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Aspects of random graphs

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Chapter 1

Introduction

The present report aims at giving a survey of my work since the end of my PhD thesis "Spectral Methods for Reconstruction Problems". Since then I focussed on the analysis of properties of different models of random graphs as well as their connection to real-world networks. This report's goal is to capture these problems in a common framework. The very last chapter of this thesis about results in bootstrap percolation is different in the sense that the given graph is deterministic and only the decision of being active for each vertex is probabilistic; since the proof techniques resemble very much results on random graphs, we decided to include them as well. We start with an overview of the five random graph models, and with the description of bootstrap percolation corresponding to the last chapter. Some properties of these models are then analyzed in the different parts of this thesis.

1.1 The binomial random graph $\mathcal{G}(n,p)$

The *binomial random graph* $\mathcal{G}(n,p)$ is defined as the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where Ω is the set of all graphs with vertex set $\{1,2,\ldots,n\}$, $\mathcal{F}$ is the family of all subsets of Ω, and $\mathbb{P}$ is the probability measure on $(\Omega,\mathcal{F})$ defined by

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{n}{2}-|E(G)|}.$$

This space may be viewed as the set of outcomes of $\binom{n}{2}$ independent coin flips, one for each pair (u,v) of vertices, where the probability of success (that is, adding edge uv) is p. Note that $p = p(n)$ may (and usually does) tend to zero as n tends to infinity. We often write $\mathcal{G}(n,p)$ when we mean a graph drawn from the distribution $\mathcal{G}(n,p)$. This model was historically the first random graph model introduced by [74] and independently by Erdős and Rényi in [64], and it is also known as the Erdős-Rényi random graph model. Many properties of this model are known, some of which are explained below when needed. For further reference, we refer to the two basic references about this model, which are the two monographs [27] and [94].

1.2 The random geometric graph $\mathscr{G}(n, r)$

Random geometric graphs were first introduced in a slightly different setting by Gilbert [75] to model the communications between radio stations. Since then, several closely related variants on these graphs have been widely used as a model for wireless communication, and have also been extensively studied from a mathematical point of view. Given a positive integer n, and a non-negative real r, we consider a random geometric graph $G \in \mathscr{G}(n, r)$ defined as follows. The vertex set V of G is obtained by choosing n points independently and uniformly at random (u.a.r.) in the square $\mathcal{S}_n = [-\sqrt{n}/2, \sqrt{n}/2]^2$ or in the unit square $[0, 1]^2$ (clearly, the two models are equivalent, and all results from one model scale to the other; we mention in each of the sections below which of the two squares is used). Note that in either case, with probability 1, no point in $\mathcal{S}_n$ is chosen more than once, and thus we assume $|V| = n$. For notational purposes, for $G \in \mathscr{G}(n, r)$, we identify each vertex $v \in V$ with its corresponding geometric position $v = (v_x, v_y) \in \mathcal{S}_n$ (or in $[0, 1]^2$), where v_x and v_y denote the usual x- and y-coordinates in $\mathcal{S}_n$. Finally, the edge set of $G \in \mathscr{G}(n, r)$ is constructed by connecting each pair of vertices u and v by an edge if and only if $d_E(u, v) \leq r$, where d_E denotes the Euclidean distance in $\mathcal{S}_n$ (or in $[0, 1]^2$). Some properties of this model are mentioned below; for further reference we refer to the monograph by Penrose [140], and also to the recent survey of Walters [165].

1.3 Random hyperbolic graphs $G_{\alpha,C}(n)$

This random graph model is more recent. Motivated by the applications of real-world networks, Krioukov et. al. [107] (see also [137]) introduced the model of random hyperbolic graphs. This model replicates many of the characteristic properties that are observed in real world networks (e.g., power law degree distributions, high clustering and small diameter), but on the other hand it is also susceptible to mathematical analysis. In fact, the authors of [137] argued only empirically and via some non-rigorous methods that random hyperbolic graphs have many of the desired properties. Actually, Boguñá, Papadopoulos and Krioukov [26] computed explicitly a maximum likelihood fit of the Internet graph, convincingly illustrating that this model is adequate for reproducing the structure of real networks with high accuracy. A bit later, however, Gugelmann, Panagiotou, and Peter [80], initiated also the rigorous study of the degree distribution, the clustering coefficient of random hyperbolic graphs, and very recently, other parameters have been studied as well (see for example [68, 38, 23]). In words, the random hyperbolic graph model is a simple variant of the uniform distribution of n vertices within a disc of radius R of the hyperbolic plane, where two vertices are connected if their hyperbolic distance is at most R. More formally, the random hyperbolic graph model $\mathcal{G}_{\alpha,C}(n)$ is defined in [80] as described next: for $\alpha > \frac{1}{2}$, $C \in \mathbb{R}$, $n \in \mathbb{N}$, set $R = 2 \ln n + C$, and build $G = (V, E)$ with vertex set $V = [n] = \{1, \ldots, n\}$ as follows:

- For each $v \in V$, polar coordinates (r_v, θ_v) are generated identically and independently distributed with joint density function $f(r, \theta)$, with θ_v chosen uniformly at random in

the interval $[0, 2\pi)$ and r_v with density

$$f(r) = \begin{cases} \dfrac{\alpha \sinh(\alpha r)}{C(\alpha, R)}, & \text{if } 0 \le r < R, \\ 0, & \text{otherwise,} \end{cases}$$

where $C(\alpha, R) = \cosh(\alpha R) - 1$ is a normalization constant.

- For $u, v \in V$, $u \neq v$, there is an edge with endpoints $u = (r, \theta)$ and $v = (r', \theta')$ provided $d(r, r', \theta - \theta') \le R$, where $d = d(r, r', \theta - \theta')$ denotes the hyperbolic distance between two vertices whose native representation polar coordinates are (r, θ) and (r', θ'), obtained by solving

$$\cosh(d) \ = \cosh(r)\cosh(r') - \sinh(r)\sinh(r')\cos(\theta - \theta').$$

1.4 The uniform random model in a restricted class of graphs

A natural random graph model is also the following: among all *labelled* graphs on the same set of vertices $\{1, \ldots, n\}$ with a certain property (say, having no cycle), choose one uniformly at random. When choosing without restriction, this model is clearly equivalent to $\mathcal{G}(n, p)$ with $p = \frac{1}{2}$, but when imposing a restriction, the model adds different flexibilities. We will study in this thesis one parameter for the uniform random graph model of trees on $\{1, \ldots, n\}$, of forests of $\{1, \ldots, n\}$, and a different parameter for the uniform random graph model on graphs not containing a cycle greater than a given length $k \ge 3$. The most studied model is clearly the uniform random graph model for trees and forests. Since the model is very different from the previous ones, different proof techniques are used, such as double counting or methods from analytic combinatorics. For further reference we refer to the monograph of [58].

1.5 Random directed acyclic graphs

In contrast to the previous random graph models, this model studied here is a model on directed graphs. Similar models as studied here were introduced for undirected graphs as graphs with a given degree sequence w by [45], and for directed graphs in [42]. Our model, based on the paper [30], is the following: there are two parameters, $n \in \mathbb{N}$, and an infinite sequence $w = (w_1, w_2, \ldots)$ of nonnegative integers that may be a function of n. The first layer, that is, the source, consists of one vertex, $L_0 = \{v\}$. The next layers are recursively defined. For the inductive hypothesis, suppose that all layers up to including layer k are created, and let us label all vertices of those layers. In particular, $L_k = \{v_{d_{k-1}+1}, v_{d_{k-1}+2}, \ldots, v_{d_k}\}$, where $d_k = \sum_{i=0}^{k} |L_i|$. We would like the vertices of L_k to have a total degree with distribution $\vec{w} = (w_{d_{k-1}+1}, w_{d_{k-1}+2}, \ldots, w_{d_k})$. It may happen that a vertex $v_i \in L_k$ has an indegree $deg^-(v_i)$ already larger than w_i, and there is no hope to obtain a total degree of w_i. If this is not the case, then the requirement can be easily fulfilled. As a result, $\vec{w}$, the desired degree

distribution, will serve as a (deterministic) lower bound for the actual degree distribution we obtain during the (random) process: let S be a new set of vertices of cardinality n. All directed edges that are created at this time-step will be from the layer L_k to a random subset of S that will form a new layer L_{k+1}. Each vertex $v_i \in L_k$ generates $\max\{w_i - deg^-(v_i), 0\}$ random directed edges from v_i to S. Therefore, we generate $e_k = \sum_{v_i \in L_k} \max\{w_i - deg^-(v_i), 0\}$ random edges at this time-step. The destination of each edge is chosen uniformly at random from S. All edges are generated independently, and so we perform e_k independent experiments. The set of vertices of S that were chosen at least once forms a new layer L_{k+1}. Note that it can happen that two parallel edges are created during this process. However, this is a rare situation for sparse random graphs we investigate here. We look at two different degree sequences: *random regular DAGs*, in which case $w_i = d$ for all $i \in \mathbb{N}$, and $d \geq 3$, as well as *random power law DAGs*: in this case, given three parameters $\beta > 2$, $d > 0$, and $0 < \alpha < 1$, set $M = n^\alpha$, $i_0 = n\left(\frac{d}{M}\frac{\beta-2}{\beta-1}\right)^{\beta-1}$ and $c = (\frac{\beta-2}{\beta-1})dn^{1/(\beta-1)}$. The degree distribution is then given by $w_i = c(i_0 + i - 1)^{-\frac{1}{\beta-1}}$. Note that in this case the degree sequence w_i is decreasing and follows a power law with exponent β. From the same observation it follows that the maximum value is $w_1 = ci_0^{-\frac{1}{\beta-1}} = M$.

1.6 Bootstrap percolation

Given a graph $G = (V, E)$, a set $A \subseteq V$, and $j \in \mathbb{N}$, the *bootstrap percolation* process $\mathbb{B}_j(G; A)$ is defined as follows: initially, a vertex $v \in V$ is *active* if $v \in A$, and *inactive* otherwise. Then, at each round, each inactive vertex becomes active if it has at least j active neighbors. The process keeps going until it reaches a stationary state in which every inactive vertex has less than j active neighbors. We call this the *final state* of the process. Note that we may slow down the process by delaying the activation of some vertices, but the final state is invariant. If G is a d-regular graph, then there is a natural characterization of the final state in terms of the k-core (i.e., the largest subgraph of minimum degree at least k): the set of inactive vertices in the final state of $\mathbb{B}_j(G; A)$ is precisely the vertex set of the $(d - j + 1)$-core of the subgraph of G induced by the initial set of inactive vertices $V \setminus A$ (see e.g. [91]). We say that $\mathbb{B}_j(G; A)$ *disseminates* if all vertices are active in the final state.

Define $\mathbb{B}_j(G; q)$ to be the same bootstrap percolation process, where the set of initially active vertices is chosen at random: each $v \in V$ is initially active with probability q, independently from all other vertices. This process (which can be regarded as a type of cellular automaton on graphs) was introduced in 1979 by Chalupa, Leath and Reich [40] on the grid $\mathbb{Z}^m$ as a simple model of dynamics of ferromagnetism, and has been widely studied ever since on many families of deterministic or random graphs. Aside from its mathematical interest, bootstrap percolation was extensively studied by physicists: it was used to describe complex phenomena in jamming transitions [163], magnetic systems [152] and neuronal activity [162], and also in the context of stochastic Ising models [66]. For more applications of bootstrap percolation, see the survey [2] and the references therein.

Notation. The properties of all random graph models are usually investigated from

an asymptotic perspective, as n grows to infinity. All asymptotics throughout are as $n \to \infty$. Unless otherwise mentioned, we use the following standard notation for the asymptotic behavior of sequences of not necessarily positive numbers[1] a_n and b_n: $a_n = O(b_n)$ if $\limsup_{n\to\infty} |a_n|/|b_n| \leq C < +\infty$; $a_n = \Omega(b_n)$ if $b_n = O(a_n)$; $a_n = \Theta(b_n)$ if $a_n = O(b_n)$ and $a_n = \Omega(b_n)$; $a_n = o(b_n)$ if $\lim_{n\to\infty} |a_n|/|b_n| = 0$, and $a_n = \omega(b_n)$ if $b_n = o(a_n)$. We also use ω without argument to denote any function tending with n to infinity, usually sufficiently slowly. We say that an event in a probability space holds *asymptotically almost surely* (or *a.a.s.*), if the probability that it holds tends to 1 as n goes to infinity. Finally, for simplicity, we will write $f(n) \sim g(n)$ if $f(n)/g(n) \to 1$ as $n \to \infty$ (that is, when $f(n) = (1 + o(1))g(n)$). All logarithms throughout this thesis are natural logarithms.

[1]We nevertheless, when an asymptotic expression is naturally negative, we may write $f(n) - o(g(n))$ to underline that something has to be subtracted from $f(n)$.

1.7 Presented papers

The results in this thesis are based on the following papers:

1. J. Barré, M. Lelarge, D. Mitsche: *On rigidity, orientability and cores of random graphs with sliders*. Preprint available at http://math.unice.fr/~dmitsche/Publications/publications.html.

2. B. Bollobás, D. Mitsche, P. Prałat: *Metric dimension for random graphs*. Electronic Journal of Combinatorics 20 (4) (2013), P1.

3. A. Bonato, M. Lozier, D. Mitsche, X. Pérez-Giménez, P. Prałat: *The domination number of on-line social networks and random geometric graphs*. TAMC 2015, 150-163.

4. A. Bonato, D. Mitsche, P. Prałat: *Vertex-pursuit in random directed acyclic graphs*. SIAM Journal on Discrete Mathematics 27 (2013), 732-756.

5. J. Díaz, D. Mitsche, G. Perarnau, X. Pérez-Giménez: *On the relation between graph distance and Euclidean distance in random geometric graphs*. Advances in Applied Probability 48.3 (2016), to appear.

6. J. Díaz, D. Mitsche, X. Pérez-Giménez: *Large connectivity for dynamic random geometric graphs*. IEEE Transactions on Mobile Computing 8 (6) (2009), 821-835.

7. J. Díaz, D. Mitsche, X. Pérez-Giménez: *On the probability of the existence of fixed-size components in random geometric graphs*. Advances in Applied Probability 41.2 (2009), 344-357.

8. J. Díaz, D. Mitsche, X. Pérez-Giménez: *Sharp Threshold for Hamiltonicity of Random Geometric Graphs*. SIAM Journal on Discrete Mathematics 21 (1) (2007), 1157-1165.

9. A. Frieze, D. Mitsche, X. Pérez-Giménez, P. Prałat: *On-line list colouring of random graphs*. Electronic Journal of Combinatorics 22 (2) (2015), P2.41.

10. O. Giménez, D. Mitsche, M. Noy: *Maximum degree in minor-closed classes of graphs*, European Journal of Combinatorics 55 (2016), 41-61.

11. W. Kinnersley, D. Mitsche, P. Prałat: *A note on the acquaintance time of random graphs*. Electronic Journal of Combinatorics 20 (3) (2013), P52.

12. M. Kiwi, D. Mitsche: *A bound for the diameter of random hyperbolic graphs*. ANALCO 2015, 26-39.

13. D. Mitsche, G. Perarnau: *On the treewidth and related parameters of Random Geometric Graphs*. STACS 2012, 408-419. Also: SIAM Journal on Discrete Mathematics, accepted subject to minor revision.

14. D. Mitsche, X. Pérez-Giménez, P. Prałat: *Strong-majority bootstrap percolation on regular graphs with low dissemination threshold*. Preprint available at http://math.unice.fr/~dmitsche/Publications/publications.html.

15. D. Mitsche, P. Prałat: *On the hyperbolicity of random graphs.* Electronic Journal of Combinatorics 21 (2) (2014), P2.39.

16. D. Mitsche, P. Prałat: *Revolutionaries and spies on random graphs.* Combinatorics, Probability and Computing 22 (2013), 417-432.

17. D. Mitsche, J. Rué: *On the limiting distribution of the metric dimension for random forests.* European Journal of Combinatorics 49 (2015), 68-89.

Chapter 2

The binomial random graph $\mathcal{G}(n,p)$

In this chapter we state our results on parameters analyzed in the model $\mathcal{G}(n,p)$. Despite its limitations for applications in real-world networks, this model is from the mathematical point of view the most analyzed and most understood one, and we investigated quite a few properties in this model. In particular, in this chapter we describe our results regarding the following properties: hyperbolicity (based on the paper [123]), acquaintance time (based on [102]), metric dimension (based for the dense case on [30] and for the sparse case on [125]), the game of revolutionaries and spies (see [124]), on-line list coloring (see [72]), and rigidity at the existence of sliders (see [19]). All results in this chapter are asymptotic results as $n \to \infty$, under some restrictions on the edge probability p, depending on the parameter investigated.

2.1 Hyperbolicity

This section is based on [123]. Hyperbolicity is a property of metric spaces that generalizes the idea of negatively curved spaces like the classical hyperbolic space or Riemannian manifolds of negative sectional curvature (see, for example, [7, 78]). Moreover, this concept can be applied to discrete structures such as trees and Cayley graphs of many finitely generated groups. The study of properties of Gromov's hyperbolic spaces from a theoretical point of view is a topic of recent and increasing interest in graph theory and computer science. Informally, in graph theory hyperbolicity measures how similar a given graph is to a tree—trees have hyperbolicity zero and graphs that are "tree-like" have "small" hyperbolicity. Formally, a connected graph $G = (V, E)$ is *δ-hyperbolic*, if for every four vertices $u, v, x, y \in V$, the two largest values in the set

$$\{d(u,v) + d(x,y), d(u,x) + d(v,y), d(u,y) + d(v,x)\}$$

differ by at most 2δ. The *hyperbolicity* of G, denoted by $\delta_H(G)$, is the smallest δ for which this property holds.

