



On the satisfiability threshold of formulas with three literals per clause^{☆,☆☆}

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A B S T R A C T

In this paper we present a new upper bound for randomly chosen 3-CNF formulas. In particular we show that any random formula over n variables, with a clauses-to-variables ratio of at least 4.4898 is, as n grows large, asymptotically almost surely unsatisfiable. The previous best such bound, due to Dubois in 1999, was 4.506. The first such bound, independently discovered by many groups of researchers since 1983, was 5.19. Several decreasing values between 5.19 and 4.506 were published in the years between. We believe that the probabilistic techniques we use for the proof are of independent interest.

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1. Introduction

Satisfiability of Boolean formulas is a problem universally believed to be hard. Determining the source of this hardness will lead, as is often stressed, to applications in domains even outside the realm of mathematics or computer science; moreover, and perhaps more importantly, it will enhance our understanding of the foundations of computing.

In the beginning of the 90s, several groups of experimentalists chose to examine the source of this hardness from the following viewpoint: consider a *random* 3-CNF formula with a given clauses-to-variables ratio, which is known as the *density* of the formula. What is the probability of it being satisfied and how does this probability depend on the density? Their simulation results led to the conclusion that, if the density is fixed and below a number approximately equal to 4.27, then for large n a randomly chosen formula is almost always satisfiable; whereas if the density is fixed and above 4.27, a randomly chosen formula is, for large n , almost always unsatisfiable. More importantly, around 4.27 the complexity of several well-known complete algorithms for checking satisfiability reaches a steep peak (see e.g. [10,15,16]). So in a certain sense, 4.27 is the point where, from an empirical statistical viewpoint, the “hard” instances of SAT are to be found. Similar results were obtained for other combinatorial problems and also for k -SAT for values of $k > 3$.

These experimental results were followed by an intense activity to provide “rigorous results” (the expression IS often used in this context to refer to theorems). Perhaps the most important advance is due to Friedgut: in [7] he proved that there is a sequence of reals $(c_n)_n$ such that, for any $\epsilon > 0$, the probability of a randomly chosen 3-CNF formula with density $c_n - \epsilon$ being satisfiable approaches 1 (as $n \rightarrow \infty$); whereas for density $c_n + \epsilon$, it approaches 0. Intuitively, this means that the transition from satisfiability to unsatisfiability is sharp. However, it is still not known if $(c_n)_n$ converges.

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Despite the fact that the convergence of $(c_n)_n$ is still an open problem, increasingly improved upper and lower bounds on its terms have been computed in a rigorous way by many groups of researchers. The current best lower bound is 3.52 [9,2].

With respect to upper bounds, which is the subject of this work, the progress was slower but better, in the sense that the experimentally established threshold is more closely bounded from above, rather than from below. A naïve application of the first-moment method yields an upper bound of 5.191 (see e.g. [6]). An important advance was made in [8], where the upper bound was improved to 4.76. In what follows, the work of several groups of researchers, based on more refined variants of the first-moment method, culminated in the value of 4.571 [4,11] (see the nice surveys [12,3] for a complete sequence of the events). The core idea in these works was to use the first-moment method by computing the expected number of not all satisfying truth assignments, but only of those among them that are local maxima in the sense of a lexicographic ordering, within a degree of locality determined by the Hamming distance between truth assignments (considered as binary sequences). For degree of locality 1, this amounts to computing the expected number of satisfying assignments that become unsatisfying assignments by flipping any of their “false” values (value 0) to “true” (value 1). Such assignments are sometimes referred to as *single-flip* satisfying assignments.

The next big step was taken by Dubois et al. [5], who showed that 4.506 is an upper bound. Instead of considering further variations of satisfiability, they limited the domain of computations to formulas that have a typical *syntactic* characteristic. Namely, they considered formulas where the cardinality of variables with given numbers of occurrences as positive and negative literals, respectively, approaches a 2D Poisson distribution. Asymptotically almost all formulas have this typical property (we say that such formulas have a Poisson 2D degree sequence). It turns out that the expectation of the number of single-flip satisfying assignments is exponentially reduced when computed for such formulas. To get the aforementioned upper bound, Dubois et al. further limited the domain of computations to formulas that are *positively unbalanced*, i.e. formulas where every variable has at least as many occurrences as a positive literal as it has as a negated one.

A completely different direction was recently taken in [13]. Their work was motivated by results on the geometry of satisfying assignments, and especially the way they form clusters (components where one can move from one satisfying assignment to another by hops of small Hamming distance). Most of these results were originally based on analytical, but non-rigorous, techniques of Statistical Physics; lately however important rigorous advances were made [1,14]. The value of the upper bound obtained by Maneva and Sinclair (see [13]) was 4.453, far below any other upper bound presently known (including the one in this paper). However it was proved assuming a conjecture on the geometry of the satisfying truth assignments which is presently proved only for k -SAT for $k \geq 8$ in the mentioned paper by Achlioptas and Ricci-Tersenghi.

In this paper, we show that 4.4898 is an upper bound. More precisely, our main result is the following:

Theorem 1.1. *Let $\gamma = 4.4898$ and $m = \lfloor \gamma n \rfloor$. A random 3-CNF formula in $\mathcal{F}_{n,m}$ (i.e. with n variables and m clauses, no repetition of clauses and no repetition of variables in a clause) is not satisfiable a.a.s.*

Our approach builds upon previous work. It makes use of (i) single-flip satisfying truth assignments, (ii) formulas with a Poisson 2D degree sequence and (iii) positively unbalanced formulas. We add to these previously known techniques two novel elements that further reduce the expectation computed. Our approach is rigorous: although we make use of computer programs, the outputs we use are formally justified. What is interesting is not that much the numerical value we get, although it constitutes a further improvement to a long series of results. The main interest lies, we believe, on the one hand in the new techniques themselves and on the other hand in the fact that putting together so many disparate techniques necessitates a delicately balanced proof structure. Let us mention, for example, that each one of the techniques we used requires a particular assumption about the type of the underlying random formula model; but then to successfully integrate them, a careful analysis of how these models are related became necessary. For example, at some points (see the proof for details) we need to work over the model we call $\mathcal{F}_{n,m}$, the uniform probability space of 3-CNF formulas with n variables and m clauses, where repetition of clauses or repetition of variables in a clause is not allowed; at other points we need to work with the non-uniform configuration model (see below); and elsewhere we need to work with multi-formulas (not configurations) where repetitions of clauses or literals are allowed and the degree sequence is fixed. To make things worse, we had to go back and forth between such models, as the first-moment method refinement techniques were successively applied. As a result, we came up with a model translating machinery that we believe has its own interest and might prove useful to researchers investigating problems that require random model transitions.

First, we start by recursively eliminating one-by-one the occurrences of pure literals from the random formula, until we get its *impure core*, i.e. the largest sub-formula with no pure literals (a pure literal is one that has at least one occurrence in the formula but whose negation has none). Obviously this elimination has no effect on the satisfiability of the formula. Since we consider random formulas with a given 2D degree sequence, we first have to determine what is the 2D degree sequence of the impure core. For this, we use the differential equation method [18]. The setting of the differential equations is more conveniently carried out in the so-called *configuration* model, where the random formula is constructed by starting with as many labelled copies of each literal as its number of occurrences and then by considering random 3D matchings of these copies. The matchings define the clauses. The change of models from the standard one to the configuration model with a Poisson 2D degree sequence is formalized in Lemma 2.1. We also take care of the fact that the configuration model allows formulas with (i) multiple clauses and (ii) multiple occurrences of the same variable in a clause, whereas we are interested in simple formulas, i.e. formulas where neither (i) nor (ii) holds. For our purposes, it is enough to bound from below the probability of getting a simple formula in the configuration model by $e^{-\Theta(n^{1/3} \log n)}$, see Lemma 3.1. The differential equations are then analytically solved, and we thus obtain the 2D degree sequence of the core, see Proposition 4.1.

Second, we require that not only the 2D degree sequence is Poisson, but also that the numbers of clauses with none, one, two and three positive literals, respectively, are close to the expected numbers. Notice that these expected numbers have to reflect the fact that we consider positively unbalanced formulas. This is formalized in Lemma 5.1.

The expectation of the number of satisfying assignments, in the framework determined by all the restrictions above, is computed in Proposition 6.1. This expectation turns out to be given by a sum of polynomially many terms of functions that are exponential in n . We estimate this sum by its maximum term, using a standard technique. However in this case, finding the maximum term entails maximizing a function of many variables whose number depends on n . To avoid a maximization that depends on n we prove a truncation result which allows us to consider formulas that have a Poisson 2D degree sequence only for *light* variables, i.e. variables whose number of occurrences, either as positive or negated literals, is at most a constant independent of n .

Then we carry out the maximization. The technique we use is the standard one by Lagrange multipliers. We get a complex 3×3 system which can be solved numerically. We formally prove that the system does not maximize on the boundary of the system and we make a sweep over the domain which confirms the results of the numerical solution. It must be pointed out that the computer-based (i.e. non-analytic) part of our argument is only used for the calculation of the numerical value of our bound. Everything else in the proof is absolutely rigorous. Moreover, our numerical technique stabilizes, i.e. refining even further the sweeping does not affect the first five digits of our bound. This approach of a rigorous proof plus numerical techniques for the final computations is often used in the literature, especially with respect to computations of lower bounds, where systems of differential equations are numerically solved to produce the final number (see, e.g., [9]). It has also been used in the seminal paper [8], where the first significant lowering of the naïve first-moment upper bound was exhibited.

As usual, *asymptotically almost surely* (a.a.s.) will mean with probability tending to 1 as $n \rightarrow \infty$. All asymptotic expressions as $1 - o(1)$ are always with respect to n .

2. Background and basic definitions

Consider a given set of n Boolean variables, and let $m = \lfloor \gamma n \rfloor$. Let $\mathcal{F}_{n,m}$ be the set of 3-CNF formulas with n variables and m clauses, where repetition of clauses or repetition of variables in a clause is not allowed. Without loss of generality, we also denote by $\mathcal{F}_{n,m}$ the probability space of formulas in $\mathcal{F}_{n,m}$ drawn with uniform probability. Throughout the paper, we fix the value $\gamma = 4.4898$, and prove for that value that a random 3-CNF formula is not satisfiable with high probability.

Throughout the paper, *scaled* will always mean divided by n , and a *scaled natural* will be a member of $\frac{1}{n}\mathbb{N}$. For a formula $\phi \in \mathcal{F}_{n,m}$, we define the following parameters which depend on ϕ : For any $i, j \in \mathbb{N}$, let $d_{i,j}$ be the scaled number of variables with i positive occurrences and j negative occurrences in ϕ . Then,

$$\sum_{i,j \in \mathbb{N}} d_{i,j} = 1. \tag{1}$$

The sequence $\mathbf{d} = (d_{i,j})_{i,j \in \mathbb{N}}$ is called the *degree sequence* of ϕ . The scaled number of clauses of ϕ is denoted by c , and can be expressed in terms of \mathbf{d} by

$$c(\mathbf{d}) = \frac{1}{3} \sum_{i,j \in \mathbb{N}} (i + j) d_{i,j}. \tag{2}$$

Note that if $\phi \in \mathcal{F}_{n,m}$, then c must additionally satisfy $c = \lfloor \gamma n \rfloor / n$.

