1}{2}, \infty)$, the expectation and variance asymptotics for the number of extreme points exhibit linear scaling in n, i.e., the renormalization is the standard one.

Theorem

We have for all $\alpha \in (\frac{1}{2}, \infty)$

$$\lim_{n \to \infty} \frac{\mathbb{E}[S^{ext}(\mathcal{P}_{\alpha,n})]}{n} = \mu,$$

and

$$\lim_{n \to \infty} \frac{\mathrm{Var}[S^{ext}(\mathcal{P}_{\alpha,n})]}{n} = \sigma^2,$$

where $\mu, \sigma^2 \in (0, \infty)$ *are in terms of expectations and covariances of scores involving extreme points of a Poisson point process on the upper half-plane, with intensity density decaying exponentially with the height.*

II Convergence to normal

Theorem

Denote by N the standard normal with mean zero and variance one. As $n \to \infty$, for any $\alpha \in (1, \infty)$ we have

$$\frac{S^{iso}(\mathcal{P}_{\alpha,n}) - \mathbb{E}[S^{iso}(\mathcal{P}_{\alpha,n})]}{\sqrt{\mathrm{Var}[S^{iso}(\mathcal{P}_{\alpha,n})]}} \xrightarrow{\mathcal{D}} N.$$

The above limit fails for $\alpha \in (\frac{1}{2}, 1)$. As $n \to \infty$, for any $\alpha \in (\frac{1}{2}, \infty)$ we have

$$\frac{S^{ext}(\mathcal{P}_{\alpha,n}) - \mathbb{E}[S^{ext}(\mathcal{P}_{\alpha,n})]}{\sqrt{\mathrm{Var}[S^{ext}(\mathcal{P}_{\alpha,n})]}} \xrightarrow{\mathcal{D}} N.$$

III Proof Ideas: A transfer map

We map the point process $\mathcal{P}_{\alpha,n}$ in the disc $\mathcal{D}_R$ to a Poisson point process hosted by a rectangle D in the upper half-plane. The rectangle has length proportional to n and height R. We set

$$I_n := \frac{\pi}{2} e^{R/2} = \frac{\pi}{2\nu} \cdot n.$$

We put

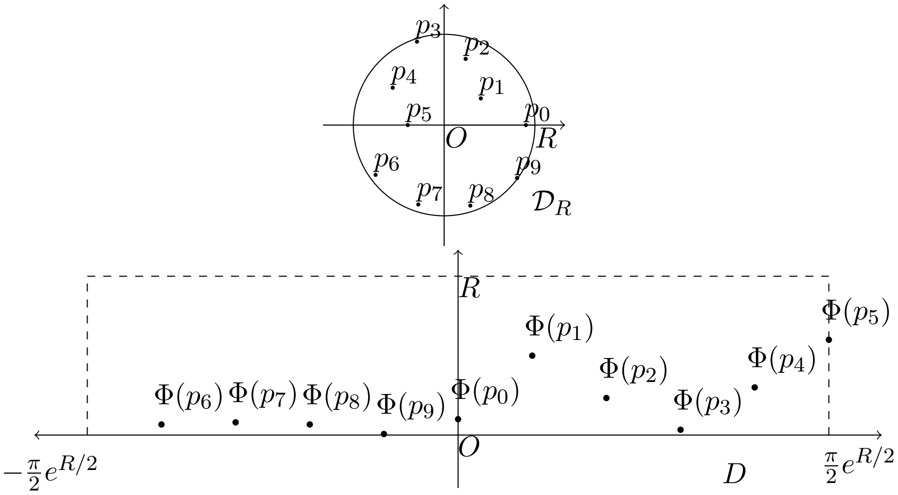
$$D := (-I_n, I_n] \times [0, R].$$

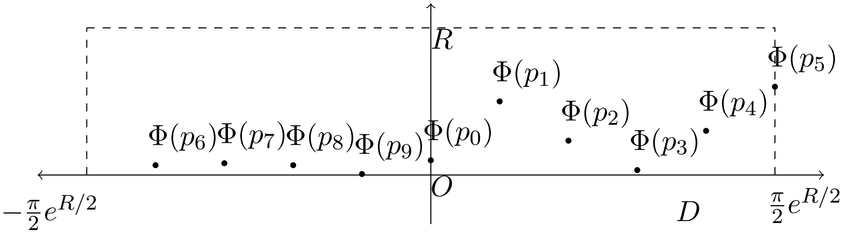
For $p \in \mathcal{D}_R$, we write $p := (\theta(p), y(p))$, with $y(p)$ the defect radius and $\theta(p)$ the angle with respect to a reference point.

Re-scale the angle $\theta(p)$ by $\frac{1}{2}e^{R/2}$, setting $x(p) := \frac{1}{2}\theta(p)e^{R/2}$. This defines the map $\Phi : \mathcal{D}_R \to D$, mapping $(\theta(p), y(p)) \mapsto (x(p), y(p))$.

III Proof Ideas

The mapping $\Phi : \mathcal{D}_R \to D$**.**





First simplification (approximating transferred point process). The point process in D has intensity density

$$\beta e^{-\alpha y} + o(n^{-1}), \quad \beta := \frac{2\nu\alpha}{\pi}; \quad y \text{ is height.}$$

The Poisson point process on upper half-plane with intensity measure

$$\mu_\alpha(S) = \int_S \beta e^{-\alpha y} dx dy, \quad S \subset D.$$

well approximates image of transferred points.

Second simplification (restriction to narrow horiontal strip). Recall $R = 2\log\frac{n}{\nu}$.