In our paper [123] we investigate the hyperbolicity for binomial random graphs. Surprisingly, previous to our paper this important graph parameter was not well investigated for random graphs: in [133], sparse random graphs ($p = c/n$ for some real number $c > 1$) are analyzed. It was shown that $\mathcal{G}(n,p)$ is, with positive probability, not δ-hyperbolic for any

positive δ. Before our results, nothing was known before for $p \gg n^{-1}$. On the other hand, it is known that for a random d-regular graph G, for $d \geq 3$, we have that a.a.s.

$$\frac{1}{2}\log_{d-1} n - \omega \leq \delta_H(G) \leq \frac{1}{2}\log_{d-1} n + O(\log\log n),$$

where ω is any function tending to infinity together with n. (In fact, almost geodesic cycles are investigated in [20], and this is an easy consequence of this result.) The hyperbolicity of the class of Kleinberg's small-world random graphs is investigated in [44].

Our contribution is the following result.

Theorem 2.1.1. *Let $G \in \mathcal{G}(n,p)$.*
Suppose first that

$$d = (n-1)p \gg \frac{\log^5 n}{(\log\log n)^2} \quad \text{and} \quad p = 1-\varepsilon \text{ with } \varepsilon \gg 1/n^2.$$

Let ω be a function tending to infinity with n sufficiently slowly. Let $j = j(n) \geq 2$ be a sequence of integers such that $d^j/n - 2\log n \geq \omega$ and $d^{j-1}/n - 2\log n < \omega$. Then, the following properties hold a.a.s.

(i) If j is even and $d^{j-1}/n \leq \frac{1}{16}\log n$, then $\delta_H(G) = j/2$.

(ii) If j is even, $d^{j-1}/n > \frac{1}{16}\log n$ (but still $d^{j-1}/n - 2\log n < \omega$), then

$$j/2 - 1 \leq \delta_H(G) \leq j/2.$$

(iii) j is odd, then $\delta_H(G) = (j-1)/2$.

Furthermore, the following complementary results hold.

(iv) For $p = 1 - 2c/n^2$ for some constant $c > 0$, a.a.s. $\delta_H(G) \in \{0, 1/2, 1\}$. More precisely,

$$\begin{aligned} \mathbb{P}(\delta_H(G) = 0) &= (1+o(1))e^{-c}, \\ \mathbb{P}(\delta_H(G) = 1/2) &= (1+o(1))ce^{-c}, \text{ and} \\ \mathbb{P}(\delta_H(G) = 1) &= (1+o(1))\left(1-(c+1)e^{-c}\right). \end{aligned}$$

(v) For $p = 1 - o(1/n^2)$, a.a.s. $\delta_H(G) = 0$.

Remark 2.1.2. *Even though the value of j depends on the value of ω, the final result only depends on ω in the sense that the smaller ω, the exacter the result. For example, if j is even and say $d^j/n - 2\log n = 2\omega$, then we are in case (i), and so we get $\delta_H(G) = j/2$. If we change ω to say 4ω, j will be replaced by $j+1$, we are then in case (iii) and get $\delta_H(G) = ((j+1)-1)/2 = j/2$. If on the other hand j is odd and $d^j/n - 2\log n = 2\omega$, we are in case (iii) and obtain $\delta_H(G) = (j-1)/2$. If we change again ω to 4ω, j will be replaced by $j+1$, we are in case (ii) and get $(j+1)/2 - 1 = (j-1)/2 \leq \delta_H(G) \leq (j+1)/2$, and hence the result is less exact than before.*

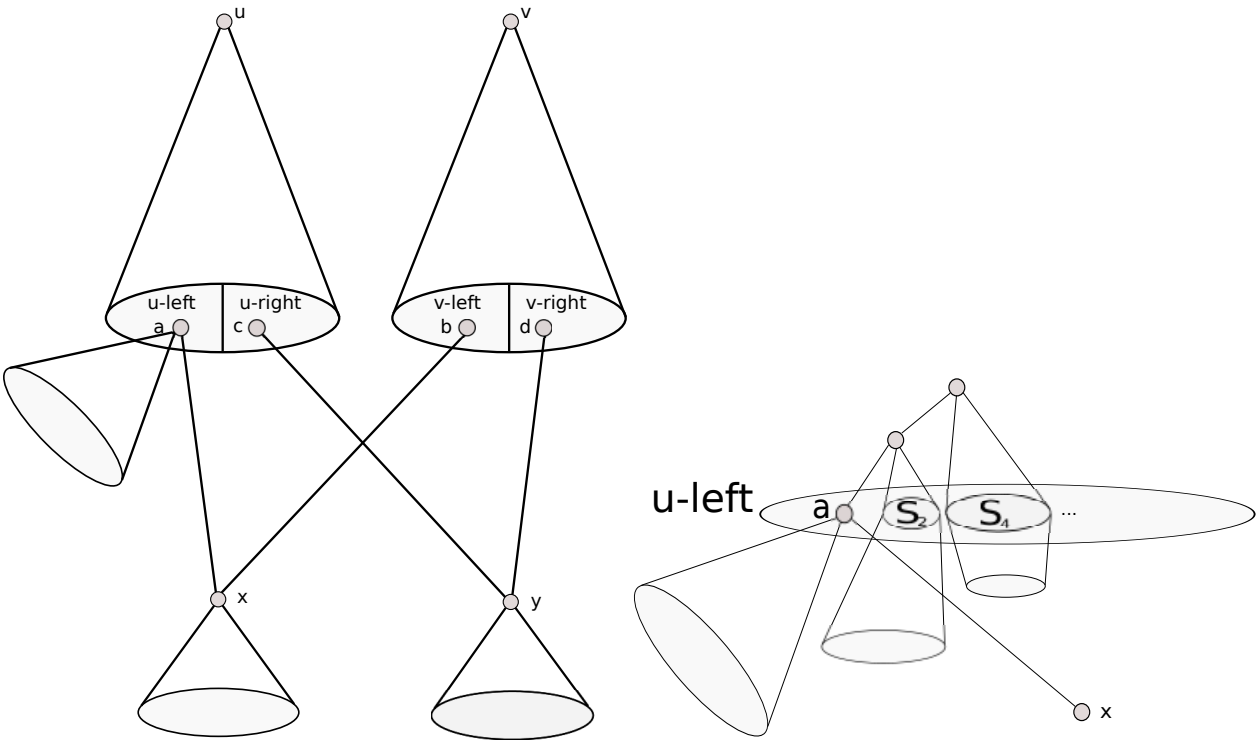


Figure 2.1: Left: HYPERBOLICITY(u, v, x, y), the big picture, Right: the neighborhood exposure around a in more detail

Parts (i), (ii) and (iii) of Theorem 2.1.1 follow easily from the following result (parts (iv) and (v) are comparably easier):

Theorem 2.1.3. *Suppose that*

$$d = (n-1)p \gg \frac{\log^5 n}{(\log\log n)^2} \qquad \textit{and} \qquad p = 1 - \omega(1/n^2).$$

Let $i = i(n) \geq 2$ be the largest even integer such that $d^{i-1} \leq \frac{1}{16} n \log n$. Let $G \in \mathcal{G}(n,p)$. Then, a.a.s., $\delta_H(G) \geq i/2$.

The proof of Theorem 2.1.3 is constructive, we provide here just the strategy without giving its correctness proof. Fix a four-tuple of different vertices u, v, x, y and consider the following process (see Figure 2.1):
HYPERBOLICITY(u, v, x, y)

1. Let $B = \{v, x, y\}$. Perform Breadth-First Search (BFS) from u in the graph induced by $V \setminus B$ to expose $N_{V\setminus B}(u, i/2-1)$. Make sure that the following properties hold (otherwise stop the process):

   (a) $N_{V\setminus B}(u, i/2-1)$ expands well.

   (b) The graph induced by $N_{V\setminus B}(u, i/2-1)$ is a tree.

   (c) There is no edge from $N_{V\setminus B}(u, i/2-2)$ to $\{v, x, y\}$.

(As a result, $N(u, i/2-1) = N_{V\setminus B}(u, i/2-1)$ and so $N(u, i/2-1)$ expands well, $N(u, i/2-1)$ is a tree, and $\{v, x, y\} \cap N(u, i/2-1) = \emptyset$.)

2. Let $D = N(u, i/2-1) \cup \{x, y\}$. Perform BFS from v in the graph induced by $V \setminus D$ to expose $N_{V\setminus D}(v, i/2-1)$. (The reason that here we restrict ourselves to the induced graph is to make sure no edge in this graph is already exposed and so, as typical, we perform BFS by exposing edges one by one, as required.) Make sure that the following properties hold (otherwise stop):

   (d) $N_{V\setminus D}(v, i/2-1)$ expands well.

   (e) There is no edge from $N_{V\setminus D}(v, i/2-1)$ to $S(u, i/2-1)$ (note that edges from vertices of $N(u, i/2-2)$ are already exposed, so that the only chance for the intersection of $N_{V\setminus D}(v, i/2-1)$ and $N(u, i/2-1)$ to be non-empty is when we reach vertices of $S(u, i/2-1)$).

   (f) The graph induced by $N_{V\setminus D}(v, i/2-1)$ is a tree.

   (g) There is no edge from $N_{V\setminus D}(v, i/2-2)$ to $\{x, y\}$.

   (As a result, $N(v, i/2-1) = N_{V\setminus D}(v, i/2-1)$ and $N(v, i/2-1)$ so expands well, $N(v, i/2-1) \cap N(u, i/2-1) = \emptyset$, $N(v, i/2-1)$ is a tree, and $\{x, y\} \cap N(v, i/2-1) = \emptyset$.)

3. Let us partition (arbitrarily) the neighbors of u into two sets, U_L and U_R, such that $||U_L| - |U_R|| \leq 1$. Let us partition the vertices of $S(u, i/2-1)$ and call vertices that are at distance $i/2-2$ from U_L to be 'u-left'; otherwise, they are called 'u-right'. Similarly, the vertices of $S(v, i/2-1)$ are partitioned into v-left and v-right ones. Expose the edges between x and $S(u, i/2-1) \cup S(v, i/2-1)$ and similarly also between y and $S(u, i/2-1) \cup S(v, i/2-1)$. Make sure that the following properties hold (otherwise stop):

   (h) The number of u-left vertices is $\frac{d^{i/2-1}}{2}(1 + o(\log^{-1} n))$, and the number of u-right vertices is also $\frac{d^{i/2-1}}{2}(1 + o(\log^{-1} n))$.

   (i) The number of v-left vertices is $\frac{d^{i/2-1}}{2}(1 + o(\log^{-1} n))$, and the number of v-right vertices is also $\frac{d^{i/2-1}}{2}(1 + o(\log^{-1} n))$.

   (j) There is exactly one edge between x and the u-left vertices; call the corresponding neighbor of x to be a. There is no edge between x and the u-right vertices.

   (k) There is exactly one edge between x and the v-left vertices; call the corresponding neighbor of x to be b. There is no edge between x and the v-right vertices.

   (l) There is exactly one edge between y and the u-right vertices; call the corresponding neighbor of y to be c. There is no edge between y and the u-left vertices.

   (m) There is exactly one edge between y and the v-right vertices; call the corresponding neighbor of y to be d. There is no edge between y and the v-left vertices.

4. In this step, the neighborhood of a is investigated. Unfortunately, this is slightly more complicated since some part of the neighborhood of a is already "buried" in

$N(u, i/2-1)$. In order to accomplish our goal, we need to perform BFS not only from a (up to level $i/2-2$), but also from some other vertices of $S(u, i/2-1)$ (this time going not as deep as $i/2-2$; the level until which the neighborhood is explored depends on the distance from a)—see Figure 2.1 (right side).

Formally, for $1 \leq k \leq i/2-2$, let S_k be the set of vertices of $S(u, i/2-1)$ that are at distance k from a in the tree induced by $N(u, i/2-1)$. (In fact, k has to be even in order for S_k to be non-empty, but we consider all values of k for simplicity.) Let

$$F = \left(N(u, i/2-1) \setminus \left(\{a\} \cup \bigcup_{k=1}^{i/2-2} S_k \right) \right) \cup N(v, i/2-1) \cup \{x\} \cup \{y\}.$$

We perform BFS from a and from vertices of $\bigcup_{k=1}^{i/2-2} S_k$ in the graph induced by $V \setminus F$; we reach vertices at distance $i/2-2$ from a and at distance $i/2-2-k$ from S_k. Make sure that the following properties hold (otherwise stop).

(n) $N_{V\setminus F}(a, i/2-2)$ expands well. Moreover, for all $1 \leq k \leq i/2-2$ and all $\ell \in S_k$ we have that $N(\ell, i/2-2-k)$ expands well. In particular,

$$|N_{V\setminus F}(a, i/2-2)| = d^{i/2-2}(1 + o(\log^{-1} n))$$
$$\sum_{k=1}^{i/2-2} \sum_{\ell \in S_k} |N_{V\setminus F}(\ell, i/2-2-k)| = o(d^{i/2-2} \log^{-1} n).$$

(o) There is no edge from $N_{V\setminus F}(a, i/2-2) \setminus \{a\}$ to F and for every $k = 1, 2, \ldots, i/2-2$ and every vertex $\ell \in S_k$, there is no edge from $N_{V\setminus F}(\ell, i/2-2-k) \setminus \{\ell\}$ to F.

(p) All graphs exposed in this step are disjoint trees. Note that this implies that $N(a, i/2-2)$ is a tree.

(q) For $1 \leq k \leq i/2-2$, let S'_k be the set of vertices of $S(v, i/2-1)$ that are at distance k from b in the tree induced by $N(v, i/2-1)$ and let

$$F' = \left(N(v, i/2-1) \setminus \left(\{b\} \cup \bigcup_{k=1}^{i/2-2} S'_k \right) \right) \cup N(u, i/2-1) \cup \{x\} \cup \{y\} \cup N(a, i/2-2).$$

Perform BFS from b and from vertices of $\bigcup_{k=1}^{i/2-2} S'_k$ in the graph induced by $V \setminus F'$; Properties **(n)**, **(o)**, and **(p)** hold when a is replaced by b, F is replaced by F', and the sets S_k are replaced by S'_k.

(r) For $1 \leq k \leq i/2-2$, let S''_k be the set of vertices of $S(u, i/2-1)$ that are at distance k from c in the tree induced by $N(u, i/2-1)$ and let

$$F'' = \left(N(u, i/2-1) \setminus \left(\{c\} \cup \bigcup_{k=1}^{i/2-2} S''_k \right) \right) \cup N(v, i/2-1) \cup \{x\} \cup \{y\} \cup N(a, i/2-2) \cup N(b, i/2-2).$$

Perform BFS from c and from vertices of $\bigcup_{k=1}^{i/2-2} S''_k$ in the graph induced by $V \setminus F''$; Properties **(n)**, **(o)**, and **(p)** hold when a is replaced by c, F is replaced by F'', and the sets S_k are replaced by S''_k.

(s) For $1 \leq k \leq i/2-2$, let S'''_k be the set of vertices of $S(v, i/2-1)$ that are at distance k from d in the tree induced by $N(v, i/2-1)$ and let

$$F''' = \left(N(v, i/2-1) \setminus \left(\{d\} \cup \bigcup_{k=1}^{i/2-2} S'''_k \right) \right) \cup N(u, i/2-1) \cup \{x\} \cup \{y\} \cup N(a, i/2-2) \cup N(b, i/2-2) \cup N(c, i/2-2).$$

Perform BFS from d and from vertices of $\bigcup_{k=1}^{i/2-2} S'''_k$ in the graph induced by $V \setminus F'''$; Properties **(n)**, **(o)**, and **(p)** hold when a is replaced by d, F is replaced by F''', and the sets S_k are replaced by S'''_k.

5. Let

$$Q = N(u, i/2-1) \cup N(v, i/2-1) \cup \{y\} \cup N(a, i/2-2) \cup N(b, i/2-2) \cup N(c, i/2-2) \cup N(d, i/2-2).$$

We perform BFS from x in the graph induced by $V \setminus Q$ to expose $N_{V \setminus Q}(x, i/2-1)$. Make sure that the following properties hold (otherwise stop):

(t) $N_{V \setminus Q}(x, i/2-1)$ expands well.

(u) There is no edge from $N_{V \setminus Q}(x, i/2-1) \setminus \{x\}$ to Q.

(v) $N_{V \setminus Q}(x, i/2-1)$ is a tree. Note that since the remaining branches accessed by the edge xa and the edge xb are already guaranteed to be trees and there are no edges between different parts, this implies that $N(x, i/2-1)$ is a tree.

(w) There is no edge between x and y.

(x) Let $R = Q \cup N(x, i/2-1) \setminus \{y\}$. Properties **(t)**, **(u)**, and **(v)** hold when x is replaced by y and Q is replaced by R.