Given $\epsilon_1 > 0$ and any sequence $\boldsymbol{\xi} = (\xi_{i,j})_{i,j \in \mathbb{N}}$ of non-negative reals with $\sum_{i,j \in \mathbb{N}} \xi_{i,j} = 1$, we define the following set of degree sequences:

$$\mathcal{N}(n, \boldsymbol{\xi}, \epsilon_1) = \left\{ \mathbf{d} = (d_{i,j})_{i,j \in \mathbb{N}} : \sum_{i,j \in \mathbb{N}} d_{i,j} = 1, \frac{n}{3} \sum_{i,j \in \mathbb{N}} (i + j) d_{i,j} \in \mathbb{N}, \forall i, j \in \mathbb{N} \quad d_{i,j} n \in \mathbb{N}, \right. \\ \left. \text{and } |d_{i,j} - \xi_{i,j}| \leq \epsilon_1, \text{ and if } i > n^{1/6} \text{ or } j > n^{1/6} \text{ then } d_{i,j} = 0 \right\}.$$

Intuitively $\mathcal{N}(n, \boldsymbol{\xi}, \epsilon_1)$ can be interpreted as the set of degree sequences \mathbf{d} which are close to the ideal sequence $\boldsymbol{\xi}$, which in general is not a degree sequence since its entries $\xi_{i,j}$ need not be scaled naturals. However, if n is large enough, then $\mathcal{N}(n, \boldsymbol{\xi}, \epsilon_1) \neq \emptyset$. Now we consider the 2D Poisson ideal sequence $\boldsymbol{\delta}$ defined by $\delta_{i,j} = e^{-3\gamma} (3\gamma/2)^{i+j} / (i!j!)$. The following lemma reflects the fact that almost all $\phi \in \mathcal{F}_{n,m}$ have a degree sequence \mathbf{d} which is close to $\boldsymbol{\delta}$. A proof of an analogous result can be found in [5] (the main difference is that we have to apply Markov's inequality in the end to show that there are no variables of degree $n^{1/6}$ or larger).

Lemma 2.1. *Let \mathbf{d} be the degree sequence of a random $\phi \in \mathcal{F}_{n,m}$. For any $\epsilon_1 > 0$, we have that $\Pr_{\mathcal{F}_{n,m}}(\mathbf{d} \in \mathcal{N}(n, \boldsymbol{\delta}, \epsilon_1)) = 1 - o(1)$.*

3. The configuration model for formulas

Given a fixed degree sequence $\mathbf{d} = (d_{i,j})_{i,j \in \mathbb{N}}$ satisfying (1) and such that $c = c(\mathbf{d})$ defined by (2) is also a scaled natural, we wish to generate 3-CNF formulas with that particular degree sequence \mathbf{d} . A natural approach to this is to use the *configuration model*. A *configuration* φ with degree sequence $\mathbf{d} = (d_{i,j})_{i,j \in \mathbb{N}}$ is constructed as follows: consider n variables

and the corresponding $2n$ literals $x_1, \bar{x}_1, \dots, x_n, \bar{x}_n$; each literal has a certain number of distinct labelled copies in a way that the scaled number of variables with i positive copies and j negative copies is $d_{i,j}$; then partition the set of copies into sets of size 3, which we call the clauses of φ . Let $\mathcal{C}_{n,\mathbf{d}}$ be the set of all configurations with degree sequence \mathbf{d} , and we also denote by $\mathcal{C}_{n,\mathbf{d}}$ the probability space on the set $\mathcal{C}_{n,\mathbf{d}}$ with the uniform distribution.

A 3-CNF *multi-formula* is a formula with possible repetition of variables in one clause and/or possible repetition of clauses. A *simple formula* is a formula in $\mathcal{F}_{n,m}$. Let π be the projection from $\mathcal{C}_{n,\mathbf{d}}$ to 3-CNF multi-formulas obtained by unlabelling the copies of each literal.

A configuration $\varphi \in \mathcal{C}_{n,\mathbf{d}}$ is satisfiable if $\phi = \pi(\varphi)$ is satisfiable. A configuration $\varphi \in \mathcal{C}_{n,\mathbf{d}}$ is *simple* iff $\phi = \pi(\varphi)$ is a simple formula, i.e. does not have repetition of variables or clauses. Notice that the number of anti-images of a simple formula ϕ with degree sequence \mathbf{d} under π does not depend on the particular choice of ϕ . Hence,

$$\Pr_{\mathcal{F}_{n,m}}(\phi \text{ is SAT} \mid \mathbf{d}) = \Pr_{\mathcal{C}_{n,\mathbf{d}}}(\varphi \text{ is SAT} \mid \text{SIMPLE}), \tag{3}$$

where \mathbf{d} also denotes the event that a random $\phi \in \mathcal{F}_{n,m}$ has degree sequence \mathbf{d} .

We need a lower bound on the probability that a configuration is simple. The following result gives a weak bound which is enough for our purposes.

Lemma 3.1. *Let $\epsilon_1 > 0$ and $\mathbf{d} \in \mathcal{N}(n, \delta, \epsilon_1)$. Then*

$$\Pr_{\mathcal{C}_{n,\mathbf{d}}}(\text{SIMPLE}) \geq e^{-\Theta(n^{1/3} \log n)},$$

where the $e^{-\Theta(n^{1/3} \log n)}$ bound is uniform for all $\mathbf{d} \in \mathcal{N}(n, \delta, \epsilon_1)$.

Proof. In order to prove the lemma, we give a lower bound on the number of simple configurations and then divide it by the total number of configurations. The total number of configurations is

$$\frac{(3cn)!}{6^{cn}(cn)!}$$

To obtain a lower bound on the number of simple configurations, we construct a subset of these. Consider the set of labelled copies of literals according to the degree sequence \mathbf{d} . Among those copies which correspond to variables with one unique occurrence, we select any $2\lceil 9n^{1/3} \rceil$ (recall that in $\mathcal{N}(n, \delta, \epsilon_1)$ there is a linear number of such variables). Call this set \mathcal{U}_0 . Let \mathcal{U} be the set of all the remaining copies. We pick an arbitrary order of the copies in \mathcal{U} and put them into a list. At each step, we remove the first copy of the list, match it with two other suitable copies in the list (which are also removed) and repeat until the list contains exactly $\lceil 9n^{1/3} \rceil$ labelled copies of literals. Let x be the variable corresponding to the first labelled copy of the list at some step of the procedure. If we want to avoid repetitions of variables in clauses then, when choosing the two other copies to be matched with that first one, we have to exclude at most $2n^{1/6}$ other copies of x (recall that in $\mathcal{N}(n, \delta, \epsilon_1)$ the degrees of all variables are at most $2n^{1/6}$). If we want to avoid multiple clauses, we have to exclude at most $8n^{1/3}$ copies of the variables which are already in some clause containing a copy of x . Therefore, at most $\lceil 9n^{1/3} \rceil$ copies are excluded in total at each choice. This gives rise to at least

$$\frac{(3cn - 3\lceil 9n^{1/3} \rceil)!}{6^{cn - \lceil 9n^{1/3} \rceil} (cn - \lceil 9n^{1/3} \rceil)!} \tag{4}$$

different ways of grouping the copies in \mathcal{U} into clauses except for the $\lceil 9n^{1/3} \rceil$ copies which remain in the list at the end of the procedure. Finally, each one of the $\lceil 9n^{1/3} \rceil$ remaining copies is matched arbitrarily with any two copies in \mathcal{U}_0 to create a clause, and this completes the construction of the configurations. The choice of \mathcal{U}_0 guarantees that all the configurations we obtained are simple. Hence, dividing (4) by the total number of configurations gives a lower bound of $e^{-\Theta(n^{1/3} \log n)}$. \square

4. Elimination of pure literals

Given $\varphi \in \mathcal{C}_{n,\mathbf{d}}$, a *pure* variable of φ is a variable which has a non-zero number of occurrences which are either all syntactically positive or all syntactically negative. The only literal occurring in φ and all its copies are also called *pure*.

Observe that if φ is satisfiable and x is a pure variable of φ , then there exists some satisfying truth assignment of φ which satisfies all copies of x in φ . Hence, in order to study the satisfiability of a configuration $\varphi \in \mathcal{C}_{n,\mathbf{d}}$, we are allowed to iteratively satisfy each pure variable in φ and remove all clauses containing a copy of that variable. For each $\varphi \in \mathcal{C}_{n,\mathbf{d}}$, let $\tilde{\varphi}$ be the configuration obtained by greedily removing all pure variables and their corresponding clauses from φ , in any arbitrary order. Notice that during this elimination procedure new pure variables are created, which will be eventually eliminated.

This $\tilde{\varphi}$ is independent of the particular elimination order of pure literals and is called the *impure core* of φ . In fact, in our analysis we will eliminate only one clause containing one copy of a pure literal at a time, as the $\tilde{\varphi}$ obtained remains the same. This fact can be proved in a completely analogous way to the standard argument that shows that the obtention of the k -core of a graph does not depend on how we remove the vertices of degree less than k .

Note that φ is satisfiable iff $\tilde{\varphi}$ is satisfiable, since we are allowed to choose a truth assignment which satisfies all literals iteratively removing during the process for obtaining $\tilde{\varphi}$. Moreover, if φ is simple then $\tilde{\varphi}$ is also simple, but the converse is not necessarily true.

Furthermore, let $\hat{\varphi}$ be the configuration obtained from $\tilde{\varphi}$ by positively unbalancing all variables, i.e. switching the syntactic sign of those variables having initially more negative than positive occurrences in $\tilde{\varphi}$. Let $\hat{\mathcal{C}}_{n,\mathbf{d}}$ denote the probability space of configurations $\hat{\varphi}$, where φ was chosen from $\mathcal{C}_{n,\mathbf{d}}$ with uniform probability. Note that the probability distribution

in $\widehat{\mathcal{C}}_{n,d}$ is not necessarily uniform. Since the simplicity and the satisfiability of a configuration are not affected by positively unbalancing the variables, we have

$$\Pr_{\mathcal{C}_{n,d}}(\varphi \text{ is SAT} \wedge \text{SIMPLE}) \leq \Pr_{\widehat{\mathcal{C}}_{n,d}}(\widehat{\varphi} \text{ is SAT} \wedge \text{SIMPLE}). \tag{5}$$

Let the random variable $\widehat{\mathbf{d}}$ be the degree sequence of a random configuration in $\widehat{\mathcal{C}}_{n,d}$. We prove in the following result that if the original \mathbf{d} is close to the ideal sequence δ , then with high probability $\widehat{\mathbf{d}}$ must be close to the ideal sequence $\widehat{\delta} = (\widehat{\delta}_{i,j})_{i,j \in \mathbb{N}}$ defined by

$$\widehat{\delta}_{i,j} = \begin{cases} 2e^{-3\gamma b \frac{(3\gamma b/2)^{i+j}}{ij!}}, & \text{if } i > j, \\ e^{-3\gamma b \frac{(3\gamma b/2)^{i+j}}{ij!}}, & \text{if } i = j, \\ 0, & \text{if } i < j, \end{cases}$$

where $b = (1 - t_{\mathcal{D}}/\gamma)^{2/3}$ and $t_{\mathcal{D}}$ is the scaled number of steps in the pure literal elimination algorithm.

Proposition 4.1. *Given $\epsilon_2 > 0$, there exist $\epsilon_1 > 0$ and $0 < \beta < 1$ such that for any $\mathbf{d} \in \mathcal{N}(n, \delta, \epsilon_1)$*

$$\Pr_{\widehat{\mathcal{C}}_{n,d}}(\widehat{\mathbf{d}} \in \mathcal{N}(n, \widehat{\delta}, \epsilon_2)) = 1 - O(\beta^{n^{1/2}}).$$

Moreover, for each $\widehat{\mathbf{d}} \in \mathcal{N}(n, \widehat{\delta}, \epsilon_2)$, the probability space $\widehat{\mathcal{C}}_{n,d}$ conditional upon having degree sequence $\widehat{\mathbf{d}}$ has the uniform distribution (i.e. $\widehat{\mathcal{C}}_{n,d}$ conditional upon a fixed $\widehat{\mathbf{d}}$ behaves exactly as $\mathcal{C}_{n,\widehat{\mathbf{d}}}$).