By the transfer map Φ, we only need to consider Poisson points in the upper half plane having height at most $H := 4\log R$.

Recall
$$\mu_\alpha(S) = \int_S \frac{2\nu\alpha}{\pi} e^{-\alpha y} dx dy, \quad S \subset D.$$

Lemma (measure of balls in Euclidean plane): If $B((x,y))$ is the image under Φ of a hyperbolic ball of radius R, then uniformly in all $y \in D$ with height at most $4 \log R$, we have

$$\mu_\alpha(B((x,y))) = \gamma e^{y/2} + o(1), \quad \gamma := \frac{8\nu\alpha}{\pi(2\alpha - 1)}.$$

III Expectation asymptotics for the number of isolated pts

Let $\tilde{\mathcal{P}}_\alpha$ be PPP on upper half-plane with intensity density μ_α. From previous lemma, recalling $D := [-I_n, I_n] \times [0, H]$,

$$\mathbb{E}[\xi^{iso}((x,y), \tilde{\mathcal{P}}_\alpha \cap D)] = \exp(-\mu_\alpha(B(x,y))) \sim \exp\left(-\gamma e^{\frac{y}{2}}\right)$$

uniformly over all $(x,y) \in D$ with height y at most $H := 4\log R$. Campbell-Mecke formula and above identity yield ($\beta = \frac{2\nu\alpha}{\pi}$):

$$\begin{aligned}
\mathbb{E} \sum_{p \in \tilde{\mathcal{P}}_\alpha \cap D} \xi^{iso}(p, \tilde{\mathcal{P}}_\alpha \cap D)] &= \beta \int_{-I_n}^{I_n} \int_0^H \mathbb{E}[\xi^{iso}((x,y), \tilde{\mathcal{P}}_\alpha \cap D)] e^{-\alpha y} dx dy \\
&\sim \beta \cdot 2I_n \int_0^H e^{-\gamma e^{y/2}} e^{-\alpha y} dy \\
&\sim \beta \cdot \pi e^{R/2} \int_0^\infty e^{-\gamma e^{y/2}} e^{-\alpha y} dy \\
&= 2\alpha n \int_0^\infty e^{-\gamma e^{y/2}} e^{-\alpha y} dy. \quad \square
\end{aligned}$$

Variance of the number of isolated points.

Let F be a functional on a space $\mathbf{S}$ hosting a Poisson process $\mathcal{P}$ of intensity measure λ.

For a point $p \in \mathbf{S}$ we define the *first order difference operator* $\nabla_p F := F(\mathcal{P} \cup \{p\}) - F(\mathcal{P})$. Then the Poincaré inequality states that

$$\mathrm{Var} F \leq \mathbb{E} \int_{\mathbf{S}} (\nabla_p F(\mathcal{P}))^2 \lambda(dp).$$

We apply this inequality letting F stand for the number of isolated points, $\mathbf{S}$ the rectangle D, and letting $\lambda(dp)$ be μ_α. We obtain

$$\mathbb{E} \int_{\mathbf{S}} (\nabla_p F(\mathcal{P}))^2 \lambda(dp) = ... = O(1) \cdot n \int_0^R e^{(1-\alpha)y} dy.$$

Variance of the number of extreme points

Variance asymptotics for $S^{ext}(\mathcal{P}_{\alpha,n})$ are handled by stabilization.

For a given point p, we define a radius of stabilization $R^{\xi} := R^{\xi^{ext}}$ for ξ^{ext}, in the sense that points distant more than R^{ξ} from p do not affect the property of p being extreme.

By stabilization, the covariance converges to the covariance of two points in the infinite upper half-plane.

Central limit theorem for the number of isolated points

Poincaré type inequality for Poisson functionals: bounds the Wasserstein distance in terms of first- and second-order difference operators.

$\alpha \in (1, \infty)$: there is a high probability event on which these difference operators may be controlled, as vertices of high degree are fewer in this regime.

$\alpha \in (1/2, 1)$: there is a high probability event A_n such that conditional on A_n, the variance is much smaller than the unconditional variance, and the convergence to the standard normal fails in this regime.

Central limit theorem for the number of extreme points

The case of extreme points is different, since the extremality status of a point is influenced only by points of larger radius lying in the ball of radius R around it.

This region is typically small and makes the corresponding score functions almost independent.

Cut the plane into rectangles and define a dependency graph on the vertex set of such rectangles, so that no points in non-adjacent vertices in this dependency graph can be connected. Use the central limit theorem for dependency graphs.

Future research

1. Limit theory for the number of isolated vertices when $\alpha \in (1/2, 1)$.

2. Extend results to dimensions three and higher.

THANK YOU

References:

1. Fountoulakis, N. and J. Yukich, Limit theory for isolated and extreme points in hyperbolic random geometric graphs, *Electronic J. Prob.*, 2020.

2. D. Krioukov, M. Kitsak, R.S. Sinkovits, D. Rideout, D. Meyer, and M. Boguñá; Network cosmology, *Nature Scientific Reports*, 2012.

3. D. Krioukov, F. Papadopoulos, M. Kitsak, A. Vahdat, and M. Boguñá; Hyperbolic geometry of complex networks, *Phys. Rev. E (3)*, 2010.

Past and Future Light Cones

The next slide depicts the situation in $1+1$-dimensional de Sitter spacetime, which is seen as a one-sheeted $1+1$-dimensional hyperboloid.

There is a map between de Sitter spacetime carrying past and future light cones of a point P into a subset of the hyperbolic ball around the image of P.
For large disc $\mathcal{D}_R$ thes past and future light cones converge to the ball around the image of the point P.

Past and Future Light Cones