2.2 Acquaintance time

In this section based on [102], we study the following graph process, which was recently introduced by Benjamini, Shinkar, and Tsur [21]. Let $G = (V, E)$ be a finite connected graph. We start the process by placing one *agent* on each vertex of G. Every pair of agents sharing an edge is declared to be *acquainted*, and remains so throughout the process. In each round of the process, we choose some matching M in G. M need not be maximal, it can be a single edge. For each edge of M, we swap the agents occupying its endpoints, which may cause more pairs of agents to become acquainted. The *acquaintance time* of G, denoted by $\mathcal{AC}(G)$, is the minimum number of rounds required for all agents to become acquainted with one another.

It is clear that

$$\mathcal{AC}(G) \geq \frac{\binom{|V|}{2}}{|E|} - 1, \tag{2.1}$$

since $|E|$ pairs are acquainted initially, and at most $|E|$ new pairs become acquainted in each round. In [21], it was shown that always $\mathcal{AC}(G) = O(\frac{n^2}{\log n/\log\log n})$, where $n = |V|$. Moreover, for all functions $f : \mathbb{N} \to \mathbb{N}$ with $1 \leq f(n) \leq n^{1.5}$, the authors constructed families $\{G_n\}$ of graphs with $|V(G_n)| = n$ for all n such that $\mathcal{AC}(G_n) = \Theta(f_n)$. The problem is similar in flavor to the problems of Routing Permutations on Graphs via Matchings [5], Gossiping and Broadcasting [86], and Target Set Selection [98, 43, 148].

Let $G \in \mathcal{G}(n,p)$ with $p \geq (1+\varepsilon)\log n/n$ for some $\varepsilon > 0$. (Recall that $\mathcal{AC}(G)$ is defined only for connected graphs, and $\log n/n$ is the threshold for connectivity in $\mathcal{G}(n,p)$—see, for example, [27, 94] for more.) Since a.a.s. $|E(G)| = (1+o(1))\binom{n}{2}p$, it follows immediately from the trivial lower bound (2.1) that a.a.s. $\mathcal{AC}(G) = \Omega(1/p)$. On the other hand, it is known that a.a.s. G has a Hamiltonian path, which implies that a.a.s. $\mathcal{AC}(G) = O(n)$ (see [21]). Despite the fact that no non-trivial upper bound on $\mathcal{AC}(G)$ was known, it was conjectured in [21] that a.a.s. $\mathcal{AC}(G) = O(\operatorname{poly}\log(n)/p)$. We confirm this conjecture.

Theorem 2.2.1. *Let $\varepsilon > 0$ and $(1+\varepsilon)\log n/n \leq p \leq 1-\varepsilon$. For $G \in \mathcal{G}(n,p)$, a.a.s.*

$$\mathcal{AC}(G) = O\left(\frac{\log n}{p}\right).$$

In hopes of improving the trivial lower bound on the acquaintance time of $\mathcal{G}(n,p)$, we consider a variant of the original process. Suppose that each agent has a helicopter and can, on each round, move to any vertex she wants. (We retain the requirement that no two agents can occupy a single vertex simultaneously.) In other words, in every step of the process, the agents choose some permutation π of the vertices, and the agent occupying vertex v flies directly to vertex $\pi(v)$, regardless of whether there is an edge or even a path between v and $\pi(v)$. (In fact, it is no longer necessary that the graph be connected.) Let $\overline{\mathcal{AC}}(G)$ be the counterpart of $\mathcal{AC}(G)$ under this new model, that is, the minimum number of rounds required for all agents to become acquainted with one another. Since helicopters make it easier for agents to get acquainted, we immediately get that for every graph G,

$$\overline{\mathcal{AC}}(G) \leq \mathcal{AC}(G). \tag{2.2}$$

On the other hand, $\overline{\mathcal{AC}}(G)$ also represents the minimum number of copies of a graph G needed to cover all edges of a complete graph of the same order. Thus inequality (2.1) can be strengthened to $\overline{\mathcal{AC}}(G) \geq \binom{|V|}{2}/|E| - 1$.

We prove the following lower bound on $\overline{\mathcal{AC}}(G)$ (and hence on $\mathcal{AC}(G)$). This result also implies that a.a.s. K_n cannot be covered with $o(\log n/p)$ copies of a dense random graph $G \in \mathcal{G}(n,p)$.

Theorem 2.2.2. *Let $\varepsilon > 0$, $p \geq n^{-1/2+\varepsilon}$ and $p \leq 1-\varepsilon$. For $G \in \mathcal{G}(n,p)$, a.a.s.*

$$\mathcal{AC}(G) \geq \overline{\mathcal{AC}}(G) \geq \frac{\varepsilon}{2}\log_{1/(1-p)} n = \Omega\left(\frac{\log n}{p}\right).$$

Theorem 2.2.1 and Theorem 2.2.2 together determine the order of growth for the acquaintance time of dense random graphs (in particular, random graphs with average degree at least $n^{1/2+\varepsilon}$ for some $\varepsilon > 0$).

Corollary 2.2.3. *Let $\varepsilon > 0$, $p \geq n^{-1/2+\varepsilon}$ and $p \leq 1-\varepsilon$. For $G \in \mathcal{G}(n,p)$, a.a.s.*

$$\overline{\mathcal{AC}}(G) = \Theta\Big(\mathcal{AC}(G)\Big) = \Theta\left(\frac{\log n}{p}\right).$$

Idea of the proof. The proof of Theorem 2.2.1 is based on finding a Hamiltonian cycle in G, and by routing everyone through some small part of this cycle, a.a.s. each pair of agents gets acquainted. The proof of Theorem 2.2.2 proceeds by a union bound over all permutations over all rounds.

2.3 Metric dimension

This section is based on [30] and on [125]. Let $G = (V, E)$ be a finite, simple graph with $|V| = n$ vertices. Suppose first that G is connected. For a subset $R \subseteq V$ with $|R| = r$, and a vertex $v \in V$, define $d_R(v)$ to be the r-dimensional vector whose i-th coordinate $(d_R(v))_i$ is the length of the shortest path between v and the i-th vertex of R. We call a set $R \subseteq V$ a *resolving set* if for any pair of vertices $v, w \in V$, $d_R(v) \neq d_R(w)$. Clearly, the entire vertex set V is always a resolving set, and so is $R = V \setminus \{z\}$ for every vertex z. The *metric dimension* $\beta(G)$ (or simply β, if the graph we consider is clear from the context) is then the smallest cardinality of a resolving set. If G is an isolated vertex, then we define $\beta(G) = 1$. Now, more generally, for a graph G with connected components $G_1, \ldots, G_k$, $k \geq 2$, none of them being an isolated vertex, we have $\beta(G) = \sum_{i=1}^{k} \beta(G_i)$: in order to distinguish two vertices from the same connected component, a minimal resolving set of this connected component has to be chosen. If on the other hand G has connected components $G_1, \ldots, G_k$ and at least one isolated vertex, then $\beta(G) = \left(\sum_{i=1}^{k} \beta(G_i)\right) - 1$, as one isolated vertex is distinguished from all others without choosing the vertex: it will be the only vertex at distance ∞ from everyone else. We have the trivial inequalities $1 \leq \beta(G) \leq n-1$, with the lower bound attained for a path, and the upper bound for the complete graph.

The problem of studying the metric dimension was proposed in the mid-1970s by Slater [157], and Harary and Melter [84]. As a start, Slater [157] determined the metric dimension of trees. Two decades later, Khuller, Raghavachari and Rosenfeld [100] gave a linear-time algorithm for computing the metric dimension of a tree, and they also characterized graphs with metric dimensions 1 and 2. Later on, Chartrand, Eroh, Johnson and Oellermann [41] gave necessary and sufficient conditions for a graph G to satisfy $\beta(G) = n-1$ or $\beta(G) = n-2$.

Denoting by $D = \mathrm{diam}(G)$ the diameter of a graph G, it was observed in [100] that $n \leq D^{\beta-1} + \beta$. Recently, Hernando, Mora, Pelayo, Seara and Wood [87] proved that $n \leq (\lfloor \frac{2D}{3} \rfloor + 1)^{\beta} + \beta \sum_{i=1}^{\lceil D/3 \rceil} (2i-1)^{\beta-1}$, and gave extremal constructions that show that this bound was sharp. Moreover, in [87] graphs of metric dimension β and diameter D were characterized.

As far as we know, previous to our paper not much was known about the metric dimension of $\mathcal{G}(n,p)$. Babai et al. [10] showed that in $G(n, 1/2)$ a.a.s. the set of $\lceil (3\log n)/\log 2 \rceil$ vertices

with the highest degrees can be used to test whether two random graphs are isomorphic (in fact, they provided an $O(n^2)$ algorithm to do it), and hence they obtained an upper bound of $\lceil (3\log n)/\log 2 \rceil$ for the metric dimension of $G(n, 1/2)$ that holds a.a.s. Frieze et al. [71] studied sets resembling resolving sets, namely *identifying codes*: a set $C \subseteq V$ is an identifying code of G, if C is a dominating set (every vertex $v \in V \setminus C$ has at least one neighbor in C) and C is also a separating set (for all pairs $u, v \in V$, one must have $N[u] \cap C \neq N[v] \cap C$, where $N[u]$ denotes the closed neighborhood of u). Observe that a graph might not have an identifying code, but note also that for random graphs with diameter 2 the concepts are very similar. The existence of identifying codes and bounds on their sizes in $\mathcal{G}(n, p)$ were established in [71]. The same problem in the model of random geometric graphs was analyzed by Müller and Sereni [131], and Foucaud and Perarnau [67] studied the same problem in random d-regular graphs.

Let us collect our results into a single theorem covering all random graphs with expected average degree $d = pn(1 + o(1)) \gg \log^5 n$ and expected average degree in the complement of the graph $(n - 1 - d) = (1 - p)n(1 + o(1)) \geq (3n \log\log n)/\log n$. For a visualization of the behavior of $\log_n \beta(G(n, n^{x-1}))$ see also Figure 2.2(a) and the description right after the statement of the theorem.

The intuition behind the theorem is the following: if a random graph is sufficiently dense, then the graph locally (that is, "observed" from a given vertex) "looks" the same. In other words, the cardinality of the set of vertices at a certain graph distance from a given vertex v does not differ much for various v. After grouping the vertices according to their graph distances from v, it turns out that for the metric dimension the ratio between the sizes of the two largest groups of vertices is of crucial importance. If these two groups are roughly of the same size, then a typical vertex added to the resolving set distinguishes a lot of pairs of vertices, and hence the metric dimension is small. If, on the other hand, these two groups are very different in size, a typical vertex distinguishes those few vertices belonging to the second largest group from the rest. The number of other pairs that are distinguished is negligible and hence the metric dimension is large.

This parameter is non-monotone: to see this, start with a random graph with constant edge probability p. For each vertex v in the graph, a constant fraction of all vertices are neighbors of v and a constant fraction of vertices are non-neighbors. When decreasing p, the number of neighbors decreases, and some vertices will appear at graph distance 3. As a result, the metric dimension increases. Continuing this process, the number of vertices at graph distance 3 increases more and more, and at some point this number is comparable to the number of vertices at graph distance 2. Then, the metric dimension is small again, and the same phenomenon appears in the next iterations.

The precise statement is the following.

Theorem 2.3.1. *Suppose that*

$$\log^5 n \ll d = (n-1)p \leq n\left(1 - \frac{3\log\log n}{\log n}\right).$$

Let $i \geq 0$ be the largest integer such that $d^i = o(n)$, let $c = c(n) = d^{i+1}/n$, and let

$$q = \begin{cases} (e^{-c})^2 + (1-e^{-c})^2 & \text{if } p = o(1) \\ p^2 + (1-p)^2 & \text{if } p = \Theta(1). \end{cases}$$

For $i \geq 1$, let $\eta = \log_n d^i$. Finally, let $G = (V, E) \in \mathcal{G}(n,p)$. Then, the following assertions hold a.a.s.

(i) If $c = \Theta(1)$, then

$$\beta(G) = (1+o(1))\frac{2\log n}{\log(1/q)} = \Theta(\log n).$$

(ii) If $c \to \infty$ and $e^c \leq (\log n)/(3\log\log n)$, then

$$\beta(G) = (1+o(1))\frac{2\log n}{\log(1/q)} = (1+o(1))e^c \log n \gg \log n.$$

(iii) If $e^c > (\log n)/(3\log\log n)$, then

$$(\eta + o(1))\left(\frac{d^i}{n} + e^{-c}\right)^{-1}(\log n) \leq \beta(G) \leq (1+o(1))\left(\frac{d^i}{n} + e^{-c}\right)^{-1}(\log n).$$

In particular,

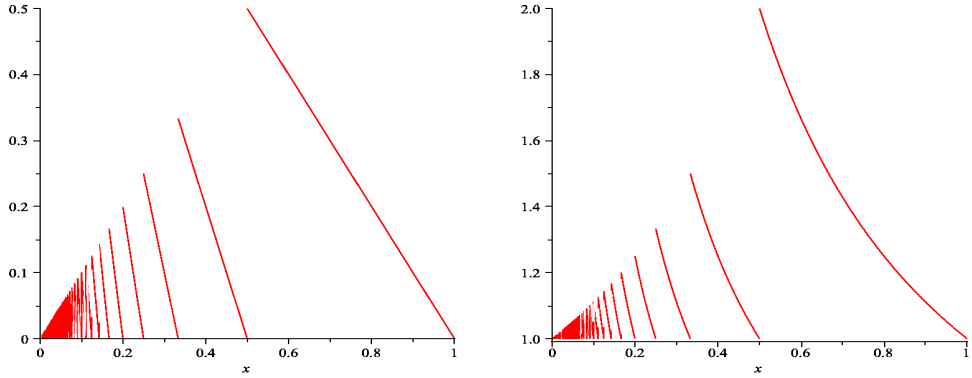
$$\beta(G) = \begin{cases} \Theta(e^c \log n) = \Theta(\frac{\log n}{\log(1/q)}) & \text{if } e^{-c} = \Omega(d^i/n) \\ \Theta(\frac{n\log n}{d^i}) & \text{if } e^{-c} \ll d^i/n, \end{cases}$$

and hence in all cases we have $\beta(G) \gg \log n$.

Observe that Theorem 2.3.1 shows that $\beta(G)$ undergoes a "zigzag" behavior as a function of p. It follows that a.a.s. $\log_n \beta(G(n, n^{x-1}))$ is asymptotic to the function $f(x) = 1 - x\lfloor 1/x \rfloor$ shown in Figure 2.2(a). Indeed, for cases (i) and (ii) we have $c = n^{o(1)}$ (that is, $d = n^{(1+o(1))/i}$ for some $i \in \mathbb{N}$) and a.a.s. $\beta(G) = n^{o(1)}$. This corresponds to a collection of points $(1/i, 0)$, $i \in \mathbb{N}$ in the figure. For ranges of p considered in case (iii), we have that a.a.s. $\beta(G)$ is of order $(d^i/n + e^{-c})^{-1}$. For $d = n^{x+o(1)}$, where $1/(i+1) < x < 1/i$ for some $i \in \mathbb{N}$, it follows that a.a.s. $\beta(G) = \Theta(n/d^i) = n^{1-ix+o(1)}$, which corresponds to linear parts of the function $f(x)$ of slope $1 - ix$. The function $f(x)$ is hence not continuous at $x = i$, $i \in \mathbb{N} \setminus \{1\}$.

The result is asymptotically tight for sparse graphs (that is, for $d = n^{o(1)}$). The ratio between our upper and lower bound is at most $2 + o(1)$ and follows another "zigzag" function $f(x) = (x\lfloor 1/x \rfloor)^{-1}$ shown in Figure 2.2(b). Indeed, for cases (i) and (ii) we obtained an asymptotic behavior of $\beta(G)$. This corresponds to a collection of points $(1/i, 0)$, $i \in \mathbb{N}$ in the figure. In case (iii) the ratio is asymptotic to η^{-1}. For $d = n^{x+o(1)}$, where $1/(i+1) < x < 1/i$ for some $i \in \mathbb{N}$, $\eta^{-1} = \eta^{-1}(x) \sim 1/(ix) \leq (i+1)/i$. Hence, $\eta^{-1} \sim (x\lfloor 1/x \rfloor)^{-1}$.

The proof of the upper bound comes from a simple application of the probabilistic method:



(a) the 'zigzag' function $f(x) = 1 - x\lfloor 1/x \rfloor$

(b) the upper/lower bound ratio $\eta^{-1}(x) = (x\lfloor 1/x \rfloor)^{-1}$

Figure 2.2:

Observation. Suppose that a (deterministic) graph $G = (V, E)$ on n vertices satisfies the following property: for a given pair of vertices $x, y \in V$, the probability that a random set W of cardinality w does *not* distinguish x and y is at most $1/n^2$. (For different pairs of vertices, the set $W = W(x, y)$ is generated independently.) Then, the metric dimension is at most w.

The observation clearly holds by the probabilistic method: indeed, since the expected number of pairs that are not distinguished by a random set W is at most $1/2$, there is at least one set W that distinguishes *all* pairs.