Proof. Given $\varphi \in \mathcal{C}_{n,d}$ with $\mathbf{d} \in \mathcal{N}(n, \delta, \epsilon_1)$, we apply the following pure literal elimination algorithm to φ : at each time step $t \geq 0$, one literal occurrence is chosen uniformly at random from all pure literal occurrences appearing in $\varphi = \varphi(t)$ at time t (if the variable of the chosen literal occurrence is not yet assigned a value, it is set such that this literal occurrence is satisfied) and the clause containing this occurrence is eliminated. The algorithm stops when there is no more pure literal occurrence left. The resulting configuration is denoted by $\widetilde{\varphi}$. It is easy to see that $\widetilde{\varphi}$ is unique and independent of the order of elimination of pure literal occurrences. Moreover, at any time step t , the algorithm retains the conditional randomness of the configuration, i.e., conditional under having any (fixed) degree sequence \mathbf{d} at time t , any configuration φ obeying this degree sequence and whose image $\phi = \pi(\varphi)$ is a simple formula, is equally likely to appear at time t in this algorithm. To analyze the expected degree distribution of $\widetilde{\varphi}$, we introduce the following variables: for any $0 \leq i, j \leq n$, $Y_{i,j}(t)$ denotes the random variable counting the number of variables with i positive and j negative occurrences at time t , and $Y_0(t)$ denotes the random variable counting the number of pure literal occurrences at time t , i.e., $Y_0(t) = \sum_{i>0} iY_{i,0}(t) + \sum_{j>0} jY_{0,j}(t)$. Also define by $L(t)$ the random variable counting the total number of literal occurrences at time t . Clearly, $L(t+1) - L(t) = -3$. Note also that $Y_0(t) = L(t) - \sum_{i,j>0} (i+j)Y_{i,j}(t)$. Defining by $G(t)$ the sequence of $Y_0(t), Y_{1,1}(t), \dots, Y_{n,n}(t)$, we have that for any t , for any $i, j > 0$ and conditional upon any values of $G(t)$

$$\begin{aligned} \mathbf{E}(Y_{i,j}(t+1) - Y_{i,j}(t) \mid G(t)) &= \frac{2}{L}((i+1)Y_{i+1,j}(t) + (j+1)Y_{i,j+1}(t) - (i+j)Y_{i,j}(t))(1 + o(1)), \\ \mathbf{E}(Y_0(t+1) - Y_0(t) \mid G(t)) &= \left(\frac{2}{L} \left(\sum_{i>0} iY_{i,1}(t) + \sum_{j>0} jY_{1,j}(t) - Y_0(t) \right) - 1 \right) (1 + o(1)). \end{aligned}$$

If we now interpolate the variables $L(t)$ and $Y_{ij}(t)$, by defining the scaled versions $y_{i,j}(t) = \frac{1}{n}Y_{i,j}(tn)$ for $i, j > 0$, $\ell(t) = \frac{1}{n}L(tn)$ and $y_0(t) = \frac{1}{n}Y_0(tn)$, (to understand the limiting behavior as n tends to infinity) this suggests the following system of differential equations:

$$\begin{aligned} \frac{d\ell}{dt} &= -3, \\ \frac{dy_{i,j}}{dt} &= \frac{2}{\ell}((i+1)y_{i+1,j} + (j+1)y_{i,j+1} - (i+j)y_{i,j}), \quad i, j > 0, \\ \frac{dy_0}{dt} &= \frac{2}{\ell} \left(\sum_{i>0} iy_{i,1} + \sum_{j>0} jy_{1,j} - y_0 \right) - 1, \end{aligned}$$

with the initial conditions $\ell(0) = 3\gamma$, $y_0(0) = \sum_{i>0} i d(i, 0) + \sum_{j>0} j d(0, j)$, and for $i, j > 0$ we have $y_{i,j}(0) = d(i, j)$, where $d(i, j)$ is the scaled number of variables of φ appearing i times positively and j times negatively. It can be easily seen that $\ell(t) = 3\gamma - 3t$. Furthermore, defining the auxiliary function $b(t) = (1 - t/\gamma)^{2/3}$, it can be checked, that a solution (its uniqueness will be proven below) for the system of differential equations involving $y_{i,j}$ is the following function:

$$y_{i,j}(t) = \sum_{k \geq i} \sum_{\ell \geq j} d(k, \ell) \binom{k}{i} \binom{\ell}{j} (b(t))^{i+j} (1 - b(t))^{k-i} (1 - b(t))^{\ell-j}.$$

Also, it follows then that

$$y_0(t) = 3\gamma - 3t - \sum_{i,j>0} (i + j)y_{i,j}(t).$$

We are interested in the smallest value of t for which $y_0(t/n) = 0$. This value of t will be the stopping time $T_{\mathcal{D}}$ of our process (for our particular system of differential equations, starting with $\varphi \in \mathcal{C}_{n,\mathbf{d}}$ with $\mathbf{d} \in \mathcal{N}(n, \delta, \epsilon_1)$, we empirically observed that $T_{\mathcal{D}}/n$ is between 0.15 and 0.16, and hence the resulting configuration $\tilde{\varphi}$ a.a.s. still contains many literal occurrences. This observation is used to simplify the statement of Wormald's theorem).

To prove the uniqueness of the solution and the concentration of the resulting distribution, we use the following version of Wormald's theorem tailored to our specific system of ODEs:

Theorem 4.2 (Wormald [18]). *Suppose that for any value of $G(t)$ and any $Y_{i,j}(t)$ (throughout the statement of the theorem $Y_{i,j}(t)$ also includes the case of $Y_0(t)$) we have*

$$\mathbf{E}[Y_{i,j}(t + 1) - Y_{i,j}(t) \mid G(t)] = f_{i,j}(t/n, \{Y_{i,j}(t)/n\}_{i,j}) + O(1/n),$$

where the $f_{i,j} : \mathbb{R}^a \rightarrow \mathbb{R}$ (a being equal to the number of variables $Y_{i,j}$ plus 1 for the variable t) are continuous functions with all Lipschitz constants uniformly bounded. Suppose also that $|Y_{i,j}(t + 1) - Y_{i,j}(t)| \leq B$, and suppose that there is an order of the functions $f_{i,j}$ such that any $f_{i,j}$ depends on at most the terms preceding it in this order (including itself). Then, for some large enough constant C and $0 < \alpha < 1/3$, the system of differential equations

$$dy_{i,j}(x)/dx = f_{i,j}(x, \{y_{i,j}(x)\})$$

with the initial conditions $y_{i,j}(0) = Y_{i,j}(0)/n$ has a unique solution and moreover, for any $t \leq T_{\mathcal{D}} - Cn^{1-\alpha}$, we have

$$Y_{i,j}(t) = ny_{i,j}(t/n) + O(n^{1-\alpha})$$

with probability at least $1 - O(n^{2+\alpha}e^{-n^{1-3\alpha}})$.

Defining the $f_{i,j}$'s to be the right-hand sides of the system of differential equations of the $y_{i,j}$, i.e.,

$$f_{i,j}(x, y_{i,j}, y_{i,j+1}, y_{i+1,j}) = \frac{2(-(i + j)y_{i,j} + (i + 1)y_{i+1,j} + (j + 1)y_{i,j+1})}{3\gamma - 3x},$$

and

$$f_0(x, y_0, y_{1,1}, y_{2,1}, \dots, y_{n,1}, y_{1,2}, \dots, y_{1,n}) = \frac{2}{3\gamma - 3x} \left(-y_0 + \sum_{i>0} iy_{i,1} + \sum_{j>0} jy_{1,j} \right) - 1,$$

it is easily seen that these functions are continuous and satisfy a uniformly bounded Lipschitz constant. Next, it is a routine calculation to show that for any $1 \leq i, j \leq n$

$$|\mathbf{E}(Y_{i,j}(t + 1) - Y_{i,j}(t) \mid G(t)) - f_{i,j}(t/n, Y_{i,j}(t/n), Y_{i,j+1}(t/n), Y_{i+1,j}(t/n))| = O(1/n),$$

and

$$|\mathbf{E}(Y_0(t + 1) - Y_0(t) \mid G(t)) - f_0(t/n, Y_0(t/n), \dots, Y_{1,n}(t/n))| = O(1/n).$$

Also, we have

$$\max_{1 \leq i,j \leq n} |Y_{i,j}(t + 1) - Y_{i,j}(t)| \leq 2 \quad \text{and} \quad |Y_0(t + 1) - Y_0(t)| \leq 3.$$

Moreover, we can rearrange the $f_{i,j}$'s in descending order of the sum of i and j (being f_0 the last element of this order), and hence we can apply Wormald's theorem. That is, for any $t \leq T_{\mathcal{D}} - Cn^{1-\alpha}$, we have for any $i, j > 0$

$$Y_{i,j}(t) = ny_{i,j}(t/n) + O(n^{1-\alpha})$$

and

$$Y_0(t) = ny_0(t/n) + O(n^{1-\alpha}),$$

with probability at least $1 - O(n^{2+\alpha}e^{-n^{1-3\alpha}})$. Since in each step Y_0 as well as $Y_{i,j}$ change by at most 3, the change of $Y_{i,j}$ and Y_0 caused by the eliminations of pure literal occurrences after time $T_{\mathcal{D}} - Cn^{1-\alpha}$ is at most $O(n^{1-\alpha})$, and thus the concentration results for $Y_{i,j}(t)$ and $Y_0(t)$ hold throughout the whole algorithm.

Now, since originally we start with $\varphi \in \mathcal{C}_{n,\mathbf{d}}$ and $\mathbf{d} \in \mathcal{N}(n, \delta, \epsilon_1)$, we have $|d(i, j) - \frac{e^{-3\gamma} (3\gamma/2)^{i+j}}{i!j!}| \leq \epsilon_1$ for any $i, j \in \mathbb{N}$. Since the functions $f_{i,j}$ and f_0 appearing in the system of ODEs are continuous and their Lipschitz constants are uniformly bounded, by the continuous dependence theorem (see e.g., [17], Theorem A.68) the solutions of the system of ODEs are continuous with respect to the initial conditions, uniformly for all n . That is, there exists an $\epsilon'_2 > 0$, such that the degree sequence $\tilde{\mathbf{d}}$ of $\tilde{\varphi}$ satisfies $|\tilde{d}(i, j) - \tilde{\delta}(i, j)| \leq \epsilon'_2$ with probability at least $1 - O(n^{2+\alpha}e^{-n^{1-3\alpha}})$, where (using the notation of b

and δ from above)

$$\tilde{\delta}(i, j) = \sum_{k \geq i} \sum_{\ell \geq j} \delta(k, \ell) \binom{k}{i} \binom{\ell}{j} (b(T_{\mathcal{D}}/n))^{i+j} (1 - b(T_{\mathcal{D}}/n))^{k-i} (1 - b(T_{\mathcal{D}}/n))^{\ell-j}.$$