The proof of the lower bound follows from an application of Suen's inequality which we describe here: Let $\mathcal{I} = \{(x, y) : x, y \in S, x \neq y\}$, and for any $(x, y) \in \mathcal{I}$, let $A_{x,y}$ be the event (with the corresponding indicator random variable $I_{x,y}$) that $d_R(x) = d_R(y)$. Let $X = \sum_{(x,y)\in\mathcal{I}} I_{x,y}$. Clearly, the probability that R is a resolving set is at most the probability that $X = 0$. The associated *dependency graph* has $\mathcal{I}$ as its vertex set, and $(x_1, y_1) \sim (x_2, y_2)$ if and only if $\{x_1, y_1\} \cap \{x_2, y_2\} \neq \emptyset$. It follows from Suen's inequality that

$$\mathbb{P}(X = 0) \leq \exp\left(-\mu + \Delta e^{2\delta}\right), \tag{2.3}$$

where

$$\begin{aligned} \mu &= \sum_{(x,y)\in\mathcal{I}} \mathbb{P}(A_{x,y}) \\ \Delta &= \sum_{(x_1,y_1)\sim(x_2,y_2)} \mathbb{E}\left[I_{x_1,y_1} I_{x_2,y_2}\right] \\ \delta &= \max_{(x_1,y_1)\in\mathcal{I}} \sum_{(x_2,y_2)\sim(x_1,y_1)} \mathbb{P}(A_{x_2,y_2}). \end{aligned}$$

For sparser graphs we have more precise results. We consider $\mathcal{G}(n, p)$ with $p = \frac{c}{n}$ with $c < 1$. This range of parameters typically has a very forest-like structure, although a few cycles might be present. In such a situation the behavior is quite regular, and indeed we can

obtain precise limiting distributions for this parameter. To make our result precise, we need the following notation. Let F_n a distribution function of a certain random variable and let Φ denote the distribution function of the standard normal law. Define the following measure of convergence

$$d(F_n, \Phi) = \sup_{h(x)} \frac{\left| \int h(x) dF_n(x) - \int h(x) d\Phi(x) \right|}{||h||},$$

where $||h|| = \sup_x |h(x)| + \sup_x |h'(x)|$, and the supremum is taken over all bounded test functions h with bounded derivative. For a random variable X denote by $\mathcal{L}(X)$ its distribution function (if it exists).

Theorem 2.3.2. *Let $G \in \mathcal{G}(n,p)$.*

(i) For $p = o(n^{-1})$, $\beta(G) = n(1+o(1))$ a.a.s.

(ii) For $p = \frac{c}{n}$ with $0 < c < 1$, the sequence of random variables

$$\frac{\beta(G) - \mathbb{E}[\beta(G)]}{\sqrt{\mathbb{V}ar\beta(G)}}$$

converges in distribution to a standard normal distribution as $n \to \infty$, and

$$d\left(\mathcal{L}\left(\frac{\beta(G) - \mathbb{E}[\beta(G)]}{\sqrt{\mathbb{V}ar\beta(G)}}\right), \Phi\right) = O\left(n^{-1/2}\right).$$

Moreover, $\mathbb{E}[\beta(G)] = Cn(1+o(1))$, where

$$C = e^{-c}\left(\frac{3}{2} + c + \frac{c^2}{2} - e^c - \frac{1}{2}e^{ce^{-c}} + \exp\left(c\frac{1-(c+1)e^{-c}}{1-ce^{-c}}\right) - c\frac{e^{-c}}{1-ce^{-c}} - \frac{c^2}{2}\left(\frac{1-(c+1)e^{-c}}{1-ce^{-c}}\right)^2\right), \tag{2.4}$$

and $\mathbb{V}ar\beta(G) = \Theta(n)$.

The plot of the constant term C given by (2.4) as a function of c is shown in Figure 2.3. For the analysis of our result in $\mathcal{G}(n,p)$ with $p = c/n$ and $c < 1$, we use the following theorem which is an adaptation of Stein's Method for the setting of random graphs (see [18]):

Theorem 2.3.3. *(Theorem 1 of [18] and its following remarks): Let I be a finite subset of $\mathbb{N}$, and let $\{X_i\}_{i\in I}$ be a family of (possibly dependent) random variables of zero expectations, and such that $W = \sum_{i\in I} X_i$ has variance 1. For each $i \in I$, let $K_i \subseteq I$, and define $Z_i = \sum_{k\in K_i} X_k$ and $W_i = \sum_{k\notin K_i} X_k$ (so that $W = W_i + Z_i$). Assume that for each $i \in I$, W_i is independent of X_i and Z_i, and that X_i, W_i, Z_i have finite second moments. Define*

$$\varepsilon = 2\sum_{i\in I}\sum_{k,\ell\in K_i}\left(\mathbb{E}[|X_iX_kX_\ell|] + \mathbb{E}[|X_iX_k|]\,\mathbb{E}[|X_\ell|]\right). \tag{2.5}$$

Then, with Φ denoting the distribution function of the standard normal law,

$$d(\mathcal{L}(W,\Phi)) \le K\varepsilon$$

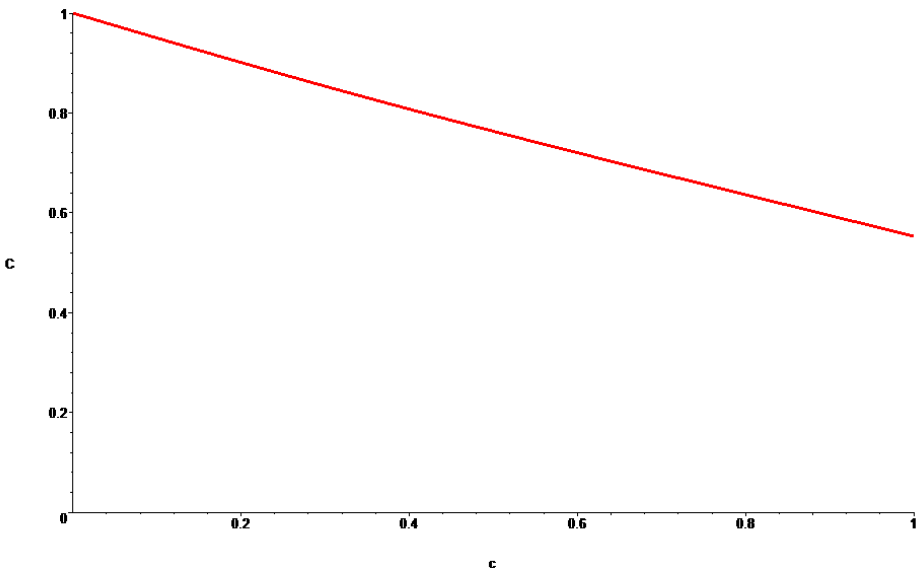


Figure 2.3: The constant term in the mean of the metric dimension when c moves from 0 to 1. For c approaching 1, we have $C \simeq 0.55339767$.

for some universal constant K. Hence, if $\{W^{(n)}\}$ is a sequence of random variables, such that each $W^{(n)}$ satisfies the conditions above, and such that the value $\varepsilon^{(n)}$ associated with $W^{(n)}$ converges to 0 as $n \to \infty$, then $\{W^{(n)}\}$ converges to the standard normal law.

We also look at the metric dimension in the uniform random tree model and compare the results, see Section 5.1 below for the comparison.

2.4 Revolutionaries and spies

This section is based on [124]. A pursuit-evasion game may be viewed as a simplified model for such network security problems. In our paper [124] we study the game of Revolutionaries and Spies invented by József Beck in the mid-1990s (according to [49]) that is played on a fixed graph G. There are two players: a team of r *revolutionaries* and a team of s *spies*. The revolutionaries want to arrange a one-time meeting consisting of m revolutionaries free of oversight by spies; the spies want to prevent this from happening. The revolutionaries start by occupying some vertices as their initial positions; more than one revolutionary is allowed to occupy some vertex. After that the spies do the same; they can start from some vertices already occupied by revolutionaries or choose brand new vertices. In each subsequent round, each revolutionary may move to an adjacent vertex or choose to stay where he is, and then each spy has the same option. This is a perfect information game, that is, both players see each other's locations and know all possible moves. Moreover, we assume that the players are perfect, that is, they can analyze the game completely and play optimally. For more on combinatorial games see, for example, [1].

A *meeting* is a set of at least m revolutionaries occupying one vertex; a meeting is *unguarded* if there is no spy at that vertex. The revolutionaries win if at the end of some

round there is an unguarded meeting. On the other hand, the spies win if they have a strategy to prevent this forever. For given r and m, the minimum number of spies required to win on a graph G is the *spy number* $\sigma(G, r, m)$. Since $\min\{|V(G)|, \lfloor r/m \rfloor\}$ meetings can be initially formed, at least that many spies are needed to have a chance to win. On the other hand, $r-m+1$ spies can create a matching with $r-m+1$ distinct revolutionaries and follow them during the whole game, preventing any unguarded meeting from taking place. If $|V(G)| < r-m+1$, this can be clearly improved since occupying all vertices clearly does the job as well. We thus get the following trivial bounds on the spy number:

$$\min\{|V(G)|, \lfloor r/m \rfloor\} \leq \sigma(G, r, m) \leq \min\{|V(G)|, r-m+1\}.$$

It is known that the lower bound is sufficient when G is a tree, and at most one additional spy is needed to win on any unicyclic graph [49]. On the other hand, the upper bound can be obtained for some chordal and bipartite graphs (such as hypercubes, for example)—see [37] for more. Moreover, grid-like graphs were studied in [89].

For $p \in (0,1)$ or $p > 0$ tending to 0 with n, define $\mathbb{L}n = \log_{\frac{1}{1-p}} n$. For constant p, clearly $\mathbb{L}n = \Theta(\log n)$, but for $p = o(1)$ we have

$$\mathbb{L}n = \frac{\log n}{-\log(1-p)} = (1+o(1))\frac{\log n}{p}.$$

Preliminary results for random graphs have been proved in [37] where it is shown that for constant $p \in (0,1)$ and for $r < c \log_{\frac{1}{\min\{p,1-p\}}} n$ with $c < 1$, and also for constant r and $p \gg n^{-1/r}$, $\sigma(G, r, m) = r - m + 1$ a.a.s. (the required condition is that $pn^r \to \infty$). In [124] we improve on these results using different techniques. We now describe our main results. Let us start with the following useful upper bound for the spy number.

Theorem 2.4.1. *Let $\varepsilon > 0$, $r = r(n) \in \mathbb{N}$, and $m = m(n) \in \mathbb{N}$. Consider a random graph $G \in \mathcal{G}(n,p)$ with $p < 1-\varepsilon$. Then, a.a.s.*

$$\sigma(G, r, m) \leq \frac{r}{m} + 2(2+\sqrt{2}+\varepsilon)\mathbb{L}n.$$

In particular, it follows from this theorem that $\sigma(G, r, m) = (1+o(1))r/m$ whenever $r/m \gg \mathbb{L}n$, and $\sigma(G, r, m) = \Theta(r/m)$ if $r/m = \Theta(\mathbb{L}n)$. The next theorem provides a lower bound.

Theorem 2.4.2. *Let $\varepsilon > 0$, $\eta \in (0, 1/3]$, $r = r(n) \in \mathbb{N}$, and $m = m(n) \in \mathbb{N}$. Consider a random graph $G \in \mathcal{G}(n,p)$ with $p = n^{-1/3+\eta+o(1)} < 1-\varepsilon$. Then, a.a.s.*

$$\sigma(G, r, m) \geq \min\{r-m, 2.99\eta\mathbb{L}n\} + 1.$$

Theorems 2.4.1 and 2.4.2 give an asymptotic behavior (up to a constant factor) of the spy number of random graphs with average degree $n^{2/3+\eta+o(1)}$ for some $\eta \in (0, 1/3]$. Therefore, we obtain the whole picture for such dense random graphs.

Corollary 2.4.3. *Let $\varepsilon > 0$, $\eta \in (0, 1/3]$, $r = r(n) \in \mathbb{N}$, and $m = m(n) \in \mathbb{N}$. Consider a random graph $G \in \mathcal{G}(n,p)$ with $p = n^{-1/3+\eta+o(1)} < 1-\varepsilon$. Then, a.a.s.*

$$\sigma(G,r,m) = \begin{cases} r-m+1 & \text{if } r-m \le 2.99\eta\mathbb{L}n \\ \Theta(\mathbb{L}n) & \text{if } r-m > 2.99\eta\mathbb{L}n \text{ and } r/m = O(\mathbb{L}n) \\ (1+o(1))r/m & \text{if } r/m \gg \mathbb{L}n. \end{cases}$$

With a bit more effort and one additional idea, one can extend this (tight) result to slightly sparser graphs.

Theorem 2.4.4. *Let $\eta \in (0, 1/6]$, $r = r(n) \in \mathbb{N}$, and $m = m(n) \in \mathbb{N}$. Consider a random graph $G \in \mathcal{G}(n,p)$ with $p = n^{-1/2+\eta+o(1)}$. Then, a.a.s.*

$$\sigma(G,r,m) = \begin{cases} r-m+1 & \text{if } r-m = O(1) \\ (1+o(1))(r-m) & \text{if } r-m \gg 1 \text{ and } r-m \le 1.99\eta\mathbb{L}n \\ \Theta(\mathbb{L}n) & \text{if } r-m > 1.99\eta\mathbb{L}n \text{ and } r/m = O(\mathbb{L}n) \\ (1+o(1))r/m & \text{if } r/m \gg \mathbb{L}n. \end{cases}$$

For very sparse graphs (that is, graphs with average degree $n^{1/2-\eta+o(1)}$ for some $\eta \in [0, 1/2]$) we are less successful. However, we make a partial progress. First of all, we managed to investigate the case $r - m = O(1)$ for graphs with average degree at least $\log^3 n$ *unless* the average degree is of order $\sqrt{n \log n}$. The sub-case $r = O(1)$ (and even r growing with n slowly) seems to be easy to deal with (the revolutionaries could go to a fixed set of vertices, say $[r]$; since a.a.s. all pairs of them have as many common neighbors as expected, and for each pair there is a common neighbor not adjacent to any spy, they can join themselves and create an unguarded meeting in the same way as in the other proofs of dense graphs). Unfortunately, no approach to solve the general case $r - m = O(1)$ and r tending (fast) to infinity with n (and so m as well) is known. It is not clear if this peculiar gap at $\sqrt{n \log n}$ is an outcome of a wrong approach used or perhaps the behavior of the spy number changes in this window. This remains to be investigated.

Theorem 2.4.5. *Let $\varepsilon > 0$, $r = r(n) \in \mathbb{N}$, and $m = m(n) \in \mathbb{N}$ such that $r - m = O(1)$. Consider a random graph $G \in \mathcal{G}(n,p)$ with $\log^3 n \le pn \ll \sqrt{n \log n}$ or $\sqrt{n \log n} \ll pn < (1-\varepsilon)n$. Then, a.a.s.*

$$\sigma(G,r,m) = r - m + 1.$$

For $r - m \gg 1$ and the average degree of $n^{\eta+o(1)}$ for some $\eta \in (0, 1/2)$ we have the following result.

Theorem 2.4.6. *Let ω be any function tending to infinity arbitrarily slowly and suppose $r - m \gg 1$. Let $np = n^{\eta+o(1)}$ for some $\eta \in (0, \frac{1}{2})$, and let $G \in \mathcal{G}(n,p)$ Then, a.a.s.*

$$\sigma(G,r,m) = (1+o(1))(r-m),$$

provided that one of the following situations occurs:

(i) $r - m = o(\min\{pn, \frac{n}{(pn)^2}\})$ *and* $pn \ge (\omega n \log n)^{1/4}$,

(ii) $r - m = o(pn/(\omega \log n))$ *and* $(\omega n \log n)^{1/6} \leq pn < (\omega n \log n)^{1/4}$,

(iii) $r - m = o(pn)$ *and* $pn < (\omega n \log n)^{1/6}$.

Let $f(\eta) = 1 - \eta$ and let

$$g(\eta) = \begin{cases} \eta & \text{if } \eta \in (0, 1/3] \\ 1 - 2\eta & \text{if } \eta \in (1/3, 1/2). \end{cases}$$

Note that for a given $\eta \in (0, 1/2)$, Theorems 2.4.5 and 2.4.6 investigate the case $r - m = n^{\alpha + o(1)}$ for $\alpha \in [0, g(\eta))$; the case $r - m = n^{\alpha+o(1)}$ for $\alpha > f(\eta)$ follows from Theorem 2.4.1. The behavior of the spy number for $r - m = n^{\alpha+o(1)}$ for some $\alpha \in [g(\eta), f(\eta)]$ is not known and remains to be analyzed.

The proof of the upper bound proceeds by calculating an upper bound of the size of the intersection of non-neighborhoods of different vertices. Next, it is not difficult to show that a.a.s. a random graph $\mathcal{G}(n, p)$ has a dominating set of size $(1+o(1))\mathbb{L}n$. Using Hall's theorem for bipartite graphs, we can show that a slightly larger set can not only dominate the rest of the graph but also can create a matching with any set of cardinality $O(\mathbb{L}n)$:

Lemma 2.4.7. *Let $\gamma, \delta > 0$ be such that $\gamma - \delta > 1 + \eta$. Let $\varepsilon > 0$ and consider a random graph $G \in \mathcal{G}(n, p)$ with $p = n^{-\eta+o(1)} < 1 - \varepsilon$ for some $\eta \in [0, 1]$. Then, a.a.s. there exists a set $A \subseteq V(G)$ of cardinality $\gamma\mathbb{L}n$ such that for all sets $T \subseteq V(G) \setminus A$ of size at most $\delta\mathbb{L}n$ there is a perfect matching from T to some subset A' of A.*

The main idea of the proof of Theorem 2.4.1 is now to split spies into three groups: the first two groups, τ_1 and τ_2, consist each of $\gamma\mathbb{L}n$ *super-spies*, and the third group τ_3 consists of $\lfloor r/m \rfloor$ *regular-spies*. Super-spies from team τ_1 will occupy the whole set A of size $\gamma\mathbb{L}n$ such that for all sets $T \subseteq V(G) \setminus A$ of size at most $\delta\mathbb{L}n$ there is a perfect matching from T to some subset A' of A (the existence of this set is guaranteed by Lemma 2.4.7) at odd times, but some of them might be sent to a mission at even times. If this is the case, then they will be back to the set A in the next round. Similarly, super-spies from team τ_2 will occupy A at even times but might be used to protect some other vertices at odd times. In particular, the set A will be constantly protected and so no unguarded meeting can take place there. Regular-spies (team τ_3) will occupy a set $B_t \subseteq V(G) \setminus A$ at time t. Moreover, no two regular-spies will occupy the same vertex, so $|B_t| = \lfloor r/m \rfloor$ for all t. The revolutionaries start the game by occupying a set R_1 and forming at most $\lfloor r/m \rfloor$ meetings. The super-spies (from both teams τ_1 and τ_2) go to the set A.

The regular-spies can easily protect the vertices of $V(G) \setminus A$ in which meetings take place by placing a spy on each vertex where there are at least m revolutionaries. The remaining regular-spies go to arbitrary vertices of $V(G) \setminus A$ not occupied by another spy. As a result, no unguarded meeting takes place in the first round. Suppose that no unguarded meeting takes place at time $t - 1$ and that the regular-spies occupy a set $B_{t-1} \subseteq V(G) \setminus A$ of cardinality $\lfloor r/m \rfloor$. At the beginning of round t, the revolutionaries might form at most $\lfloor r/m \rfloor$ meetings at vertices of $M_t \subseteq V(G) \setminus A$ (as we already pointed out, meetings at A are constantly protected by super-spies, so we do not have to worry about them). Let $B = M_t \cap B_{t-1}$ be the set of vertices in which meetings are already guarded by regular-spies. It remains to show

that there exists a perfect matching between $M_t \setminus B$ and some subset S of $A \cup (B_{t-1} \setminus B)$. Indeed, if this is the case, then the regular-spies that do not protect any meeting as well as the super-spies from the team protecting A in the previous round, move from S to $M_t \setminus B$. The Super-spies from another team come back to A to guard this set and prepare themselves for another mission. No unguarded meeting takes place at time t, $B_t \subseteq V(G) \setminus A$, and no two regular-spies occupy the same vertex. The result will follow by induction on t. In order to show that a perfect matching can be formed we use Hall's theorem.

For the proofs of the lower bounds in Theorem 2.4.2, we employ the following adjacency property and its generalizations. For fixed positive integers k and l, we say that a graph G is (l,k)-*existentially closed* (or (l,k)-*e.c.*) if for any two disjoint subsets of $V(G)$, $A \subseteq V(G), B \subseteq V(G)$, with $|A| = l$ and $|B| = k$, there exists a vertex $z \in V(G) \setminus (A \cup B)$ not joined to any vertex in B and joined to every vertex in A. We will use the following simple observation.

Observation 2.4.8. *Let r, m, s be positive integers such that $s \leq r - m$, and let G be any $(2,s)$-e.c. graph. Then*

$$\sigma(G, r, m) \geq s + 1.$$

In particular, if G is $(2, r-m)$-e.c., then $\sigma(G, r, m) = r - m + 1$.

In order to deal with $r - m$ tending to infinity with n faster than before, that is, for the proof of Theorem 2.4.4, we need to relax slightly the $(2,s)$-e.c. property at the price of obtaining a bit weaker lower bound for the spy number. For fixed positive integers l and k, we say that a graph G is $(1,l,k)$-*existentially closed* (or $(1,l,k)$-*e.c.*) if for any vertex $v \in V(G)$ and two disjoint subsets of $A, B \subseteq V(G) \setminus \{v\}$ with $|A| = l$ and $|B| = k$, there exists a vertex $z \in V(G) \setminus (\{v\} \cup A \cup B)$ not joined to any vertex in B, joined to v, and joined to *some* vertex in A. Note that $(1,l,k)$-e.c. is a natural generalization of the $(2,k)$-e.c. property, which is equivalent to $(1,1,k)$-e.c. Generalizing this once again by looking at the same properties at distance i instead of distance 1, we finally obtain the results of Theorem 2.4.5 and Theorem 2.4.6.

2.5 On-line list coloring

This section is based on the paper [72]. Consider the following combinatorial game played by two players, named Mr. Paint and Mrs. Correct. They play on a finite, undirected graph in which each vertex has assigned a non-negative number representing the number of erasers at the particular vertex. We assume for simplicity that this number is initially the same for each vertex. In each round, first Mr. Paint selects a subset of the vertices and paints them all the same color; he cannot use this color in subsequent rounds. Mrs. Correct then has to erase the color from some of the vertices in order to prevent adjacent vertices having the same color. Whenever the color at a vertex is erased, the number of erasers at that vertex decreases by 1, but naturally, Mrs. Correct cannot erase the color if she has no erasers left at that vertex. Vertices whose colors have not been erased can be considered as being permanently colored and can be removed from the game. The game has two possible endings: (i) all vertices have been permanently colored, in which case Mrs. Correct wins, or (ii) at some point of the game, Mr. Paint presents two adjacent vertices u and v and neither u nor v has any eraser

left, in which case Mr. Paint wins. If, regardless of which sets she gets presented, there is a strategy for Mrs. Correct to win the game having initially $k-1$ erasers at each vertex, we say that the graph is *k-paintable*. The smallest k for which the graph is k-paintable is called the *paintability* number of G, and denoted by $\chi_P(G)$. Note that this parameter is indeed well defined: for any graph on n vertices, $n-1$ erasers at each vertex always guarantee Mrs. Correct to win, as she can always choose one vertex from a set presented to her and erase colors on the remaining ones. This problem is also known as the *on-line list coloring* and the corresponding graph parameter is also called the *on-line choice number of G*—see, for example, [167] and below for the relation to the (off-line) list coloring. We briefly mention here the relation to other known graph parameters. A *proper coloring* of a graph is a labeling of its vertices with colors such that no two adjacent vertices have the same color. A coloring using at most k colors is called a (proper) *k-coloring*. The smallest number of colors needed to color a graph G is called its *chromatic number*, and it is denoted by $\chi(G)$. Let L_k be an arbitrary function that assigns to each vertex of G a list of k different colors. We say that G is *L_k-list-colorable* if there exists a proper coloring of the vertices such that every vertex is colored with a color from its own list. A graph is *k-choosable*, if for every such function L_k, G is L_k-list-colorable. The minimum k for which a graph is k-choosable is called the *list chromatic number*, or the *choice number*, and denoted by $\chi_L(G)$. Since the choices for L_k contain the special case where each vertex is assigned the list of colors $\{1, 2, \ldots, k\}$, it is clear that a k-choosable graph has also a k-coloring, and so $\chi(G) \leq \chi_L(G)$. It is also known that if a graph is k-paintable, then it is also k-choosable [156], that is, $\chi_L(G) \leq \chi_P(G)$. Indeed, if there exists a function L_k so that G is not L_k-list-colorable, then Mr. Paint can easily win by fixing some permutation of all colors present in L_k and presenting at the i-th step all vertices containing the i-th color of the permutation on their lists (unless the vertex was already removed before). Finally, it was shown in [167] that the paintability of a graph G on n vertices is at most $\chi(G)\log n + 1$. Combining all inequalities we get the following:

$$\chi(G) \leq \chi_L(G) \leq \chi_P(G) \leq \chi(G)\log n + 1. \tag{2.6}$$

It follows from the well-known results of Bollobás [28], Łuczak [115] (see also McDiarmid [116]) that the chromatic number of $\mathcal{G}(n,p)$ a.a.s. satisfies

$$\chi(\mathcal{G}(n,p)) \sim \frac{\log(1/(1-p))n}{2\log(np)}, \tag{2.7}$$

for $np \to \infty$ and p bounded away from 1. The study of the choice number of $\mathcal{G}(n,p)$ was initiated in [3], where Alon proved that a.a.s., the choice number of $\mathcal{G}(n,1/2)$ is $o(n)$. Kahn then showed (see [4]) that a.a.s. the choice number of $\mathcal{G}(n,1/2)$ equals $(1+o(1))\chi_{\mathcal{G}(n,1/2)}$. In [108], Krivelevich showed that this holds for $p \gg n^{-1/4}$, and Krivelevich, Sudakov, Vu, and Wormald [109] improved this to $p \gg n^{-1/3}$. On the other hand, Alon, Krivelevich, Sudakov [6] and Vu [164] showed that for any value of p satisfying $2 < np \leq n/2$, the choice number is $\Theta(np/\log(np))$. Later, Krivelevich and Vu [110] generalized this to hypergraphs; they also improved the leading constants and showed that the choice number for $C \leq np \leq 0.9n$ (where C is a sufficiently large constant) is at most a multiplicative factor of $2+o(1)$ away from the chromatic number, the best known factor for $p \leq n^{-1/3}$. Our results below

(see Theorem 2.5.1, Theorem 2.5.2, and Theorem 2.5.3) show that even for the on-line case, for a wide range of p, we can asymptotically match the best known constants of the off-line case. Moreover, if $np \geq \log^{\omega} n$ (for some function ω tending to infinity as $n \to \infty$), then we get the same multiplicative factor of $2 + o(1)$.

Our main results are the following theorems. The first one deals with dense random graphs.

Theorem 2.5.1. *Let $\varepsilon > 0$ be any constant, and suppose that*

$$(\log\log n)^{1/3}(\log n)^2 n^{-1/3} \ll p \leq 1 - \varepsilon.$$

Let $G \in \mathcal{G}(n,p)$. Then, a.a.s.,

$$\chi_P(G) \sim \frac{n}{2\log_b(np)} \sim \chi(G),$$

where $b = 1/(1-p)$.

Note that if $p = o(1)$, then

$$\frac{n}{2\log_b(np)} = \frac{n\log(1/(1-p))}{2\log(np)} \sim \frac{np}{2\log(np)} = \Theta\left(\frac{np}{\log(np)}\right).$$

For constant p it is not true that $\log(1/(1-p)) \sim p$ but the order is preserved, provided $p \leq 1 - \varepsilon$ for some $\varepsilon > 0$.

For sparser graphs we are less successful in determining the asymptotic behavior of $\chi_P(\mathcal{G}(n,p))$. Nevertheless, we can prove the following two theorems that determine the order of the graph parameter we study.

Theorem 2.5.2. *Let $\varepsilon > 0$ be any constant, and suppose that*

$$\frac{(\log n)^{2+\varepsilon}}{n} \leq p = O((\log\log n)^{1/3}(\log n)^2 n^{-1/3}).$$

Let $G \in \mathcal{G}(n,p)$. Then, a.a.s.,

$$\chi_P(G) = \Theta\left(\frac{np}{\log(np)}\right) = \Theta(\chi(G)).$$

Moreover, if $np = (\log n)^{C+o(1)}$, a.a.s.

$$\chi(G) \leq \chi_P(G) \leq (1+o(1)) \begin{cases} 2\chi(G) & \text{if } C \to \infty \\ \frac{2C}{C-2}\chi(G) & \text{if } C \in [4,\infty) \\ 4\chi(G) & \text{if } C \in (2,4). \end{cases}$$

Finally, for very sparse graphs we have the following.

Theorem 2.5.3. *Let $G \in \mathcal{G}(n,p)$ with $p = O(1/n)$. Then, a.a.s., $\chi_P(G) = \Theta(1) = \Theta(\chi(G))$.*

Idea of the proofs. The proof of Theorem 2.5.3 is comparably the easiest one: first it is shown that for trees and unicyclic graphs the on-line coloring number is deterministically at most 3, and then we show that for $p = \Theta(1/n)$, the induced subgraph of all vertices of degree at least C for C being sufficiently large consists a.a.s. only of trees and unicyclic components. On the remaining graph one can play greedily: add a vertex to a chosen set unless at least one neighbor of it is there, and hence for the remainder at most C erasers are needed. For the proofs of Theorem 2.5.1 and 2.5.2 we first show by adding vertices greedily that every subset has a 'large' independent set, where 'large' depends on the size of the subset; the necessary strong concentration that allows us to take a union bound over all subsets of a certain size is provided by an inequality of [109]. Then any presented set can be classified according to its size: for bigger sets, we find a relatively large independent set, and for small sets, choosing one random vertex is enough.

2.6 Rigidity with sliders

This section is based on the paper [19]. The basic idea is the following: consider a set of points, some of them allowed to move freely in the Euclidean plane, and some constrained to move on fixed lines, called sliders. The free points have two degrees of freedom, the points attached to sliders have only one. Now, add bars between pairs of these points; a bar fixes the length between the two end-points. The points and bars form a *framework*. A framework is said to be rigid if it cannot be deformed (but can possibly be translated and rotated in the plane); equivalently, it is rigid if the distance between any pair of points, connected by a bar or not, is fixed. Characterizing the rigidity of a framework is very difficult in general. In the absence of sliders, a celebrated theorem by Laman [111] ensures that for a generic framework, its rigidity properties only depend on its underlying graph, where points are vertices and bars are edges: the geometry does not enter. This theorem has been generalized to frameworks with sliders in [160]. We will implicitly assume that all frameworks are generic, so that rigidity has a purely graph theoretical characterization and we can deal with vertices and edges instead of points and bars.

We will call a vertex of *type 1* (respectively *type 2*) if it is (respectively is not) connected to a slider. Consider now a percolation setting: take a set of n vertices, a fraction q of which are type 2, and add edges randomly. The questions are: when does a giant (that is: including a positive fraction of the vertices) rigid structure emerge? What is its size? When edges are sampled independently at random between pairs of vertices, the resulting graph is an Erdős-Rényi random graph $\mathcal{G}(n, p = c/n)$. In this case and for $q = 1$ (no slider), Kasiviswanathan et al. [97] showed that the threshold for a giant rigid component is $c \simeq 3.588$, and that the transition is discontinuous: as soon as the giant rigid component appears, it already includes a positive fraction of all n vertices. This recovers numerical and heuristic results found earlier in the physics literature [129, 62], and contrasts with the emergence of a giant connected component at $c = 1$, which is continuous.

When $q = 0$, rigidity is closely related to the emergence of the giant connected component. Our goal is to investigate the case where $q \in [0, 1]$. We are thus interested in situations

interpolating between standard connectivity percolation and rigidity percolation as studied in [97]. We obtain the following results:

- We compute the threshold for rigidity percolation as a function of q.
- We show that the transition is continuous for $q \leq 1/2$ and discontinuous for $q > 1/2$, thus uncovering what is called a "tricritical" point in statistical mechanics, for $q = 1/2$.
- On the way, we obtain new results on cores for $\mathcal{G}(n, p)$ and their generalization to two types of vertices. We prove in particular a conjecture on the size of the $3 + 2$-core in [97].

Rigidity percolation has physical motivations: it is a model to understand some properties of network glasses and proteins [161, 144, 36, 146]. Thus, problems related to ours have been investigated by theoretical physicists. We have already cited investigations on random graphs starting with [129, 62], with only one type of vertex (type 2, or more generally type k). In [128], Moukarzel heuristically studied a model with two types of vertices: a fraction of the vertices are pinned to the plane, instead of being allowed to move in one direction; they could be called "type 0" vertices. In this case, the transition disappears as the fraction of pinned vertices increases: there is no tricritical point, but rather a critical point.

In order to compute the threshold for rigidity, we use the same connection as [97] between orientability and rigidity. We then use recently introduced and powerful methods to compute the orientability threshold (see [113]). To investigate the continuous or discontinuous character of the transition, we rely on various refinements of a method introduced in [93] to investigate the cores of a random graph.

Vertices are either of type 1 or of type 2, and for $i \in \{1, 2\}$. By n_i we denote the number of vertices of type i, and so $n = n_1 + n_2$.

We denote subgraphs by G', with $n_i(G')$ vertices of type $i \in \{1, 2\}$, $n(G') = n_1(G') + n_2(G')$ vertices in total, and $m(G')$ edges. When the context is clear, we use the following notations: $n' = n(G')$, $n'_i = n_i(G')$ and $m' = m(G')$.

Definition 2.6.1. *Let G be a graph with $n = n_1 + n_2$ vertices and m edges. G is* sparse *if for every subgraph $G' \subseteq G$ on $n' = n'_1 + n'_2 \geq 2$ vertices and m' edges, we have:*

$$m' \leq n'_1 + 2n'_2 + \min(0, n'_1 - 3) = 2n' - \max(n'_1, 3).$$

In terms of physics, a sparse graph represents a structure without redundant constraint. The special treatment needed for subgraphs with 0, 1 or 2 vertices of type 1, i.e. for $n'_1 < 3$, can then be understood: a structure which is not connected at all to the underlying plane (that is $n'_1 = 0$) cannot be pinned, and always keeps at least three degrees of freedom, hence the -3; a structure with one slider (that is $n'_1 = 1$) always keeps at least two degrees of freedom, hence the -2; and similarly for $n'_1 = 2$. If $n'_1 \geq 3$, the structure can be completely pinned to the underlying plane, and thus has zero degrees of freedom.