When plugging in the initial condition $\delta(k, \ell) = e^{-3\gamma} \frac{(3\gamma/2)^{k+\ell}}{k!\ell!}$, the expression for $\tilde{\delta}(i, j)$ can be simplified and we obtain

$$\tilde{\delta}(i, j) = e^{-3\gamma b} \frac{(3\gamma b/2)^{i+j}}{i!j!},$$

where $b = b(t) = (1 - t/\gamma)^{2/3}$. Setting $\alpha = 1/6$ and unbalancing $\tilde{\varphi}$ (i.e., changing the syntactic sign of a variable appearing more often negatively than positively) we obtain a configuration $\hat{\varphi}$ whose degree sequence $\hat{\mathbf{d}}$ satisfies for all $i, j \in \mathbb{N}$

$$|\hat{\mathbf{d}}(i, j) - \tilde{\delta}(i, j)| \leq \epsilon_2$$

with probability at least $1 - O(e^{-\sqrt{n}})$, where $\epsilon_2 = 2\epsilon'_2$ and $\hat{\delta}(i, j)$ is defined as follows:

$$\hat{\delta}(i, j) = \begin{cases} 2\tilde{\delta}(i, j), & \text{if } i > j, \\ \tilde{\delta}(i, j), & \text{if } i = j, \\ 0, & \text{if } i < j. \end{cases}$$

Setting $\beta = 1/e$ concludes the proof of the proposition. \square

5. Clause typicality

Let $\hat{\mathbf{d}} \in \mathcal{N}(n, \hat{\delta}, \epsilon_2)$. Then, each $\varphi \in \mathcal{C}_{n, \hat{\mathbf{d}}}$ has a scaled number of clauses of $\hat{\mathbf{c}} = c(\hat{\mathbf{d}})$ (see (2)). Moreover, let ℓ_p and ℓ_n be the scaled number of copies in φ of positive and of negative literals respectively. Then

$$\ell_p(\hat{\mathbf{d}}) = \sum_{i, j \in \mathbb{N}} i \hat{d}_{i, j}, \quad \ell_n(\hat{\mathbf{d}}) = \sum_{i, j \in \mathbb{N}} j \hat{d}_{i, j}. \tag{6}$$

Given any fixed $\varphi \in \mathcal{C}_{n, \hat{\mathbf{d}}}$ and for $k \in \{0, \dots, 3\}$, let \hat{c}_k be the scaled number of clauses in φ containing exactly k positive clauses (clauses of syntactic type k). We call $\hat{\mathbf{c}} = (\hat{c}_0, \dots, \hat{c}_3)$ to the clause-type sequence of φ . By definition

$$\hat{c}_1 + 2\hat{c}_2 + 3\hat{c}_3 = \ell_p, \quad 3\hat{c}_0 + 2\hat{c}_1 + \hat{c}_2 = \ell_n, \tag{7}$$

and by adding the equations in (7), $\hat{c}_0 + \dots + \hat{c}_3 = \hat{c}$. The $\hat{c}_0, \dots, \hat{c}_3$ are random variables in $\mathcal{C}_{n, \hat{\mathbf{d}}}$, but the next result shows that if $\hat{\mathbf{d}}$ is close enough to $\hat{\delta}$, then $\hat{c}_0, \dots, \hat{c}_3$ as well as their sum $\hat{c}_0 + \dots + \hat{c}_3 = \hat{c}$ are concentrated with high probability. In order to see this, we need to define $\hat{\gamma} = c(\hat{\delta})$, $\lambda_p = \ell_p(\hat{\delta})$ and $\lambda_n = \ell_n(\hat{\delta})$ (see (2) and (6)), which can be interpreted as the limit of \hat{c} , ℓ_p and ℓ_n respectively when $\hat{\mathbf{d}}$ approaches $\hat{\delta}$. In terms of these numbers, we thus define for all $k \in \{0, \dots, 3\}$

$$\hat{\gamma}_k = \binom{3}{k} \frac{\lambda_p^k \lambda_n^{3-k}}{(\lambda_p + \lambda_n)^3} \hat{\gamma}, \tag{8}$$

and let $\hat{\gamma} = (\hat{\gamma}_0, \dots, \hat{\gamma}_3)$. Then we have $\hat{\gamma}_1 + 2\hat{\gamma}_2 + 3\hat{\gamma}_3 = \lambda_p$, $3\hat{\gamma}_0 + 2\hat{\gamma}_1 + \hat{\gamma}_2 = \lambda_n$ and $\hat{\gamma}_0 + \hat{\gamma}_1 + \hat{\gamma}_2 + \hat{\gamma}_3 = \hat{\gamma}$.

The next result shows that when $\hat{\mathbf{d}}$ is close enough to $\hat{\delta}$, then each \hat{c}_k is close to the corresponding $\hat{\gamma}_k$. Indeed, given $\epsilon > 0$ and for any $\hat{\mathbf{d}} \in \mathcal{N}(n, \hat{\delta}, \epsilon_2)$, let $\mathcal{C}_{n, \hat{\mathbf{d}}}^\epsilon$ be the set of all $\varphi \in \mathcal{C}_{n, \hat{\mathbf{d}}}$ such that for $k \in \{0, \dots, 3\}$, $|\hat{c}_k - \hat{\gamma}_k| \leq \epsilon$. We also denote by $\mathcal{C}_{n, \hat{\mathbf{d}}}^\epsilon$ the corresponding uniform probability space.

Lemma 5.1. *Given $\epsilon > 0$, there is $\epsilon_2 > 0$ and $0 < \beta < 1$ such that for any $\hat{\mathbf{d}} \in \mathcal{N}(n, \hat{\delta}, \epsilon_2)$,*

$$\Pr_{\mathcal{C}_{n, \hat{\mathbf{d}}}^\epsilon}(\mathcal{C}_{n, \hat{\mathbf{d}}}^\epsilon) = 1 - O(\beta^n).$$

Proof. Let us fix $k \in \{0, 1, 2, 3\}$. We will show that for a small enough $\epsilon_2 > 0$ the probability that $|\hat{c}_k - \hat{\gamma}_k| > \epsilon$ is $O(\beta^n)$ with $0 < \beta < 1$. Given a $\varphi \in \mathcal{C}_{n, \hat{\mathbf{d}}}$, we choose an ordering of the \hat{c}_n clauses uniformly at random. For each $t = 1, \dots, \hat{c}_n$, define the random variable Y_t to be 1 if the t th clause has syntactic type k and 0 otherwise. These random variables are all identically distributed, and we have that

$$\Pr(Y_t = 1) = \frac{\binom{\ell_p n}{k} \binom{\ell_n n}{3-k}}{\binom{\ell_p n + \ell_n n}{3}} = \binom{3}{k} \frac{[\ell_p n]_k [\ell_n n]_{3-k}}{[\ell_p n + \ell_n n]_3}, \quad \forall t, 1 \leq t \leq \hat{c}_n.$$

Let $S = \sum_{t=1}^{\hat{c}_n} Y_t$ and consider the following Doob martingale

$$S_i = \mathbf{E}(S \mid Y_1, \dots, Y_i), \quad \forall i, 0 \leq i \leq \hat{c}_n.$$

We have $S_{\hat{c}_n} = S$ and also $S = \hat{c}_k n$. Moreover,

$$S_0 = \mathbf{E}S = \binom{3}{k} \frac{[\ell_p n]_k [\ell_n n]_{3-k}}{[\ell_p n + \ell_n n]_3} \hat{c}_n \sim \binom{3}{k} \frac{\ell_p^k \ell_n^{3-k}}{(\ell_p + \ell_n)^3} \hat{c}_n.$$

Since $\hat{\mathbf{d}} \in \mathcal{N}(n, \hat{\delta}, \epsilon_2)$, by choosing ϵ_2 small enough, we can guarantee that \hat{c} is arbitrarily close to $\hat{\gamma}$ and thus

$$|S_0/n - \gamma_k| < \epsilon/2. \tag{9}$$

Now we show that $|S_i - S_{i+1}| \leq C$ for some constant $C > 0$, for any $i \in \{0, \dots, \widehat{cn} - 1\}$. For such a fixed i , define by $g = \widehat{cn} - i$ the number of clauses remaining at step i (i.e., when the value of Y_i but not yet Y_{i+1} is known), and denote by $3gp$ ($0 \leq p \leq 1$) the number of positive literals remaining (by $3gq$ the number of negative literals, respectively, where $q = 1 - p$) at step i . Furthermore, for such a fixed $i \geq 1$, we define $r \in \{0, \dots, 3\}$ to be the random variable indicating the syntactic type of the i th clause. Then,

$$|S_i - S_{i+1}| = \left| \frac{\binom{3gp}{k} \binom{3gq}{3-k}}{\binom{3g}{3}} g - \frac{\binom{3gp-r}{k} \binom{3gq-3+r}{3-k}}{\binom{3g-3}{3}} (g-1) + D \right|, \tag{10}$$

where $D = 1$ if $k = r$ and 0 otherwise. Since for our purposes we do not need to calculate the exact constant C , we ignore D . The right-hand side of (10) excluding D can be written as

$$\left| \frac{g(3gp)!(3gq)!3!(3g-3)! - (g-1)(3gp-r)!(3gq-3+r)!3!(3g-6)!3g(3g-1)(3g-2)F_1F_2F_3}{(3gp-k)!k!(3-k)!(3g)!(3gq-3+k)!} \right|, \tag{11}$$

where F_1, F_2 and F_3 are altogether 3 factors, such that in r of them the leading term is $3gp$, and in $3 - r$ of them the leading term is $3gq$. (11) can then be rewritten as

$$\left| \frac{1}{k!(3-k)!} \left(\frac{3!F_4F_5F_6}{3(3g-1)(3g-2)} - \frac{3!F_1F_2F_3G}{3(3g-4)(3g-5)} \right) \right|, \tag{12}$$

where F_4, F_5 and F_6 are altogether 3 factors, such that in k of them the leading term is $3gp$, and in $3 - k$ of them the leading term is $3gq$. G consists of $k - r$ factors which contain $3gp$ as leading term and $r - k$ factors which contain $3gq$ as leading term (negative values appear in the denominator). Hence, both terms of (12) contain a total of k factors containing $3gp$ as leading term, and $3 - k$ factors containing $3gq$ as a leading term. When bringing these terms to a common denominator, we obtain

$$\left| \frac{1}{k!(3-k)!} \left(\frac{3!F_4F_5F_6(3g-4)(3g-5) - 3!(3g-1)(3g-2)F_1F_2F_3G}{3(3g-1)(3g-2)(3g-4)(3g-5)} \right) \right|. \tag{13}$$

Observe that after expanding the products in the numerator the leading terms cancel out and the remaining terms are all $O(g^5)$. Since the denominator is clearly $\Omega(g^5)$, we obtain that $|S_i - S_{i+1}| \leq C$. Thus, by Azuma's inequality,

$$\Pr(|S_{\widehat{cn}} - S_0| \geq R) \leq e^{-\frac{R^2}{2C^2\widehat{cn}}}.$$

Setting $R = \epsilon n/2$, and by (9), we get

$$\Pr(|\widehat{c}_k - \gamma_k| \geq \epsilon) = \Pr(|S_{\widehat{cn}} - \gamma_k n| \geq \epsilon n) \leq \Pr(|S_{\widehat{cn}} - S_0| \geq \epsilon n/2) \leq e^{-\Theta(n)} = \Theta(\beta^n),$$

for some $0 < \beta < 1$. Taking a union bound over all k completes the proof of the lemma. \square

6. Counting simple and satisfiable configurations

All the previous results establish a connection between the uniform probability spaces $\mathcal{F}_{n,m}$ and $\mathcal{C}_{n,\widehat{a}}^\epsilon$. In order to prove Theorem 1.1, it remains to bound the probability that a configuration $\varphi \in \mathcal{C}_{n,\widehat{a}}^\epsilon$ is simple and satisfiable. More precisely, the goal of this section is to prove the following proposition.

Proposition 6.1. *There exist $\epsilon > 0$ and $0 < \beta < 1$ such that for any $\widehat{a} \in \mathcal{N}(n, \widehat{\delta}, \epsilon)$,*

$$\Pr_{\mathcal{C}_{n,\widehat{a}}^\epsilon}(\text{SAT} \wedge \text{SIMPLE}) = O(\beta^n).$$

We now build up the machinery for its proof. Let $\mathcal{N}(n, \widehat{\delta}, \widehat{\gamma}, \epsilon)$ be the set of tuples $(\widehat{a}, \widehat{c})$ such that $\widehat{a} \in \mathcal{N}(n, \widehat{\delta}, \epsilon)$ and $\widehat{c} = (\widehat{c}_k)_{0 \leq k \leq 3}$ is a tuple of scaled naturals satisfying (7) (recall also from (6) the definition of ℓ_p and ℓ_n), and moreover $|\widehat{c}_k - \widehat{\gamma}_k| \leq \epsilon$. For each $(\widehat{a}, \widehat{c}) \in \mathcal{N}(n, \widehat{\delta}, \widehat{\gamma}, \epsilon)$, we define $\mathcal{C}_{n,\widehat{a},\widehat{c}}$ to be the uniform probability space of all configurations with degree sequence \widehat{a} and clause-type sequence \widehat{c} . It suffices to show that for any $(\widehat{a}, \widehat{c}) \in \mathcal{N}(n, \widehat{\delta}, \widehat{\gamma}, \epsilon)$ we have

$$\Pr_{\mathcal{C}_{n,\widehat{a},\widehat{c}}}(\text{SAT} \wedge \text{SIMPLE}) = O(\beta^n).$$

Hence, we consider \widehat{a}, \widehat{c} and the probability space $\mathcal{C}_{n,\widehat{a},\widehat{c}}$ to be fixed throughout this section, and we try to find a suitable bound for $\Pr(\text{SAT} \wedge \text{SIMPLE})$.

We need some definitions. Let us fix any given configuration $\varphi \in \mathcal{C}_{n,\widehat{a},\widehat{c}}$. A *light* variable of φ is a variable with $i \leq M$ positive occurrences and $j \leq M$ negative occurrences in φ . We will use here and in the numerical calculations that follow the value $M = 23$. The other variables are called *heavy*. We consider a weaker notion of satisfiability in which heavy variables are treated as jokers and are always satisfied regardless of their sign in the formula and their assigned value. Given a configuration $\varphi \in \mathcal{C}_{n,\widehat{a},\widehat{c}}$ and a truth assignment A , we say that $A \models^b \varphi$ iff each clause of φ contains at least one heavy variable or at least one satisfied occurrence of a light variable. Let SAT^b be the set of configurations $\varphi \in \mathcal{C}_{n,\widehat{a},\widehat{c}}$ for which there exists at least one truth assignment A such that $A \models^b \varphi$. Clearly, if $A \models \varphi$, then also $A \models^b \varphi$, and hence $\text{SAT} \subset \text{SAT}^b$. We still introduce a further restriction to satisfiability in a way similar to [11,4], in order to decrease the number of satisfying truth assignments of each configuration without altering the set of simple satisfiable configurations. Given a configuration $\varphi \in \mathcal{C}_{n,\widehat{a},\widehat{c}}$ and a truth assignment A , we say that $A \models^{b'} \varphi$ iff $A \models^b \varphi$ and moreover each light variable which is assigned the value zero by A appears at least once as the only satisfied literal of a *blocking clause*, i.e. a clause with one satisfied negative

literal and two unsatisfied ones. Let $SAT^{b'}$ be the set of configurations which are satisfiable according to this latter notion. Notice that if $\varphi \in SIMPLE$, then $\varphi \in SAT^{b'}$ iff $\varphi \in SAT^b$, which can be proved by an argument analogous to the one in [11] and [4]. Therefore, we have

$$\Pr(SAT \wedge SIMPLE) \leq \Pr(SAT^b \wedge SIMPLE) = \Pr(SAT^{b'} \wedge SIMPLE) \leq \Pr(SAT^{b'}).$$

Let X be the random variable counting the number of satisfying truth assignments of a randomly chosen configuration $\varphi \in \mathcal{C}_{n,\hat{a},\hat{c}}$ in the $SAT^{b'}$ sense. We need to bound

$$\Pr(SAT^{b'}) = \Pr(X > 0) \leq EX = \frac{|\{(\varphi, A) : \varphi \in \mathcal{C}_{n,\hat{a},\hat{c}}, A \models^{b'} \varphi\}|}{|\mathcal{C}_{n,\hat{a},\hat{c}}|}. \tag{14}$$

In the following subsection, we obtain an exact but complicated expression for EX by a counting argument, and then we give a simple asymptotic bound which depends on the maximization of a particular continuous function over a bounded polytope. The next subsection contains the maximization of that function.

6.1. Asymptotic bound on EX

First, we compute the denominator of the rightmost member in (14).

$$\begin{aligned} |\mathcal{C}_{n,\hat{a},\hat{c}}| &= \binom{n}{(\hat{d}_{i,j}n)_{i,j}} \binom{\ell_p n}{\hat{c}_1 n, 2\hat{c}_2 n, 3\hat{c}_3 n} \binom{\ell_n n}{3\hat{c}_0 n, 2\hat{c}_1 n, \hat{c}_2 n} \frac{(3\hat{c}_0 n)!}{(\hat{c}_0 n)! 6^{60n}} \frac{(2\hat{c}_1 n)!}{2^{\hat{c}_1 n}} \frac{(2\hat{c}_2 n)!}{2^{\hat{c}_2 n}} \frac{(3\hat{c}_3 n)!}{(\hat{c}_3 n)! 6^{\hat{c}_3 n}} \\ &= \frac{n!}{\prod_{i,j} (\hat{d}_{i,j} n)!} \frac{(\ell_p n)! (\ell_n n)!}{2^{\hat{c}_1 n} 3^{(\hat{c}_0 + \hat{c}_3)n} (\hat{c}_0 n)! (\hat{c}_1 n)! (\hat{c}_2 n)! (\hat{c}_3 n)!}. \end{aligned}$$

In order to deal with the numerator in (14), we need some definitions. Let us consider any fixed $\varphi \in \mathcal{C}_{n,\hat{a},\hat{c}}$ and any assignment A such that $A \models^{b'} \varphi$. We will classify the variables, the clauses and the copies of literals in φ into several types, and define parameters counting the scaled number of items of each type. Variables are classified according to their degree. A variable is said to have degree (i, j) if it appears i times positively and j times negatively in φ . Let \mathcal{L} and \mathcal{H} , respectively, be the set of possible degrees for light and heavy variables.

$$\mathcal{L} = \{(i, j) \in \mathbb{N}^2 : 0 \leq i, j \leq M\}, \quad \mathcal{H} = \{(i, j) \in \mathbb{N}^2 : i > M \text{ or } j > M\}.$$

We also consider an extended notion of degree for light variables which are assigned 0 by A . One of such variables has extended degree (i, j, k) if it has degree (i, j) and among its j negative occurrences k appear in a blocking clause, being the only satisfied literal of the clause. Let

$$\mathcal{L}' = \{(i, j, k) \in \mathbb{N}^3 : 0 \leq i \leq M, 1 \leq k \leq j \leq M\},$$

be the set of possible extended degrees for these light 0-variables. For each $(i, j) \in \mathcal{L}$, let $t_{i,j}$ be the scaled number of light variables assigned 1 by A with degree (i, j) in φ . For each $(i, j, k) \in \mathcal{L}'$, let $f_{i,j,k}$ be the scaled number of light variables assigned 0 by A with extended degree (i, j, k) in φ . We must have

$$t_{i,j} + \sum_{k=1}^j f_{i,j,k} = \hat{d}_{i,j}, \quad \forall (i, j) \in \mathcal{L}. \tag{15}$$

On the other hand, we classify the copies of literals occurring in φ into five different types depending on their sign in φ , their assignment by A and whether they belong or not to a blocking clause. Each copy receives a label from the set

$$\mathcal{S} = \{\text{ps, ns1, ns2, pu, nu}\},$$

where the labels ps, pu, ns1, ns2 and nu denote positive-satisfied, positive-unsatisfied, negative-satisfied inside a blocking clause, negative-satisfied inside a non-blocking clause and negative-unsatisfied, respectively. It is useful to consider as well coarser classifications of the copies of literals in φ and thus we define the types p, n and ns which correspond to positive, negative and negative-satisfied copies, respectively. Also, let

$$\mathcal{S}' = \{\text{ps, ns, pu, nu}\} \quad \text{and} \quad \mathcal{S}'' = \{\text{p, n}\}.$$

For each of the types $\sigma \in \mathcal{S} \cup \mathcal{S}' \cup \mathcal{S}''$ that we defined, let ℓ_σ be the scaled number of copies of type σ . Note that ℓ_p and ℓ_n were already defined (see Eqs. (6) and (7)). Also, let h_σ be the scaled number of copies of type σ which come from heavy variables. Recall that these copies are always satisfied by definition regardless of their sign. In view of Eqs. (6) and (7), and of

$$h_{\text{ps}} = \sum_{\mathcal{H}} i \hat{d}_{i,j}, \quad h_{\text{ns}} = \sum_{\mathcal{H}} j \hat{d}_{i,j},$$

we observe that $\ell_p, \ell_n, h_{\text{ps}}$ and h_{ns} are constants which do not depend on the particular choice of (φ, A) . The parameters h_{ns1} and h_{ns2} depend on the particular (φ, A) and satisfy

$$h_{\text{ns1}} + h_{\text{ns2}} = h_{\text{ns}}. \tag{16}$$

The parameters ℓ_{ps} , ℓ_{pu} , ℓ_{ns1} , ℓ_{ns2} and ℓ_{nu} also depend on (φ, A) and can be expressed as

$$\begin{aligned} \ell_{ps} &= \sum_{\mathcal{L}} it_{i,j} + h_{ps}, & \ell_{pu} &= \sum_{\mathcal{L}'} if_{i,j,k}, & \ell_{ns1} &= \sum_{\mathcal{L}'} kf_{i,j,k} + h_{ns1}, \\ \ell_{ns2} &= \sum_{\mathcal{L}'} (j - k)f_{i,j,k} + h_{ns2}, & \ell_{nu} &= \sum_{\mathcal{L}} jt_{i,j}. \end{aligned} \tag{17}$$

Finally, the clauses of φ are classified into 16 extended types (not to be mistaken with the four syntactic types defined immediately before (7)). Each type is represented by a 2×2 matrix from the set

$$\mathcal{A} = \left\{ \alpha = \begin{pmatrix} ps(\alpha) & ns(\alpha) \\ pu(\alpha) & nu(\alpha) \end{pmatrix} \in \mathbb{N}^4 : \sum_{\sigma \in \mathcal{S}'} \sigma(\alpha) = 3, ps(\alpha) + ns(\alpha) > 0 \right\}.$$

A clause is said to be of extended type $\alpha = \begin{pmatrix} ps(\alpha) & ns(\alpha) \\ pu(\alpha) & nu(\alpha) \end{pmatrix}$ if for each $\sigma \in \mathcal{S}'$ the clause contains $\sigma(\alpha)$ copies of literals of type σ . Notice that all clauses of extended type α also contain the same number of copies of type σ for all other $\sigma \in \mathcal{S} \cup \mathcal{S}''$ and thus we can define c_α to be this number. For each $\alpha \in \mathcal{A}$, let c_α be the scaled number of clauses of extended type α (while \widehat{c}_k , $0 \leq k \leq 3$ is the number of clauses of syntactic type k , i.e. with k positive literals). We have

$$\sum_{\substack{\alpha \in \mathcal{A} \\ p(\alpha)=k}} c_\alpha = \widehat{c}_k. \tag{18}$$

The parameters ℓ_{ps} , ℓ_{pu} , ℓ_{ns1} , ℓ_{ns2} and ℓ_{nu} can also be expressed in terms of the c_α by

$$\ell_\sigma = \sum_{\alpha \in \mathcal{A}} \sigma(\alpha)c_\alpha, \quad \forall \sigma \in \mathcal{S}. \tag{19}$$

We now consider the following equations:

$$\ell_{ps} + \ell_{pu} = \ell_p, \quad \ell_{ns1} + \ell_{ns2} + \ell_{nu} = \ell_n \tag{20}$$

$$\ell_{ps} = \sum_{\mathcal{L}} it_{i,j} + h_{ps}, \quad \ell_{ns1} = \sum_{\mathcal{L}'} kf_{i,j,k} + h_{ns1}, \quad \ell_{ns2} = \sum_{\mathcal{L}'} (j - k)f_{i,j,k} + h_{ns2} \tag{21}$$

$$\ell_{ps} = \sum_{\alpha \in \mathcal{A}} ps(\alpha)c_\alpha, \quad \ell_{ns1} = \sum_{\alpha \in \mathcal{A}} ns1(\alpha)c_\alpha, \quad \ell_{ns2} = \sum_{\alpha \in \mathcal{A}} ns2(\alpha)c_\alpha. \tag{22}$$

In view of (6) and (7), the system of equations $\{(15), (16), (17), (18), (19)\}$ is equivalent to $\{(15), (16), (18), (20), (21), (22)\}$.