We recall the following standard definition:

Definition 2.6.2. *G is Laman-sparse if for every subgraph $G' \subseteq G$ with $n' \geq 2$, $m' \leq 2n' - 3$.*

Laman-sparsity and sparsity are equivalent if there are only vertices of type 2, i.e. $n = n_2$. Moreover, a sparse graph is always Laman-sparse.

Definition 2.6.3. *G is* minimally rigid *if either $n = 1$, or G is sparse and*

$$m = n_1 + 2n_2 + \min(0, n_1 - 3). \tag{2.8}$$

Remark 2.6.4. *For $n_1 \geq 6$, a minimally rigid graph G does not need to be connected as can be seen by considering the disjoint union of two cliques of size three with all nodes of type 1.*

Recall that a spanning subgraph is one that includes the entire vertex set V.

Definition 2.6.5. *A graph is* rigid *if it contains a spanning subgraph which is minimally rigid. A* rigid block *in G is defined to be a vertex-induced rigid subgraph. A* rigid component *of G is an inclusion-wise maximal block.*

Remark 2.6.6. *Note that for a sparse graph G, a rigid block is always minimally rigid.*

Note that a rigid component does not need to be connected. By definition, it is clear that rigidity is preserved under addition of edges and that the size of the largest (in terms of vertices covered) rigid component of a graph can only increase when edges are added.

We now describe our probabilistic setting: consider for the following statements the random graph $G \in \mathcal{G}(n, c/n)$ where each edge is present independently with probability c/n, with $c > 0$. Each vertex gets type 1 with probability $1 - q$ and type 2 with probability q, where $q \in [0, 1]$.

To state our result, we need some notations. Let $Q(x, y) = e^{-x} \sum_{j \geq y} \frac{x^j}{j!}$. We define the function $c^*(q)$ as follows:

- for $q \leq 1/2$, we set $c^*(q) = \frac{1}{1-q}$;
- for $q > 1/2$, let $\xi^* = \xi^*(q)$ be the positive solution to:

$$\xi \frac{(1-q)Q(\xi, 1) + qQ(\xi, 2)}{(1-q)Q(\xi, 2) + 2qQ(\xi, 3)} = 2.$$

  In this case we set

$$c^*(q) = \frac{\xi^*}{(1-q)Q(\xi^*, 1) + qQ(\xi^*, 2)}.$$

It will follow from the proof that the equation for ξ^* has indeed a unique positive solution and that for $q > 1/2$, $c^*(q) < \frac{1}{1-q}$.

We can now state our first theorem:

Theorem 2.6.7. *Let $G \in \mathcal{G}(n, c/n)$ with $c > 0$, and let $q \in [0, 1]$. Let $R_n(q, c)$ ($R_n^C(q, c)$, respectively) be the number of vertices covered by the largest rigid component (connected rigid block, respectively) of G.*

- *For $c > c^*(q)$, there is a giant rigid component in G, i.e., there exists $\alpha = \alpha(q, c) > 0$ such that*

$$\mathbb{P}\left(\frac{R_n(q, c)}{n} \geq \alpha\right) \to 1 \text{ as } n \to \infty$$

- *For $c < c^*(q)$, there is no giant rigid component in G; i.e.,*

$$\forall \alpha > 0 \ , \ \mathbb{P}\left(\frac{R_n(q,c)}{n} \geq \alpha\right) \to 0 \text{ as } n \to \infty$$

The above results also hold true for $R_n^C(q,c)$. Moreover, for $c > c^*(q)$, a.a.s., there is one unique giant rigid component (one unique giant connected rigid block, respectively).

Our next theorem states that the transition as c varies and q is held fixed is continuous for $q \leq 1/2$ and discontinuous for $q > 1/2$. More precisely, we have the following:

Theorem 2.6.8. *Let $G \in \mathcal{G}(n, c/n)$ with $c > 0$, and let $q \in [0,1]$, and let $R_n(q,c)$ as in the previous theorem.*
- *The transition is discontinuous for $q > 1/2$: let $q > 1/2$; there is $\alpha(q) = \alpha > 0$ such that for any $c > c^*(q)$,*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{R_n(q,c)}{n} \geq \alpha\right) = 1$$

- *The transition is continuous for $q \leq 1/2$: let $q \leq 1/2$; for any $\alpha > 0$,*

$$\lim_{c \to \frac{1}{1-q}} \lim_{n\to\infty} \mathbb{P}\left(\frac{R_n(q,c)}{n} \geq \alpha\right) = 0$$

We now relate rigidity and orientability. We start with the following definition of 1.5-*orientability.*

Definition 2.6.9. *A graph is 1.5-orientable if there exists an orientation of the edges such that type 1 vertices have in-degree at most 1 and type 2 vertices have in-degree at most 2.*

A standard argument in the context of network flows gives (see Proposition 3.3 in [134])

Proposition 2.6.10. *A graph G is* 1.5*-orientable if and only if for every induced subgraph G' of G, $m' \leq n_1' + 2n_2'$.*

As a corollary, we see that a sparse graph G is always 1.5-orientable. Moreover, we see that if G is 1.5-orientable, then G will remain 1.5-orientable after removing some edges and if G is not 1.5-orientable then adding edges cannot make it 1.5-orientable.

Our next theorem shows that the threshold for being 1.5-orientable for the random graph $G \in \mathcal{G}(n, c/n)$ is the same as the one for the appearance of a giant rigid component.

Theorem 2.6.11. *Let $G \in \mathcal{G}(n, c/n)$ with $c > 0$, and let $q \in [0,1]$.*

(a) if $c < c^*(q)$, G is 1.5-orientable a.a.s.

(b) if $c > c^*(q)$, G is not 1.5-orientable a.a.s.

We now relate the notion of rigidity and 1.5-orientability with a new notion of core.

Definition 2.6.12. *For a graph with type* 1 *and type* 2 *vertices, the* 2.5-core *is the largest induced subgraph with all type* 1 *vertices with degree at least* 2 *and all type* 2 *vertices with degree at least* 3*.*

Note that this definition coincides with the 2-core (3-core, respectively) if the graph contains only type 1 vertices (type 2, respectively).

One can show that we can construct the 2.5-core by removing recursively type 1 vertices with degree at most 1 and type 2 vertices with degree at most 2. Note that the 2.5-core can be empty and in this case, the graph is 1.5-orientable. More generally, a graph G is 1.5-orientable if and only if its 2.5-core is orientable.

Clearly the size of the 2.5-core can only increase with the addition of edges. In our probabilistic setting, it turns out that for a fixed q, the 2.5-core appears at a value $\tilde{c}(q) \leq c^*(q)$.

Let $Q(x, y)$ as before. We define

$$\tilde{c}(q) = \inf_{\xi>0} \frac{\xi}{(1-q)Q(\xi,1) + qQ(\xi,2)}. \tag{2.9}$$

Note that as $\xi \to 0$, we have $\frac{\xi}{(1-q)Q(\xi,1)+qQ(\xi,2)} \to \frac{1}{1-q}$, in particular $\tilde{c}(q) \leq \frac{1}{1-q}$. Let $\tilde{\xi}(q, c)$ be the largest solution to

$$\xi = c(1-q)Q(\xi,1) + cqQ(\xi,2). \tag{2.10}$$

We can now state the theorem:

Theorem 2.6.13. *Let $G \in \mathcal{G}(n, c/n)$ with $c > 0$ and let $q \in [0, 1]$. Let Core be the 2.5-core of G, $n_1(Core)$ ($n_2(Core)$, respectively) be the number of nodes of type 1 (type 2, respectively) in the core and $m(Core)$ be the number of edges in the core. We have*

(a) if $c < \tilde{c}(q)$ and $q > 0$, then a.a.s. the 2.5-core has $o_p(n)$ vertices.

(b) if $c > \tilde{c}(q)$, then a.a.s. the 2.5-core is such that $n_1(Core)/n \to (1-q)Q(\tilde{\xi}(q,c)), 2)$, $n_2(Core)/n \to qQ(\tilde{\xi}(q,c)), 3)$, and $2m(Core)/n \to \tilde{\xi}(q,c)\left((1-q)Q(\tilde{\xi}(q,c)), 1) + qQ(\tilde{\xi}(q,c)), 2)\right)$.

Remark 2.6.14. *If the core is not $o_p(n)$, i.e., when $c > \tilde{c}(q)$, we have*

$$\frac{m(Core)}{n_1(Core) + 2n_2(Core)} \to \frac{\tilde{\xi}(q)}{2} \frac{(1-q)Q(\tilde{\xi}(q)), 1) + qQ(\tilde{\xi}(q)), 2)}{(1-q)Q(\tilde{\xi}(q)), 2) + 2qQ(\tilde{\xi}(q)), 3)}.$$

In particular, if this ratio is larger than one, then the 2.5-core is not 1.5-orientable. A simple computation shows that this ratio becomes larger than one exactly for $c > c^*(q) \geq \tilde{c}(q)$. Moreover, we have $c^*(q) = \tilde{c}(q) = \frac{1}{1-q}$ for $q \leq 0.5$ and $c^*(q) > \tilde{c}(q)$ for $q > 0.5$.

Remark 2.6.15. *When q is fixed and we increase c from 0 to infinity, it is easy to note the following from the previous theorem: for $q \leq 1/2$, the size of the 2.5-core is continuous in c whereas for $q > 1/2$, the 2.5-core appears discontinuously.*

In the absence of sliders ($q = 0$), the largest rigid component is closely related to the $3+2$-core (see [97]). This led the authors of [97] to formulate a conjecture on the size of the $3+2$-core. We introduce now a generalization of the $3+2$-core which will play a role in our proof of Theorem 2.6.8.

Definition 2.6.16. *Starting from the 2.5-core, one constructs a larger subgraph as follows: add recursively type 1 vertices which are linked by one edge with the current subgraph, and type 2 vertices which are linked by two edges with the current subgraph. The resulting subgraph is called the* $2.5 + 1.5$-core.

Note that this definition coincides with the 2+1-core (3+2-core, respectively) if the graph contains only type 1 vertices (type 2, respectively).

Furthermore, we also compute the threshold and the size of the 2.5+1.5-core. This proves a conjecture in [97] on the 3+2-core. The proof follows again the ideas in [93]. We use the same definitions of $\tilde{c}(q)$ and $\tilde{\xi}(q)$ as before and state the following theorem:

Theorem 2.6.17. *Let* $G \in \mathcal{G}(n, c/n)$ *with* $c > 0$ *and* $q \in [0, 1]$. *Let* $Core+$ *be the* $2.5 + 1.5$*-core of* G, *and* $n(Core+)$ *the number of vertices inside the* $2.5 + 1.5$*-core. If* $c > \tilde{c}(q)$, *where* $\tilde{c}(q)$ *is defined by (2.9), then a.a.s.,* $n(Core+)/n \to 1 - e^{-\tilde{\xi}} - q\tilde{\xi}e^{-\tilde{\xi}}$, *where* $\tilde{\xi}$ *is defined in (2.10).*

Remark 2.6.18. *For* $q \leq 1/2$, *we have* $\tilde{c}(q) = \frac{1}{1-q}$, *and if* $c \searrow \frac{1}{1-q}$, *then we have* $\tilde{\xi} \to 0$, *and thus* $n(Core+)/n \to 0$.

For the proof of the aforementioned theorems, the following lemma plays a crucial role, and hence we state it here. Let $X_{n'}$ denote the number of subgraphs of size n' with more than $n'_1 + 2n'_2$ edges. We have:

Lemma 2.6.19. *Let* $q \in (0, 1)$, *and let* $G \in \mathcal{G}(n, p)$ *with* $p = c/n$ *and* $c < \frac{1}{1-q}$. *A.a.s., there exists a strictly positive constant* $\alpha = \alpha(q, c - \frac{1}{1-q}) > 0$ *such that* $\sum_{1 \leq n' \leq \alpha n} X_{n'} = 0$.

Idea of the proofs. The proof of Lemma 2.6.19 is based on a standard but tedious first moment method argument. This lemma rules out small subgraphs that are too dense, and this result is used in the proofs of the different theorems of this section. The orientability threshold is then proven using an application of the objective method as introduced in [113]. The rigidity result is then, similarly to the work of [97], obtained using the previous result about orientability and the connections between orientability and rigidity. The results about the cores are proven in the configuration model using exponential clocks for each half-edge, as introduced by [93].

2.7 Future work

On the one hand, a tightening of the given bounds, a wider range of the parameter p, as well as more precise results about the distribution of many of the parameters in question (metric dimension, acquaintance time, the game of revolutionaries and spies) is clearly part of future work. On the other hand, together with Paweł Prałat (Ryerson Univ.) and Colin McDiarmid (Univ. Oxford) we started to analyze the clique covering number of such graphs, that is, the minimum number of colors needed to color the vertices of a graph in such a way that there is no monochromatic maximal clique. We have preliminary results that show that for sufficiently sparse graphs the clique covering number is a.a.s. equal to the chromatic number, whereas for p sufficiently large the clique covering number is much smaller.

Chapter 3

Random geometric graphs

In this chapter we deal with different parameters of $\mathscr{G}(n,r)$. Compared to $\mathcal{G}(n,p)$, this model is less understood, but in recent years quite a few properties were also analyzed from a mathematical point of view (see [140]). We analyzed several properties of $\mathscr{G}(n,r)$ and mention results about the following parameters in this chapter: treewidth (based on [121], where we used the model $\mathcal{S}_n$), the relation between the graph distance and the Euclidean distance (see [52], again using $\mathcal{S}_n$ as the underlying model), Hamiltonicity (see [55], using $[0,1]^2$ as underlying square), domination number ([31], again using $[0,1]^2$ as underlying square), the existence of small-size components (see [54], using the unit torus $[0,1)^2$ as underlying space) and a dynamic version of random geometric graphs (see [53], again using the unit torus $[0,1)^2$ as underlying space). Whereas the choice of the size of the underlying square clearly does not affect the results, the restriction in the latter two sections to consider the unit torus $[0,1)^2$ is done in order to avoid otherwise tedious complications around the boundary of $[0,1]^2$; we conjecture that both results can be extended to the unit square $[0,1]^2$ as well. All results in this section are asymptotic as $n \to \infty$, and under some restrictions on the radius $r = r(n)$, depending on the parameter investigated.

3.1 Treewidth of random geometric graphs

In this section, based on [121], we use the model $\mathcal{S}_n$. It is well known that the property of the existence of a giant component of order $\Theta(n)$ undergoes a sharp threshold in $\mathscr{G}(n,r)$ (see e.g. [77]), this is, there exists a constant value r_c such that for any $\varepsilon > 0$, a.a.s. the largest component of $G \in \mathcal{G}(n, r_c - \varepsilon)$ is of order $O(\log n)$, whereas in $G \in \mathcal{G}(n, r_c + \varepsilon)$, a single component of order $\Theta(n)$ is present, while the others have order $O(\log n)$ (see [140, Chapter 10]). The exact value of r_c is not yet determined. Since random geometric graphs have been heavily used for modeling communication networks, it is natural to analyze the expected complexity of different algorithms applied to this class. Courcelle's Theorem [47] states that any problem that can be expressed in monadic second order logic, can be solved in linear time for the class of graphs with bounded treewidth. This motivates the study of this parameter and other tree-like parameters on random geometric graphs.

The parameter treewidth was introduced independently by Halin in [83] and by Robertson and Seymour in [150].

For a graph $G = (V, E)$ on n vertices, we call $(T, \mathcal{W})$ a *tree decomposition* of G, where $\mathcal{W}$ is a set of vertex subsets $W_1, \ldots, W_s \subseteq V$, called bags, and T is a forest with vertices in $\mathcal{W}$, such that

1. $\bigcup_{i=1}^{s} W_i = V$.
2. For any $e = uv \in E$ there exists a set $W_i \in \mathcal{W}$ such that $u, v \in W_i$.
3. For any $v \in V$, the subgraph induced by the $W_i \ni v$ is connected as a subgraph of T.

The *width* of a tree-decomposition is $\mathrm{w}(T, \mathcal{W}) = \max\limits_{1 \le i \le s} |W_i| - 1$, and the *treewidth* of a graph G can be defined as

$$\mathrm{tw}(G) = \min_{(T,\mathcal{W})} \mathrm{w}(T, \mathcal{W}).$$

Observe that if G is a graph with connected components $H_1, \ldots, H_m$, then

$$\mathrm{tw}(G) = \max_{1 \le i \le m} \mathrm{tw}(H_i) \ . \tag{3.1}$$

The concept of treedepth has been introduced under different names in the literature. In our paper [121] we follow the definition given by Nešetřil and Ossona de Mendez as a tree-like parameter in the scope of homomorphism theory, where it provides an alternative definition of bounded expansion classes [135]. For the sake of completeness, we note that the treedepth is also equivalent to the height of an elimination tree (used for instance in the parallel Cholesky decomposition [145]). Furthermore, analogous definitions can be found using the terminology of rank function [136], vertex ranking number (or ordered coloring) [51] or weak coloring number [101]. We now give the precise definition of treedepth. Let T be a rooted tree. The *height* of T is defined as the number of vertices of the longest rooted path. The *closure* of T is the graph that has the same set of vertices and a pair of vertices is connected by an edge if one is an ancestor of the other in T. We say that the tree T is an *elimination tree* of a connected graph G if G is a subgraph of the closure of T. The *treedepth* of a connected graph G, $\mathrm{td}(G)$, is defined to be the minimum height of an elimination tree of G.