So far we verified that the constraints $\{(15), (16), (18), (20), (21), (22)\}$ express necessary conditions for the parameters of any particular (φ, A) , with $\varphi \in \mathcal{C}_{n, \widehat{\mathbf{d}}, \widehat{\mathbf{c}}}$ and $A \models^{br} \varphi$. Now we will see that they are also sufficient, in the sense that for each tuple of parameters satisfying the above-mentioned constraints we will be able to construct pairs (φ, A) .

Let $\bar{t} = (t_{i,j})_{\mathcal{L}}$, $\bar{f} = (f_{i,j,k})_{\mathcal{L}'}$, $\bar{h} = (h_{ns1}, h_{ns2})$, $\bar{c} = (c_\alpha)_{\alpha \in \mathcal{A}}$, $\bar{\ell} = (\ell_\sigma)_{\sigma \in \mathcal{S}}$ and

$$K = |\mathcal{L}| + |\mathcal{L}'| + 2 + |\mathcal{A}| + |\mathcal{S}| = (M + 1)^2(1 + M/2) + 23.$$

We define the bounded polytope $\mathcal{P}(\widehat{\mathbf{d}}, \widehat{\mathbf{c}}) \subset \mathbb{R}^K$ as the set of tuples $\bar{x} = (\bar{t}, \bar{f}, \bar{h}, \bar{c}, \bar{\ell})$ of non-negative reals satisfying $\{(15), (16), (18), (20), (21), (22)\}$, and consider the following set of lattice points in $\mathcal{P}(\widehat{\mathbf{d}}, \widehat{\mathbf{c}})$:

$$\mathcal{I}(n, \widehat{\mathbf{d}}, \widehat{\mathbf{c}}) = \mathcal{P}(\widehat{\mathbf{d}}, \widehat{\mathbf{c}}) \cap \left(\frac{1}{n} \mathbb{N} \right)^K.$$

For any tuple of parameters $\bar{x} \in \mathcal{I}(n, \widehat{\mathbf{d}}, \widehat{\mathbf{c}})$, we proceed to count the number of pairs (φ, A) , with $\varphi \in \mathcal{C}_{n, \widehat{\mathbf{d}}, \widehat{\mathbf{c}}}$ and $A \models^{br} \varphi$, satisfying these parameters. We denote this number by $T(\bar{x}, n, \widehat{\mathbf{d}}, \widehat{\mathbf{c}})$. In order to do the counting, we first consider the number of ways of assigning to each variable a truth value and a degree (and an extended degree as well for light 0-variables). This gives

$$2^{\sum_{\mathcal{H}} \widehat{d}_{i,j} n} \binom{n}{(t_{i,j} n)_{\mathcal{L}}, (f_{i,j,k} n)_{\mathcal{L}'}, (\widehat{d}_{i,j} n)_{\mathcal{H}}}.$$

Notice that the only constraints required at this stage for the parameters are those in (15). Then we have to choose for each light 0-variable which k of the j negatively satisfied copies contribute to blocking clauses, and also (taking into account (16)) which h_{ns1} of the h_{ns} negative copies of heavy variables contribute to blocking clauses:

$$\left(\prod_{\mathcal{L}'} \binom{j}{k}^{f_{i,j,k} n} \right) \binom{h_{ns} n}{h_{ns1} n, h_{ns2} n}.$$

So far, we have assigned labels in \mathcal{J} to all copies of literals in φ . This construction is compatible with (17). Now we have to decide which specific copies of each type $\sigma \in \mathcal{J}$ will be used in the construction of clauses of each extended type $\alpha \in \mathcal{A}$. The number of ways of doing that is

$$\prod_{\sigma \in \mathcal{J}} \binom{\ell_\sigma n}{(\sigma(\alpha) c_\alpha n)_{\alpha \in \mathcal{A}}},$$

where it is sufficient that the \bar{c} parameters satisfy (18) and that the $\bar{\ell}$ parameters are expressible as in (19). Finally, the number of ways of constructing c_α clauses of each extended type $\alpha \in \mathcal{A}$ is $\prod_{\alpha \in \mathcal{A}} W(\alpha)$, where

$$W(\alpha) = \frac{(w(\alpha) c_\alpha n)! (c_\alpha n)!^{2-w(\alpha)}}{(w(\alpha)!)^{c_\alpha n}} = \begin{cases} (c_\alpha n)!^2 & \text{if } w(\alpha) = 1, \\ \frac{(2c_\alpha n)!}{2^{c_\alpha n}} & \text{if } w(\alpha) = 2, \\ \frac{(3c_\alpha n)!}{(c_\alpha n)! 6^{c_\alpha n}} & \text{if } w(\alpha) = 3, \end{cases}$$

and $w(\alpha)$ is the number of 0's in the matrix α . Putting everything together, we have that

$$T(\bar{x}, n, \hat{\mathbf{d}}, \hat{\mathbf{c}}) = 2^{\sum_{\mathcal{H}} \hat{d}_{i,j} n} \binom{n}{(t_{i,j} n)_{\mathcal{L}}, (f_{i,j,k} n)_{\mathcal{L}'}, (\hat{d}_{i,j} n)_{\mathcal{H}}} \times \left(\prod_{\mathcal{L}'} \binom{j}{k}^{f_{i,j,k} n} \right) \binom{h_{ns} n}{h_{ns1} n, h_{ns2} n} \prod_{\sigma \in \mathcal{J}} \binom{\ell_\sigma n}{(\sigma(\alpha) c_\alpha n)_{\alpha \in \mathcal{A}}} \prod_{\alpha \in \mathcal{A}} W(\alpha).$$

Hence

$$EX = \frac{1}{|\mathcal{C}_{n, \hat{\mathbf{d}}, \hat{\mathbf{c}}}|} \sum_{\bar{x} \in \mathcal{I}(n, \hat{\mathbf{d}}, \hat{\mathbf{c}})} T(\bar{x}, n, \hat{\mathbf{d}}, \hat{\mathbf{c}}).$$

To characterize the asymptotic behavior of $T(\bar{x}, n, \hat{\mathbf{d}}, \hat{\mathbf{c}}) / |\mathcal{C}_{n, \hat{\mathbf{d}}, \hat{\mathbf{c}}}|$ with respect to n , we define

$$F(\bar{x}) = \frac{\prod_{\sigma \in \mathcal{J}} \ell_\sigma^{\ell_\sigma}}{\prod_{\mathcal{L}} t_{i,j}^{t_{i,j}} \prod_{\mathcal{L}'} \left(f_{i,j,k} / \binom{j}{k} \right)^{f_{i,j,k}} h_{ns1}^{h_{ns1}} h_{ns2}^{h_{ns2}} \prod_{\alpha \in \mathcal{A}} ((w(\alpha)! / 2) c_\alpha)^{c_\alpha}}$$

and

$$B(\hat{\mathbf{d}}, \hat{\mathbf{c}}) = 2^{\sum_{\mathcal{H}} \hat{d}_{i,j} h_{ns}} \prod_{\mathcal{L}} \hat{d}_{i,j} \frac{3^{c_0+c_3} c_0^{c_0} c_1^{c_1} c_2^{c_2} c_3^{c_3}}{\ell_p^{\ell_p} \ell_n^{\ell_n}}.$$

Then we use the following form of Stirling's inequality which holds for any $k \in \mathbb{N}$:

$$\sqrt{2\pi(k+1/8)}(k/e)^k \leq k! \leq \sqrt{2\pi(k+1/4)}(k/e)^k,$$

and we obtain

$$\frac{T(\bar{x}, n, \hat{\mathbf{d}}, \hat{\mathbf{c}})}{|\mathcal{C}_{n, \hat{\mathbf{d}}, \hat{\mathbf{c}}}|} \leq \text{poly}_1(n) (B(\hat{\mathbf{d}}, \hat{\mathbf{c}}) F(\bar{x}))^n,$$

where $\text{poly}_1(n)$ is some fixed polynomial in n which can be chosen to be independent of $\bar{x}, \hat{\mathbf{d}}$ and $\hat{\mathbf{c}}$ (as long as $\bar{x} \in \mathcal{I}(n, \hat{\mathbf{d}}, \hat{\mathbf{c}})$ and $(\hat{\mathbf{d}}, \hat{\mathbf{c}}) \in \mathcal{N}(n, \hat{\delta}, \hat{\gamma}, \epsilon)$). Moreover, since the size of $\mathcal{I}(n, \hat{\mathbf{d}}, \hat{\mathbf{c}})$ is also polynomial in n , we can write

$$EX \leq \text{poly}_2(n) \left(B(\hat{\mathbf{d}}, \hat{\mathbf{c}}) \max_{\bar{x} \in \mathcal{I}(n, \hat{\mathbf{d}}, \hat{\mathbf{c}})} F(\bar{x}) \right)^n \leq \text{poly}_2(n) \left(B(\hat{\mathbf{d}}, \hat{\mathbf{c}}) \max_{\bar{x} \in \mathcal{P}(n, \hat{\mathbf{d}}, \hat{\mathbf{c}})} F(\bar{x}) \right)^n,$$

for some other fixed polynomial $\text{poly}_2(n)$. By continuity, if we choose ϵ to be small enough, we can guarantee that

$$EX \leq \left((1 + 10^{-7}) B \max_{\bar{x} \in \mathcal{P}(n, \hat{\delta}, \hat{\gamma})} F(\bar{x}) \right)^n, \tag{23}$$

where (recall the definition in (8))

$$\begin{aligned} B &= B(\hat{\delta}, \hat{\gamma}) = 2^{\sum_{\mathcal{H}} \hat{d}_{i,j}} \left(\sum_{\mathcal{H}} j \hat{\delta}_{i,j} \right)^{\sum_{\mathcal{H}} j \hat{d}_{i,j}} \prod_{\mathcal{L}} \hat{\delta}_{i,j} \frac{3^{\hat{\gamma}_0+\hat{\gamma}_3} \hat{\gamma}_0^{\hat{\gamma}_0} \hat{\gamma}_1^{\hat{\gamma}_1} \hat{\gamma}_2^{\hat{\gamma}_2} \hat{\gamma}_3^{\hat{\gamma}_3}}{\lambda_p^{\lambda_p} \lambda_n^{\lambda_n}} \\ &= 2^{\sum_{\mathcal{H}} \hat{d}_{i,j}} \left(\sum_{\mathcal{H}} j \hat{\delta}_{i,j} \right)^{\sum_{\mathcal{H}} j \hat{d}_{i,j}} \frac{\prod_{\mathcal{L}} \hat{\delta}_{i,j}}{(3\hat{\gamma})^{2\hat{\gamma}}}. \end{aligned} \tag{24}$$

6.2. Maximization of $F(\bar{x})$

We wish to maximize F or equivalently $\log F$ over the domain $\mathcal{P}(n, \widehat{\delta}, \widehat{\gamma})$. We first show the following lemma:

Lemma 6.2. $F(\bar{x})$ does not maximize on the boundary of $\mathcal{P}(n, \widehat{\delta}, \widehat{\gamma})$.

Proof. We show the equivalent statement that $\log(F)$ does not maximize on the boundary of $\mathcal{P} = \mathcal{P}(n, \widehat{\delta}, \widehat{\gamma})$. Let \bar{x} be a fixed point on the boundary of \mathcal{P} . Since the boundary of \mathcal{P} consists only of non-negativity constraints of some parameters, this means that some values of $\ell_\sigma, t_{i,j}, f_{i,j,k}, c_\alpha, h_{ns1}$ or h_{ns2} of \bar{x} have to be zero. It turns out that the only ℓ_σ which can be zero is ℓ_{ns2} : for all other ℓ_σ one can consider the linear program which contains all constraints and as objective function the minimization of the particular ℓ_σ . Then, the value of the objective function is in all cases except for ℓ_{ns2} strictly greater than zero (this is shown using CPLEX). In the following we call all c_α for which $ns2(\alpha) > 0$, all $f_{i,j,k}$ with $M \geq j > k \geq 1$ as well as h_{ns2} (cf. (21) and (22)) to be forced. We now make a case analysis depending on which of the remaining parameters is zero.

Case 1: None of the ℓ_σ is equal to zero.

In this case we know that at least one of the $t_{i,j}, f_{i,j,k}, c_\alpha, h_{ns1}$ or h_{ns2} is equal to zero. First observe that the interior of \mathcal{P} contains at least one feasible point \bar{x}_0 : since all constraints are linear, one can add additional linear constraints to ensure that all values of $\ell_\sigma, t_{i,j}, f_{i,j,k}, c_\alpha, h_{ns1}$ and h_{ns2} are strictly positive and then check the feasibility of the constraints (we did this using CPLEX). Since \mathcal{P} is convex, we can find a direction that points towards \bar{x}_0 . In this direction all the $t_{i,j}, f_{i,j,k}, c_\alpha, h_{ns1}$ and h_{ns2} components which were zero must increase (perhaps at different rates), while the non-zero variables can either increase or decrease (and remain non-zero). Hence, the directional derivative in this direction contains at least one term which is $-\log(0)$ (i.e. $+\infty$) (plus/minus some constants) and hence it is an increasing direction.

Case 2: $\ell_{ns2} = 0$, and in addition to the forced values of $c_\alpha, f_{i,j,k}$ and h_{ns2} at least one of the other values of $c_\alpha, t_{i,j}, f_{i,j,k}$ or h_{ns1} is zero as well.

Using a similar argument as in the previous case, we can find a feasible point \bar{x}_1 for which $\ell_{ns2} = 0$, all forced parameters are zero, but no other parameters are zero (we did this using CPLEX). Again, by convexity of \mathcal{P} , we can find a direction that points towards \bar{x}_1 . In this direction ℓ_{ns2} and all the forced parameters remain zero, but all other $c_\alpha, t_{i,j}, f_{i,j,k}$ and h_{ns1} which were zero before increase (there is at least one of them). Hence, the directional derivative in this direction contains at least one term which is $-\log(0)$ (plus/minus some constants), and hence it is an increasing direction.

Case 3: $\ell_{ns2} = 0$, all the forced $c_\alpha, f_{i,j,k}$ and h_{ns2} are zero, but nothing else is zero.

In this case we move infinitesimally in the following direction: we increase ℓ_{ns2} and decrease ℓ_{ns1} by the same infinitesimally small amount $=: \zeta$, we increase c_α for $\alpha \in \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \right\}$ by ζ , we decrease c_α for $\alpha \in \left\{ \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ by ζ , and we also increase $f_{2,2,1}$ and h_{ns2} by $\zeta/2$ and decrease $f_{2,2,2}$ and h_{ns1} by $\zeta/2$ (note that since all values of $f_{i,j,k}, c_\alpha$ and h_{ns1} which are not forced are strictly positive, all the changes are allowed). It can be checked that when performing all these changes, all constraints remain valid, and by convexity of \mathcal{P} , we can move into this direction. In this direction the variables ℓ_{ns2}, c_α with $\alpha = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, f_{2,2,1}$ and h_{ns2} which were zero before move away from zero, all other zero variables remain zero, and all other non-zero variables remain non-zero. Hence, the directional derivative in this direction contains one term $\log(0)$ and three terms $-\log(0)$ (plus/minus some constants), and hence it is an increasing direction. \square

Since $\log F$ does not maximize on the boundary of its domain, the maximum must be attained at a critical point of $\log F$ in the interior of $\mathcal{P}(n, \widehat{\delta}, \widehat{\gamma})$. Next, we use the Lagrange multipliers technique to characterize each critical point of $\log F$ in terms of the solution of a 3×3 system.

In order to characterize the critical points of $\log F$ and the value of the maximum of F , we need some definitions. Let us consider the functions

$$\begin{aligned} \Psi_{i,j} &= v_{ps}^i + (v_{ns1} + v_{ns2})^j - v_{ns2}^j \quad \forall (i,j) \in \mathcal{L}, \\ \Psi_k &= \sum_{\substack{\alpha \in \mathcal{A} \\ p(\alpha)=k}} \frac{2}{w(\alpha)!} \mu_{ps}^{ps(\alpha)} \mu_{ns1}^{ns1(\alpha)} \mu_{ns2}^{ns2(\alpha)} \quad \forall k \in \{0, \dots, 3\}, \\ \Psi &= v_{ns1} + v_{ns2}, \quad \Psi_p = v_{ps} \mu_{ps} + 1, \quad \Psi_n = v_{ns1} \mu_{ns1} + v_{ns2} \mu_{ns2} + 1, \end{aligned}$$

defined on tuples $(\bar{v}, \bar{\mu}) = (v_{ps}, v_{ns1}, v_{ns2}, \mu_{ps}, \mu_{ns1}, \mu_{ns2})$ of positive reals, and let

$$L = v_{ps}^{h_{ps}} \left(\prod_{\mathcal{L}} \Psi_{i,j}^{\widehat{\delta}_{i,j}} \right) \Psi^{h_{ns}} \left(\prod_{k=0}^3 \Psi_k^{\widehat{\gamma}_k} \right) \Psi_p^{-\lambda_p} \Psi_n^{-\lambda_n}.$$

Lemma 6.3. There is a bijective correspondence between critical points of $\log F$ in the interior of $\mathcal{P}(n, \widehat{\delta}, \widehat{\gamma})$ and positive solutions of the 6×6 system of equations

$$\nabla_{\bar{v}, \bar{\mu}} (\log L) = 0. \tag{25}$$

Moreover, the value of F at each of these critical points can be expressed in terms of the corresponding solution by

$$v_{ps}^{h_{ps}} \left(\frac{\lambda_p}{\Psi_p}\right)^{\lambda_p} \left(\frac{\lambda_n}{\Psi_n}\right)^{\lambda_n} \prod_{\mathcal{L}} \left(\frac{\Psi_{i,j}}{\widehat{\delta}_{i,j}}\right)^{\widehat{\delta}_{i,j}} \left(\frac{\Psi}{h_{ns}}\right)^{h_{ns}} \prod_{k=0}^3 \left(\frac{\Psi_k}{\widehat{\gamma}_k}\right)^{\widehat{\gamma}_k}. \tag{26}$$

Proof. We use the Lagrange multipliers technique to characterize the critical points of $\log F$ in the interior of $\mathcal{P}(n, \widehat{\delta}, \widehat{\gamma})$. Let $(\rho'_{i,j})_{\mathcal{L}}, \rho', (\rho'_k)_{k \in \{0, \dots, 3\}}$ and (ρ'_p, ρ'_n) be the Lagrange multipliers of the equations in (15), (16), (18) and (20) respectively. These equations are called *partition constraints*. Moreover, let $(v'_{ps}, v'_{ns1}, v'_{ns2})$ and $(\mu'_{ps}, \mu'_{ns1}, \mu'_{ns2})$ be the Lagrange multipliers of the equations in (21) and (22) respectively. After a few manipulations, the standard Lagrange multiplier equations can be written as

$$\begin{aligned} t_{i,j} &= \rho_{i,j} v_{ps}^i \quad \forall (i,j) \in \mathcal{L}, & f_{i,j,k} &= \binom{j}{k} \rho_{i,j} v_{ns1}^k v_{ns2}^{j-k} \quad \forall (i,j,k) \in \mathcal{L}', \\ h_{ns1} &= \rho v_{ns1}, & h_{ns2} &= \rho v_{ns2}, \\ c_\alpha &= \frac{2}{w(\alpha)!} \rho_{p(\alpha)} \mu_{ps}^{ps(\alpha)} \mu_{ns1}^{ns1(\alpha)} \mu_{ns2}^{ns2(\alpha)} \quad \forall \alpha \in \mathcal{A}, \\ \ell_{ps} &= \rho_p v_{ps} \mu_{ps}, & \ell_{pu} &= \rho_p, & \ell_{ns1} &= \rho_n v_{ns1} \mu_{ns1} \\ \ell_{ns2} &= \rho_n v_{ns2} \mu_{ns2}, & \ell_{nu} &= \rho_n, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \rho_{i,j} &= e^{\rho'_{i,j}-1} \quad \forall (i,j) \in \mathcal{L}, & \rho &= e^{\rho'-1} \\ \rho_k &= e^{\rho'_k-1} \quad \forall k \in \{0, \dots, 3\}, & \rho_\sigma &= e^{-\rho'_\sigma-1} \quad \forall \sigma \in \{p, n\}, \\ v_\sigma &= e^{v'_\sigma} \quad \forall \sigma \in \{ps, ns1, ns2\}, & \mu_\sigma &= e^{\mu'_\sigma} \quad \forall \sigma \in \{ps, ns1, ns2\}. \end{aligned}$$

Notice that the new variables defined above must be strictly positive. By combining (27) with the partition constraints (15), (16), (18) and (20), the variables $(\rho_{i,j})_{\mathcal{L}}, \rho, (\rho_k)_{k \in \{0, \dots, 3\}}$ and (ρ_p, ρ_n) are eliminated and $(\bar{t}, \bar{f}, \bar{h}, \bar{c}, \bar{\ell})$ can be expressed just in terms of v_σ and μ_σ ($\sigma \in \{ps, ns1, ns2\}$):

$$\begin{aligned} t_{i,j} &= \widehat{\delta}_{i,j} \frac{v_{ps}^i}{\Psi_{i,j}} \quad \forall (i,j) \in \mathcal{L}, & f_{i,j,k} &= \widehat{\delta}_{i,j} \frac{\binom{j}{k} v_{ns1}^k v_{ns2}^{j-k}}{\Psi_{i,j}} \quad \forall (i,j,k) \in \mathcal{L}' \\ h_{ns1} &= h_{ns} \frac{v_{ns1}}{\Psi}, & h_{ns2} &= h_{ns} \frac{v_{ns2}}{\Psi}, \\ c_\alpha &= \widehat{\gamma}_{p(\alpha)} \frac{2}{w(\alpha)!} \mu_{ps}^{ps(\alpha)} \mu_{ns1}^{ns1(\alpha)} \mu_{ns2}^{ns2(\alpha)} \quad \forall \alpha \in \mathcal{A}, \\ \ell_{ps} &= \lambda_p \frac{v_{ps} \mu_{ps}}{\Psi_p}, & \ell_{pu} &= \lambda_p \frac{1}{\Psi_p}, & \ell_{ns1} &= \lambda_n \frac{v_{ns1} \mu_{ns1}}{\Psi_n}, \\ \ell_{ns2} &= \lambda_n \frac{v_{ns2} \mu_{ns2}}{\Psi_n}, & \ell_{nu} &= \lambda_n \frac{1}{\Psi_n}. \end{aligned} \tag{28}$$