The definition of treedepth can also be extended to non-connected graphs. If G is a graph with connected components $H_1, \ldots, H_m$,

$$\mathrm{td}(G) = \max_{1 \le i \le m} \mathrm{td}(H_i) \ . \tag{3.2}$$

Hence, if $S \subset V(G)$ separates G into two subsets A and B, we have

$$\mathrm{td}(G) \le |S| + \max\{\mathrm{td}(A), \mathrm{td}(B)\} \ . \tag{3.3}$$

Observe that if H is a subgraph of G, then

$$\mathrm{td}(H) \le \mathrm{td}(G) \text{ and } \mathrm{tw}(H) \le \mathrm{tw}(G) \ . \tag{3.4}$$

Both parameters are closely connected: while the treewidth of a graph G is a parameter that measures the similarity between G and the class of trees in general, the treedepth of G measures how close G is to a star. In other words, the treedepth also takes into account the

diameter of the tree we are comparing the graph with. The two parameters are related by the following inequalities:

$$\mathrm{tw}(G) \leq \mathrm{td}(G) \leq (\mathrm{tw}(G)+1)\log_2 n,$$

both bounds being sharp (see [135]). Note also that $\mathrm{tw}(G) \geq \omega(G)-1$, where $\omega(G)$ denotes the size of the largest clique in G.

Our main results on treewidth and related parameters are the following two theorems:

Theorem 3.1.1. *Let $0 < r < r_c$ and let $G \in \mathscr{G}(n,r)$. Then, a.a.s., $tw(G) = \Theta(\frac{\log n}{\log\log n})$, and also a.a.s., $td(G) = \Theta(\frac{\log n}{\log\log n})$.*

Theorem 3.1.2. *Let c be a sufficiently large constant. Let $r = r(n) \geq c$ and $G \in \mathscr{G}(n,r)$. Then, a.a.s., $tw(G) = \Theta(r\sqrt{n})$, and also a.a.s., $td(G) = \Theta(r\sqrt{n})$.*

Remark 3.1.3. *Other width parameters that are sandwiched between the treewidth and the treedepth clearly then also have the same asymptotic behavior in $\mathscr{G}(n,r)$. For instance, the* pathwidth *of a graph, introduced by Robertson and Seymour [149], measures the similarity between a graph and a path. Since the pathwidth is well known to be bounded from below by the treewidth and bounded from above by the treedepth (see Theorem 5.3 and Theorem 5.11 of [154]), the former theorems imply that for those values of $r = r(n)$ the pathwidth of the graph is of the same order.*

Remark 3.1.4. *Whereas intuitively it might be clear that around the threshold of the existence of a giant component there should be a jump for parameters like treewidth or treedepth in $\mathscr{G}(n,r)$, the orders of magnitude of these parameters are not so obvious (for us). Moreover, we point out that there are differences between $\mathscr{G}(n,r)$ and $\mathcal{G}(n,p)$: it is known that in the Erdős-Rényi random graph model $\mathcal{G}(n,p)$, as soon as the giant component appears, the graph has linear treewidth (see [112]). In contrast to this, Theorem 3.1.2 shows that a random geometric graph with a linear number of edges containing a giant component only has treewidth $\Theta(\sqrt{n})$. This different behavior of the two models can be explained by their different expansion properties and the connection between balanced separators and treewidth. Classical random graphs have very good expansion properties, and thus it is difficult to find small separators of large sets of vertices. The geometric properties of the model $\mathscr{G}(n,r)$ imply the lack of large expanders. For this reason, in the latter case one can construct a tree decomposition with smaller bags. On the other hand, in the subcritical regime (with a linear number of edges, but before the existence of a giant component) the treedepth of $\mathcal{G}(n,p)$ is $\Theta(\log\log n)$ (see [142]), whereas by Theorem 3.1.1, for random geometric graphs it is already $\Theta(\frac{\log n}{\log\log n})$. (In fact, a lower bound of this order is very easy, since the largest clique is of that order, and an upper bound of $O(\log n)$ is also easy, since $O(\log n)$ is an upper bound for the size of the largest component). Furthermore, in this range, in classical random graphs the treewidth is bounded by a constant (see [142]), whereas our theorems show that in $\mathscr{G}(n,r)$ both treewidth and treedepth are asymptotically of the same order for a wide range of parameters r. The fact that for random geometric graphs the treedepth and treewidth are always asymptotically of the same order implies that $\mathscr{G}(n,r)$ is more similar to a star–shaped tree than to a path–shaped tree, which in general is not true for $\mathcal{G}(n,p)$.*

In order to simplify calculations, we will use the well-known idea of Poissonization (see [140, Section 1.7]): let V be a set of points obtained as a homogeneous Poisson point process $\mathcal{G}(P_1, r)$ of intensity 1 in $\mathcal{S}_n$. In other words, V consists of N points in the square $\mathcal{S}_n$ chosen independently and uniformly at random, where N is a Poisson random variable of mean n. Exactly as in $\mathscr{G}(n, r)$, two points $u, v \in V$ are connected by an edge if their Euclidean distance in $\mathcal{S}_n$ is at most r. The main advantage of the Poisson point process is that the number of points of V that lie in any region $A \subseteq \mathcal{S}_n$ of area a has a Poisson distribution with mean a; and the number of points of V in disjoint regions of $\S_n$ are independently distributed. Moreover, by conditioning $\mathcal{G}(P_1, r)$ upon the event $N = n$, we recover the original distribution of $\mathscr{G}(n, r)$. Therefore, since $\mathbb{P}(N = n) = \Theta(1/\sqrt{n})$, any event holding in $\mathcal{G}(P_1, r)$ with probability at least $1 - o(f_n)$ must hold in $\mathscr{G}(n, r)$ with probability at least $1 - o(f_n\sqrt{n})$. In particular, an event holding with probability $1 - o(n^{-1/2})$ in $\mathcal{G}(P_1, r)$ holds a.a.s. in $\mathscr{G}(n, r)$. We make extensive use of this property, and we perform the proofs of Theorem 3.1.1 and Theorem 3.1.2 for a graph $G \in \mathcal{G}(P_1, r)$.

Ideas of the proofs. The proof of the lower bound of Theorem 3.1.2 makes use of the following fact proven by [105]: a vertex partition $V = (A, S, B)$ is a balanced k-partition, if $|S| = k + 1$, S separates A and B, and $\frac{1}{3}(n - k - 1) \leq |A|, |B| \leq \frac{2}{3}(n - k - 1)$. If G is such that $\mathrm{tw}(G) \leq k$, then there exists a balanced k-partition. We show by taking a union bound over all possible separators that separators have to be large, since they must span a relatively large region. For the upper bound, we provide an elimination tree as sketched in Figure 3.1. Regarding the proof of Theorem 3.1.1, the lower bound follows directly from the size of the largest clique. For the upper bound, we construct an elimination tree using an auxiliary cell graph as in the proof of Theorem 3.1.2.

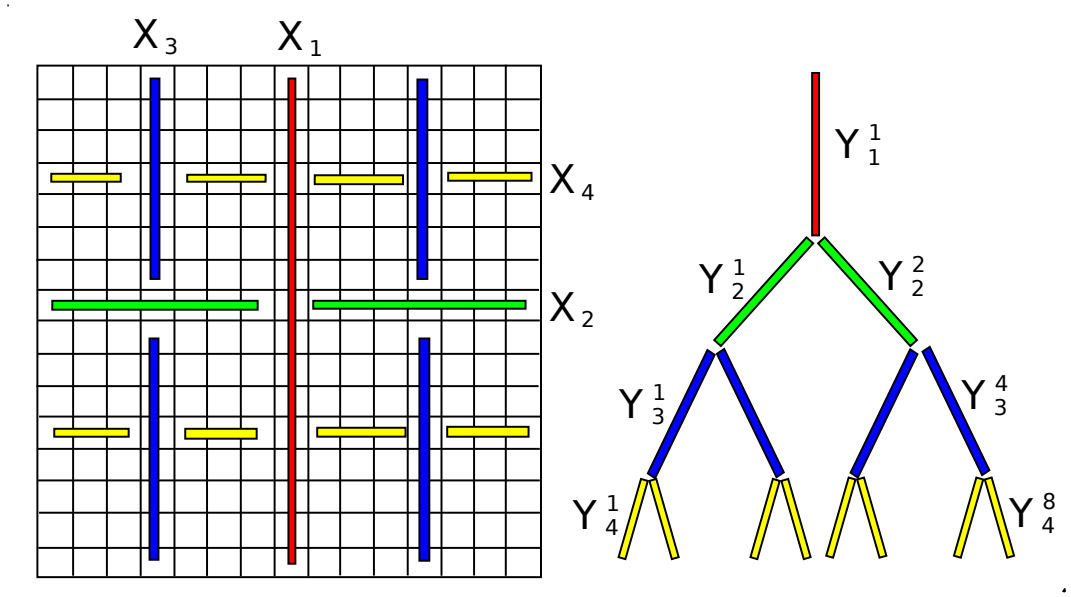


Figure 3.1: Construction of the sets X_i^j.

Remark 3.1.5. *We showed that in the case $r \geq c$, for some sufficiently large c, $tw(G) = \Theta(r\sqrt{n})$ and conjectured that the latter can be extended to the whole supercritical regime, that*

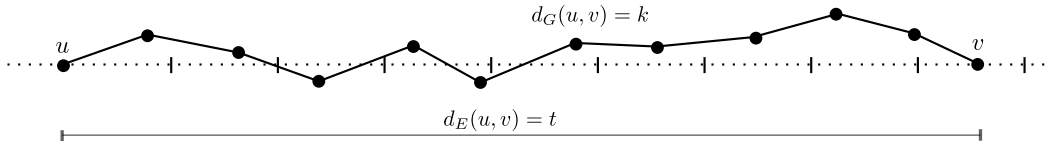


Figure 3.2: Graph distance vs. Euclidean distance between two points u and v in V

is, we conjecture that for every $r > r_c$, $tw(G) = \Theta(r\sqrt{n})$. Recently, Li and Müller, using refinements of our arguments, proved the conjecture (see [114]).

3.2 On the relation between the graph distance and the Euclidean distance

This section is based on the paper [52], again using $\mathcal{S}_n$ as the underlying square. Given a connected graph G, we define the *graph distance* between two vertices u and v, denoted by $d_G(u,v)$, as the number of edges on a shortest path from u to v. Observe first that any pair of vertices u and v must satisfy $d_G(u,v) \geq d_E(u,v)/r$ deterministically by the triangle inequality, since each edge of a geometric graph has length at most r. The goal of our paper [52] is to provide upper and lower bounds that hold a.a.s. for the graph distance of two vertices in terms of their Euclidean distance and in terms of r (see Figure 3.2).

Related work. This particular problem has risen quite a bit of interest in recent years. Given any two vertices $u, v \in V$, most of the work related to this problem has been devoted to study upper bounds on $d_G(u,v)$ in terms of $d_E(u,v)$ and r. Ellis, Martin and Yan [63] showed that there exists some large constant K such that for every $r \geq (1+\varepsilon)r_c$, $G \in \mathscr{G}(n,r)$ satisfies a.a.s. the following property: for every $u, v \in V$ such that $d_E(u,v) > r$,

$$d_G(u,v) \leq K \cdot \frac{d_E(u,v)}{r} \,. \tag{3.5}$$

Their result is stated in the unit ball random geometric graph model, but it can be easily adapted into our setting. This result was extended by Bradonjic et al. [35] for the range of r for which $\mathscr{G}(n,r)$ has a giant component a.a.s., under the extra condition that $d_E(u,v) = \Omega(\log^{7/2} n/r^2)$. Friedrich, Sauerwald and Stauffer [70] improved this last result by showing that the result holds a.a.s. for every u and v satisfying $d_E(u,v) = \omega(\log n/r)$. They also proved that if $r = o(r_c)$, a linear upper bound of $d_G(u,v)$ in terms of $d_E(u,v)/r$ is no longer possible. In particular, a.a.s. there exist vertices u and v with $d_E(u,v) \leq 3r$ and $d_G(u,v) = \Omega(\log n/r^2)$.

The motivation for the study of this problem stems from the fact that these results provide upper bounds for the diameter of $G \in \mathscr{G}(n,r)$, denoted by $\mathrm{diam}(G)$, that hold a.a.s., and the runtime complexity of many algorithms can often be bounded from above in terms of the diameter of G. For a concrete example, we refer to the problem of broadcasting information (see [35, 70]).

We show that a.a.s. one can take the constant K for which (3.5) holds as $K = 1 + o(1)$,

provided that $r = \omega(r_c)$. By the aforementioned result in [70], we know that the statement is false if $r = o(r_c)$.

A similar problem has been studied by Muthukrishnan and Pandurangan [132]. They proposed a new technique to study several problems on random geometric graphs — the so called *Bin-Covering* technique — which tries to cover the endpoints of a path by bins. They consider, among others, the problem of determining $D_G(u, v)$, which is the length of the shortest Euclidean path connecting u and v. Recently, Mehrabian and Wormald [119] studied a similar problem to the one in [132]. They deploy n points uniformly in $[0, 1]^2$, and connect any pair of points with probability p, independently of their distance. In this model, they determine the ratio of $D_G(u, v)$ and $d_E(u, v)$ as a function of p.

The following theorem is the main result of [52].

Theorem 3.2.1. *Let $G \in \mathscr{G}(n, r)$ be a random geometric graph on n vertices and radius $0 < r \leq \sqrt{2n}$. A.a.s., for every pair of vertices $u, v \in V(G)$ with $d_E(u, v) > r$ (as otherwise the statement is trivial) we have:*

(i) if $d_E(u, v) \geq \max\left\{12(\log n)^{3/2}/r,\, 21r \log n\right\}$, then

$$d_G(u, v) \geq \left\lfloor \frac{d_E(u, v)}{r} \left(1 + \frac{1}{2\left(r d_E(u, v)\right)^{2/3}} \right) \right\rfloor ;$$

(ii) if $r \geq 224\sqrt{\log n}$, then

$$d_G(u, v) \leq \left\lceil \frac{d_E(u, v)}{r} \left(1 + \gamma r^{-4/3}\right) \right\rceil ,$$

where

$$\gamma = \gamma(u, v) = \max\left\{ 1358 \left(\frac{3r \log n}{r + d_E(u, v)} \right)^{2/3},\ \frac{4 \cdot 10^6 \log^2 n}{r^{8/3}},\ 30000^{2/3} \right\} .$$

In order to prove (i), we first observe that all short paths between two points must lie in a certain rectangle. Then we show that, by restricting the construction of the path onto that rectangle, no very short path exists. For the proof of (ii) we proceed similarly. We restrict our problem to finding a path contained in a narrow strip. In this case, we show that a relatively short path can be constructed. We believe that the ideas in the proof can be easily extended to show the analogous result for d-dimensional random geometric graphs for all fixed $d \geq 2$.

Remark 3.2.2. *1. Note that the condition $d_E(u, v) \geq \max\{12(\log n)^{3/2}/r, 21r \log n\}$ in the lower bound of (i) can be replaced by $d_E(u, v) \geq 21r \log n$ if $r \geq \sqrt{4/7}(\log n)^{1/4}$, and by $d_E(u, v) \geq 12(\log n)^{3/2}/r$ if $r \leq \sqrt{4/7}(\log n)^{1/4}$. We do not know whether this condition can be made less restrictive, besides improving the multiplicative constants involved (which we did not attempt to optimize).*

2. *Similarly, the constant* 224 *in the condition* $r \geq 224\sqrt{\log n}$ *of (ii) (as well as those in the definition of* γ*) is not optimized either, and could be made slightly smaller. However, our method as is cannot be extended all the way down to* $r \geq \sqrt{\log n/\pi} = r_c$.

3. *Moreover, the error term in part (i) is* $\left(2(rd_E(u,v))^{2/3}\right)^{-1} = O(1/\log n) = o(1)$.

4. *Finally, the error term in (ii) is*

$$\gamma r^{-4/3} = \Theta\left(\max\left\{\left(\frac{\log n}{r^2 + rd_E(u,v)}\right)^{2/3},\ \left(\frac{\sqrt{\log n}}{r}\right)^4,\ r^{-4/3}\right\}\right),$$

which is $o(1)$ *iff* $r = \omega(\sqrt{\log n}) = \omega(r_c)$. *Hence, for* $r = \omega(r_c)$, *statement (ii) implies that a.a.s.*

$$d_G(u,v) \leq \left\lceil (1+o(1))\,\frac{d_E(u,v)}{r}\right\rceil,$$

thus improving the result in [63].

Theorem 3.2.1 gives an upper bound on the diameter as a corollary. First, observe that $d_E(u,v) \leq \sqrt{2n}$. From Theorem 10 in [63] for the particular case $d = 2$, one can deduce that if $r \geq (1+\varepsilon)r_c$ a.a.s.

$$\operatorname{diam}(G) \leq \frac{\sqrt{2n}}{r}\left(1 + O\left(\sqrt{\frac{\log\log n}{\log n}}\right)\right). \tag{3.6}$$

Directly from Theorem 3.2.1 we have that, for $r \geq 224\sqrt{\log n}$,

$$\operatorname{diam}(G) \leq \left\lceil \frac{\sqrt{2n}}{r}\left(1 + \widehat{\gamma} r^{-4/3}\right)\right\rceil, \tag{3.7}$$

where

$$\widehat{\gamma} = \Theta\left(\left(\frac{r\log n}{\sqrt{n}}\right)^{2/3} + \frac{\log^2 n}{r^{8/3}} + 1\right).$$

(In fact, (3.7) holds for all $r \geq (1+\varepsilon)r_c$ as a consequence of (3.6)). From straightforward computations, one can check that (3.7) improves (3.6) provided that $r = \Omega\left(\frac{\log^{5/8} n}{(\log\log n)^{1/8}}\right)$.