Then we plug these expressions into the constraints (21) and (22) and after a few manipulations we obtain the following 6×6 system on variables v_σ, μ_σ

$$\begin{aligned} v_{ps} \left(\lambda_p \frac{\partial \Psi_p / \partial v_{ps}}{\Psi_p} \right) &= v_{ps} \left(\sum_{\mathcal{L}} \widehat{\delta}_{i,j} \frac{\partial \Psi_{i,j} / \partial v_{ps}}{\Psi_{i,j}} \right) + h_{ps}, \\ v_{ns1} \left(\lambda_n \frac{\partial \Psi_n / \partial v_{ns1}}{\Psi_n} \right) &= v_{ns1} \left(\sum_{\mathcal{L}} \widehat{\delta}_{i,j} \frac{\partial \Psi_{i,j} / \partial v_{ns1}}{\Psi_{i,j}} + h_{ns} \frac{\partial \Psi / \partial v_{ns1}}{\Psi} \right), \end{aligned} \tag{29a}$$

$$\begin{aligned} v_{ns2} \left(\lambda_n \frac{\partial \Psi_n / \partial v_{ns2}}{\Psi_n} \right) &= v_{ns2} \left(\sum_{\mathcal{L}} \widehat{\delta}_{i,j} \frac{\partial \Psi_{i,j} / \partial v_{ns2}}{\Psi_{i,j}} + h_{ns} \frac{\partial \Psi / \partial v_{ns2}}{\Psi} \right), \\ \mu_{ps} \left(\lambda_p \frac{\partial \Psi_p / \partial \mu_{ps}}{\Psi_p} \right) &= \mu_{ps} \left(\sum_{k=0}^3 \widehat{\gamma}_k \frac{\partial \Psi_k / \partial \mu_{ps}}{\Psi_k} \right), \\ \mu_{ns1} \left(\lambda_n \frac{\partial \Psi_n / \partial \mu_{ns1}}{\Psi_n} \right) &= \mu_{ns1} \left(\sum_{k=0}^3 \widehat{\gamma}_k \frac{\partial \Psi_k / \partial \mu_{ns1}}{\Psi_k} \right), \\ \mu_{ns2} \left(\lambda_n \frac{\partial \Psi_n / \partial \mu_{ns2}}{\Psi_n} \right) &= \mu_{ns2} \left(\sum_{k=0}^3 \widehat{\gamma}_k \frac{\partial \Psi_k / \partial \mu_{ns2}}{\Psi_k} \right), \end{aligned} \tag{29b}$$

which can be rewritten as in (25). (26) then follows by direct substitution. \square

By restricting our attention to (29b), we observe that we can easily express v_{ps}, v_{ns1} and v_{ns2} in terms of μ_{ps}, μ_{ns1} and μ_{ns2} . By plugging this expression into (29a), we obtain a 3×3 system in terms of the variables μ_{ps}, μ_{ns1} and μ_{ns2} (recall that

only positive solutions of the μ_σ which lead to positive ν_σ are considered). We numerically solved this 3×3 system with the help of Maple, using 50 digits of precision and $M = 23$. The unique solution obtained by Maple is

$$\begin{aligned} \mu_{ps} &\approx 1.9972796, & \mu_{ns1} &\approx 0.45029358, & \mu_{ns2} &\approx 0.33794030, \\ \nu_{ps} &\approx 1.2782018, & \nu_{ns1} &\approx 0.33277280, & \nu_{ns2} &\approx 0.95336927. \end{aligned}$$

In view of (23) and Lemma 6.3, we evaluate (26) at these values and multiply this by B given in (24), and we obtain the bound

$$EX \leq ((1 + 10^{-7})0.9999998965)^n. \tag{30}$$

Once we have proved the system does not maximized at the boundary and characterized the critical points, we go into solving numerically the system with the help of Maple, which finds just one solution. We express the maximum of F over $\mathcal{P}(n, \hat{\delta}, \hat{\nu})$ in terms of this solution, and multiply it by B given in (24), and from (23) we obtain the bound

$$EX \leq ((1 + 10^{-7})0.9999998965)^n, \tag{31}$$

which concludes the proof of Proposition 6.1, since $(1 + 10^{-7})0.9999998965 < 1$.

Note that the validity of our approach relies on the assumption that the solution of the 3×3 system found by Maple is unique, which implies that the critical point of $\log F$ we found is indeed the global maximum (if an alternative solution exists it could happen that at the corresponding critical point the function F attains a value greater than the maximum obtained).

In order to be more certain about the correctness of (31) we performed the following alternative experiment: Let $\mathcal{P}_{\bar{\ell}}$ be the polytope obtained by restricting $\mathcal{P}(n, \hat{\delta}, \hat{\nu})$ to the coordinates $l_{ps}, l_{pu}, l_{ns1}, l_{ns2}, l_{nu}$. Observe that this is a 3D polytope in \mathbb{R}^5 , since its elements are determined by the values of the coordinates l_{ps}, l_{ns1}, l_{ns2} . We performed a sweep over this polytope by considering a grid of 100 equispaced points in each of the three dimensions. For each of the 100^3 fixed tuples of $(l_{ps}, l_{ns1}, l_{ns2})$ which correspond to the points on the grid, we determine the remaining two coordinates of $\mathcal{P}_{\bar{\ell}}$, and maximize $\log F$ restricted to those fixed values of $\bar{\ell}$. Observe that in this case $\log F$ is strictly concave and thus has a unique maximum which can be efficiently found by some iterative variant of Newton algorithm. We checked, again using Maple, that the value obtained for each fixed tuple of $\bar{\ell}$ is below the maximum in (31).

Therefore, through this section we have proved Proposition 6.1.

7. Proof of Theorem 1.1

Choose $\epsilon, \epsilon_2, \epsilon_1 > 0$ and $0 < \beta < 1$ which satisfy the statements of Proposition 4.1, Lemma 5.1 and Proposition 6.1. We can assume that $\epsilon_1 < \epsilon_2 < \epsilon$. Let $\mathcal{N}_1 = \mathcal{N}(n, \hat{\delta}, \epsilon_2) \subset \mathcal{N}(n, \hat{\delta}, \epsilon)$. Using Lemma 5.1 and Proposition 6.1, for any $\hat{\mathbf{d}} \in \mathcal{N}_1$

$$\Pr_{\widehat{\mathcal{C}}_{n,\hat{\mathbf{d}}}}(\text{SAT} \wedge \text{SIMPLE}) \leq \Pr_{\widehat{\mathcal{C}}_{n,\hat{\mathbf{d}}}}^{\epsilon}(\text{SAT} \wedge \text{SIMPLE}) + O(\beta^n) = O(\beta^n). \tag{32}$$

Let $\mathcal{N}_2 = \mathcal{N}(n, \hat{\delta}, \epsilon_1)$, and consider any given $\mathbf{d} \in \mathcal{N}_2$. Then from (5), (32) and Proposition 4.1,

$$\begin{aligned} \Pr_{\mathcal{C}_{n,\mathbf{d}}}(\text{SAT} \wedge \text{SIMPLE}) &\leq \Pr_{\widehat{\mathcal{C}}_{n,\mathbf{d}}}(\text{SAT} \wedge \text{SIMPLE}) \\ &= \Pr_{\widehat{\mathcal{C}}_{n,\mathbf{d}}}(\hat{\mathbf{d}} \notin \mathcal{N}_1) + \sum_{\hat{\mathbf{d}} \in \mathcal{N}_1} \Pr_{\widehat{\mathcal{C}}_{n,\hat{\mathbf{d}}}}(\text{SAT} \wedge \text{SIMPLE}) \Pr_{\widehat{\mathcal{C}}_{n,\mathbf{d}}}(\hat{\mathbf{d}}) \\ &\leq O(\beta^{n^{1/2}}) + O(\beta^n) \Pr_{\widehat{\mathcal{C}}_{n,\mathbf{d}}}(\hat{\mathbf{d}} \in \mathcal{N}_1) = O(\beta^{n^{1/2}}), \end{aligned} \tag{33}$$

where the events $(\hat{\mathbf{d}})$ and $(\hat{\mathbf{d}} \in \mathcal{N}_1)$ on the probability space $\widehat{\mathcal{C}}_{n,\mathbf{d}}$ denote that the degree sequence is $\hat{\mathbf{d}}$ and the degree sequence is in \mathcal{N}_1 . Finally, from (3), Lemmas 2.1, 3.1 and (33) we get

$$\begin{aligned} \Pr_{\mathcal{F}_{n,m}}(\text{SAT}) &= \Pr_{\mathcal{F}_{n,m}}(\mathbf{d} \notin \mathcal{N}_2) + \sum_{\mathbf{d} \in \mathcal{N}_2} \Pr_{\mathcal{C}_{n,\mathbf{d}}}(\text{SAT} \mid \text{SIMPLE}) \Pr_{\mathcal{F}_{n,m}}(\mathbf{d}) \\ &= o(1) + \sum_{\mathbf{d} \in \mathcal{N}_2} \frac{\Pr_{\mathcal{C}_{n,\mathbf{d}}}(\text{SAT} \wedge \text{SIMPLE})}{\Pr_{\mathcal{C}_{n,\mathbf{d}}}(\text{SIMPLE})} \Pr_{\mathcal{F}_{n,m}}(\mathbf{d}) \\ &\leq o(1) + e^{\Theta(n^{1/3} \log n)} \sum_{\mathbf{d} \in \mathcal{N}_2} \Pr_{\mathcal{C}_{n,\mathbf{d}}}(\text{SAT} \wedge \text{SIMPLE}) \Pr_{\mathcal{F}_{n,m}}(\mathbf{d}) \\ &\leq o(1) + e^{\Theta(n^{1/3} \log n)} O(\beta^{n^{1/2}}) \Pr_{\mathcal{F}_{n,m}}(\mathbf{d} \in \mathcal{N}_2) = o(1), \end{aligned}$$

where (\mathbf{d}) and $(\mathbf{d} \in \mathcal{N}_2)$ denote respectively the event that a random formula $\phi \in \mathcal{F}_{n,m}$ has degree sequence \mathbf{d} and the event that the degree sequence of ϕ belongs to \mathcal{N}_2 .

8. Conclusion and further work

In this paper, we presented an improvement to the existing upper bound on the conjectured 3-SAT-threshold of 4.27. We believe that further improvements based on thinning the space of satisfying assignments will have to deal with very

complicated mathematics. In fact, we first tried the same counting approach to analyze double flips instead of single flips, but we were unable to handle the resulting function.

One promising direction is the recent paper by Maneva and Sinclair [13], mentioned in the introduction. Their approach to work with non-trivial core assignments instead of satisfying assignments could make the mathematics a lot more feasible. However, to obtain completely rigorous results using this method, additional work is needed.

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