On the other hand, for the lower bound on the diameter, observe the following: for any function ω growing arbitrarily slowly with n, we can a.a.s. find two vertices u and v, each at distance at most ω from one corner (opposite from each other) of the square $\mathcal{S}_n$. For such two vertices, we trivially and deterministically have (since the sign is important here, we point out that all expressions inside asymptotic notation are positive numbers here)

$$\operatorname{diam}(G) \geq d_G(u,v) \geq \left\lceil \frac{\sqrt{2n}\left(1 - \Theta(\omega/\sqrt{n})\right)}{r}\right\rceil \tag{3.8}$$

Assuming that $\sqrt{\log^3 n/n} \ll r \ll \sqrt{n}/\log n$, our bound from Theorem 3.2.1 applied to these vertices gives that a.a.s.

$$\begin{aligned}\operatorname{diam}(G) \geq d_G(u,v) &\geq \left\lfloor \frac{\sqrt{2n}\big(1-\Theta(\omega/\sqrt{n})\big)}{r}\left(1+\Theta(r^{-2/3}n^{-1/3})\right)\right\rfloor \\ &\geq \left\lceil \frac{\sqrt{2n}\left(1-\Theta(\omega/\sqrt{n})+\Theta(r^{-2/3}n^{-1/3})\right)}{r} - 1\right\rceil. \end{aligned} \tag{3.9}$$

Assuming the additional constraint $r \ll n^{1/10}$, we have that $r^{-2/3}n^{-1/3} \gg r/\sqrt{n}$, and also $r^{-2/3}n^{-1/3} \gg \omega/\sqrt{n}$ (for ω tending to infinity sufficiently slowly). In this case, our bound in (3.9) improves upon the trivial lower bound (3.8), and can be written as

$$\operatorname{diam}(G) \geq \left\lfloor \frac{\sqrt{2n}}{r}\left(1+\Theta(r^{-2/3}n^{-1/3})\right)\right\rfloor. \tag{3.10}$$

Note that this is a.a.s. still valid if we drop the constraint $r \gg \sqrt{\log^3 n/n}$, since for $r = O\left(\sqrt{\log^3 n/n}\right)$ the random geometric graph is a.a.s. disconnected (and has infinite diameter). Hence, by (3.7) and (3.10), we obtain the following corollary:

Corollary 3.2.3. *Let $G \in \mathscr{G}(n,r)$ be a random geometric graph on n vertices and radius $0 < r \leq \sqrt{2n}$. A.a.s. we have:*

(i) if $r \geq (1+\varepsilon)r_c$, then

$$\mathit{diam}(G) \leq \left\lceil \frac{\sqrt{2n}}{r}\left(1+\widehat{\gamma}r^{-4/3}\right)\right\rceil,$$

where

$$\widehat{\gamma} = \Theta\left(\left(\frac{r\log n}{\sqrt{n}}\right)^{2/3} + \frac{\log^2 n}{r^{8/3}} + 1\right).$$

(ii) if $r \ll n^{1/10}$, then

$$\mathit{diam}(G) \geq \left\lfloor \frac{\sqrt{2n}}{r}\left(1+\Theta(r^{-2/3}n^{-1/3})\right)\right\rfloor.$$

3.3 Hamiltonicity

In this section, based on [55], we consider $\mathscr{G}(n,r)$ on the unit square $[0,1]^2$. Given a graph G on n vertices, a *Hamiltonian cycle* is a simple cycle that visits each vertex of G exactly once. A graph is said to be *Hamiltonian* if it contains a Hamiltonian cycle. Two known facts for the Hamiltonicity of random graphs are that almost all d-regular graphs ($d \geq 3$) are Hamiltonian [151], and that in the $\mathcal{G}(n,p)$ model, if $p = (\log n + \log\log n + \omega(n))/n$, then a.a.s. $\mathcal{G}(n,p)$ is Hamiltonian [106] (see also chapter 8 of [27]). A natural issue to

study is the existence of Hamiltonian cycles in $\mathscr{G}(n,r)$. Penrose in his book [140] posed it as an open problem whether exactly at the point where $\mathscr{G}(n,r)$ gets 2-connected, the graph also becomes Hamiltonian a.a.s. Petit in [143] proved that for $r=\omega(\sqrt{\log n/n})$, $\mathscr{G}(n,r)$ is Hamiltonian a.a.s. and he also gave a distributed algorithm to find a Hamiltonian cycle in $\mathscr{G}(n,r)$ with his choice of radius. We find the sharp threshold for this property in any ℓ_p metric. In fact, let p $(1\leq p\leq\infty)$ be arbitrary but fixed throughout this section, and let $\mathscr{G}(n,r)$ be a random geometric graph with respect to ℓ_p. We show the following theorem:

Theorem 3.3.1. *The property that a random geometric graph $\mathcal{G}=\mathscr{G}(n,r)$ contains a Hamiltonian cycle exhibits a sharp threshold at $r=\sqrt{\frac{\log n}{\alpha_p n}}$, where α_p is the area of the unit disk in the ℓ_p norm. More precisely, for any $\epsilon>0$,*

- *if $r=\sqrt{\frac{\log n}{(\alpha_p+\epsilon)n}}$, then a.a.s. $\mathcal{G}$ contains no Hamiltonian cycle,*
- *if $r=\sqrt{\frac{\log n}{(\alpha_p-\epsilon)n}}$, then a.a.s. $\mathcal{G}$ contains a Hamiltonian cycle,*

Idea of the proof. The lower bound of the threshold is trivial, since if $r=\sqrt{\frac{\log n}{(\alpha_p+\epsilon)n}}$, then a.a.s. $\mathcal{G}$ is disconnected [139] and hence it cannot contain any Hamiltonian cycle. The proof of the upper bound uses an auxiliary cell graph obtained from a tessellation into squares of length $\Theta(r)$: we first show that most cells contain a sufficiently large (constant) number of vertices, and that the cell graph obtained from these cells is connected, so that a Hamilton cycle on them can be constructed. Vertices of sparser cells are then inserted into the previous Hamilton cycle by using unused edges emanating from dense cells.

Remark 3.3.2. *The conjecture of Penrose (see [140]) that exactly at the point where $\mathscr{G}(n,r)$ gets 2-connected the graph a.a.s. gets Hamiltonian was after publication of our result then proved independently by [13] and [130].*

3.4 Domination number

This section is based on [31] with the underlying square being $[0,1]^2$. A *dominating set* in a graph G is a set of nodes S in G such that every node not in S is adjacent to at least one node in S. The *domination number* of G, written $\gamma(G)$, is the minimum cardinality of a dominating set in G. Dominating sets appear in numerous applications such as: network controllability [48], as a centrality measure for efficient data routing [159], and detecting biologically significant proteins in protein-protein interaction network [120]. We derive the following result.

Theorem 3.4.1. *Let $G\in\mathscr{G}(n,r)$ and let ω be any function tending to infinity as $n\to\infty$. Then a.a.s. the following holds:*

(a) Denote by $N(x)$ the minimal number of balls of radius x needed to cover $[0,1]^2$. If $r=\Theta(1)$, then

$$\Omega(1)=N(r+\sqrt{\omega\log n/n})\leq\gamma(G)\leq N(r-\omega/\sqrt{n}).$$

(b) Define $C = 2\pi\sqrt{3}/9 \approx 1.209$. *If* $\omega\sqrt{\log n/n} \leq r = o(1)$, *then*

$$\gamma(G) = (C/\pi + o(1))r^{-2}.$$

(c) If $1/\sqrt{n} \leq r < \omega\sqrt{\log n/n}$, *then*

$$\gamma(G) = \Theta(r^{-2}).$$

(d) If $r < 1/\sqrt{n}$, *then*

$$\gamma(G) = \Theta(n).$$

Remark 3.4.2. *It is straightforward to verify that the bounds on* $\gamma(G)$ *in part (a) differ by at most* 1 *if* ω *is sufficiently small, but in general we do not give accurate estimations of* $N(r)$ *for* $r = \Theta(1)$.

Overview of the proof. We relate the domination number to the problem of covering the plane with circles. Given $x \in \mathbb{R}^2$ and $\rho > 0$, we denote by $\mathcal{B}(x, \rho)$ the ball with centre x and of radius ρ. The following theorem is well known (see [99]):

Theorem 3.4.3 ([99])**.** *Given a bounded subset of the plane* M, *for* $\varepsilon > 0$ *let* $N(\varepsilon)$ *be the minimum number of balls of radius* ε *that can cover* M. *Then we have that*

$$\lim_{\varepsilon \to 0} \pi\varepsilon^2 N(\varepsilon) = C \operatorname{Area}(\overline{M}),$$

where $\overline{M}$ *denotes the closure of* M.

Observe that $(C - 1)$ can therefore, be seen as measuring the proportion of unavoidable overlapping. Moreover, [99] shows that an optimal covering of the square $[0, 1]^2$ using balls of radius ε corresponds to arranging the balls in such a way that their centers are the centers of the cells of a hexagonal tiling of length ε. More precisely, consider the lattice

$$\mathcal{L}_\varepsilon = \{i\varepsilon(\sqrt{3}, 0) + j\varepsilon(\sqrt{3}/2, 3/2) : i, j \in \mathbb{Z}\}. \tag{3.11}$$

Then the set of balls of radius ε and centre in $\mathcal{L}_\varepsilon$ that intersect $[0, 1]^2$ form a covering of $[0, 1]^2$ that gives the limit in Theorem 3.4.3. Note that for all G with maximum degree Δ, we trivially have $\gamma(G) \geq n/(1 + \Delta(G))$ (for further relations between $\gamma(G)$ and other graph parameters, see, for example, [85]). Given any constant $c > 0$, for $G \in \mathscr{G}(n, r)$ with $r \geq c\sqrt{\log n/n}$, it is easy to show, by Chernoff bounds together with union bounds, that a.a.s. $\Delta(G) = O(r^2 n)$. Therefore, a.a.s. we have $\gamma(G) = \Omega(r^{-2})$. On the other hand, we can trivially construct a dominating set of $\mathscr{G}(n, r)$ by tessellating $[0, 1]^2$ into square cells of side length $r/\sqrt{2}$ and picking one node from each cell (if the cell is not empty). This holds deterministically for any geometric graph (not necessarily random), with no restriction on r, and gives $\gamma(G) = O(r^{-2})$. It follows that, for $G \in \mathscr{G}(n, r)$ with $r \geq c\sqrt{\log n/n}$, a.a.s. $\gamma(G) = \Theta(r^{-2})$. To prove the lower bound in part (b), we fix an arbitrarily small constant $\delta > 0$, and show using an appropriate tessellation into squares of size $\alpha = \sqrt{\omega \log n/n} = o(r)$, that no dominating set of size $\lfloor (C/\pi - \delta)r^{-2} \rfloor$ can exist: if that were the case, then $[0, 1]^2$ could be covered by $\lfloor (C/\pi - \delta)r^{-2} \rfloor$ many balls of radius $r + \alpha\sqrt{2}$. For the upper bound,

we find a covering of $[0,1]^2$ with $(C/\pi + o(1))r^{-2}$ balls of radius r that are centered at some nodes of G. Again, fix some arbitrarily small constant $\delta > 0$. Let $r' = (1-\delta)r$, and consider the lattice $\mathcal{L}_{r'}$, as defined in (3.11). Let $\mathcal{L}'_{r'}$ be the set of all points $x \in \mathcal{L}_{r'}$ such that the ball with center x and radius r' intersects $\mathcal{S}_n$. Since $\mathcal{L}'_{r'}$ gives the optimal covering of $\mathcal{S}_n$ with balls of radius r', we just have to show that for every such x the ball of radius δr around x contains at least one node. The other results use similar techniques and are comparably easier.

3.5 Small-size components

In this section, based on [54], we suppose that n vertices are distributed uniformly at random in the unit torus $[0,1)^2$. Although the toroidal model is less natural, on the boundary of $[0,1]^2$ tedious issues arise complicating calculations, and we leave it as an open conjecture that the same results hold for the unit square model as well. As before, two vertices are connected if their Euclidean distance (now in the torus) is at most a given threshold $r = r(n)$. Let K_1 be the random variable counting the number of isolated vertices in $\mathscr{G}(n,r)$. By multiplying the probability that one vertex is isolated by the number of vertices one obtains

$$\mathbb{E}K_1 = n(1-r^2\pi)^{n-1} = ne^{-\pi r^2 n - O(r^4 n)}.$$

Define $\mu = ne^{-\pi r^2 n}$. Observe from the previous expression that μ is closely related to $\mathbb{E}K_1$. In fact, $\mu = o(1)$ iff $\mathbb{E}K_1 = o(1)$, and if $\mu = \Omega(1)$, then $\mathbb{E}K_1 \sim \mu$. It is well known that the asymptotic behavior of μ characterizes the connectivity of $\mathscr{G}(n,r)$ (see [140]).

From the definition of μ we have that $\mu = \Theta(1)$ iff $r = \sqrt{\frac{\log n \pm O(1)}{\pi n}}$. Therefore we conclude that the property of connectivity of $\mathscr{G}(n,r)$ exhibits a sharp threshold at $r = \sqrt{\frac{\log n}{\pi n}}$. Note that the previous classification of the connectivity of $\mathscr{G}(n,r)$ indicates that if $\mu = \Theta(1)$, the components of size 1 are predominant and those components have the main contribution to the connectivity of $\mathscr{G}(n,r)$. In fact if $\mu = \Theta(1)$, the probability that $\mathscr{G}(n,r)$ has some component of size greater than 1 other than the giant component is $o(1)$. On the other hand, Penrose ([140]) studied the number of components in $\mathscr{G}(n,r)$ that are isomorphic to a given fixed graph; equivalently, he studied the probability of finding components of a given size in $\mathscr{G}(n,r)$. However the range of radii r covered by Penrose does not exceed the thermodynamical threshold $\Theta(\sqrt{1/n})$ where a giant component appears at $\mathscr{G}(n,r)$, which is below the connectivity threshold treated here. In fact, a percolation argument in [140] only shows that with probability $1 - o(1)$ no components other than isolated vertices and the giant one exist at the connectivity threshold, without giving accurate bounds on this probability (see Section 1.4 of [140] and Proposition 13.12 and Proposition 13.13 of [140]). Throughout this section we shall consider $\mathscr{G}(n,r)$ with $r = \sqrt{\frac{\log n \pm O(1)}{\pi n}}$. We prove that for such a choice of r, given a fixed $\ell > 1$, the probability of having components of size exactly ℓ is $\Theta(\frac{1}{\log^{\ell-1} n})$. Moreover, in the process of the proof we characterize the different types of components that could exist for such a value of r.

Given a component Γ of $\mathscr{G}(n,r)$, Γ is *embeddable* if it can be mapped into the square $[r, 1-r]^2$ by a translation in the torus. Embeddable components do not wrap around the

torus. Components which are not embeddable must have a large size (at least $\Omega(1/r)$). Sometimes several non-embeddable components can coexist together. However, there are some non-embeddable components which are so spread around the torus, that they do not allow any room for other non-embeddable ones. Call these components *solitary*. Clearly, we can have at most one solitary component. We cannot disprove the existence of a solitary component, since with probability $1-o(1)$ there exists a giant component of this nature (see Corollary 2.1 of [82], implicitly it is also in Theorem 13.11 of [140]). For components which are not solitary, we give asymptotic bounds on the probability of their existence according to their size.

Given a fixed integer $\ell \geq 1$, let K_ℓ be the number of components in $\mathscr{G}(n,r)$ of size exactly ℓ. For large enough n, we can assume these to be embeddable, since $r = o(1)$. Moreover, for any fixed $\varepsilon > 0$, let $K'_{\varepsilon,\ell}$ be the number of components of size exactly ℓ which have all their vertices at distance at most εr from their leftmost one. Finally, $\tilde{K}_\ell$ denotes the number of components of size at least ℓ and which are not solitary. Notice that

$$K'_{\varepsilon,\ell} \leq K_\ell \leq \tilde{K}_\ell.$$

However, in the following we show that asymptotically all the weight in the probability that $\tilde{K}_\ell > 0$ comes from components which also contribute to $K'_{\varepsilon,\ell}$ for ε arbitrarily small. This means that the more common components of size at least ℓ are cliques of size exactly ℓ with all their vertices close together. Our main theorem is the following:

Theorem 3.5.1. *Let $\ell \geq 2$ be a fixed integer. Let $0 < \varepsilon < 1/2$ be fixed. Assume that $\mu = \Theta(1)$. Then*

$$\mathbb{P}(\tilde{K}_\ell > 0) \sim \mathbb{P}(K_\ell > 0) \sim \mathbb{P}(K'_{\varepsilon,\ell} > 0) = \Theta(\frac{1}{\log^{\ell-1} n}).$$

Idea of the proof. We first show that for any $0 < \varepsilon < 1/2$ and any integer $\ell \geq 2$, for $\mu = \Theta(1)$ we have $\mathbb{E}K'_{\varepsilon,\ell} = \Theta(\frac{1}{\log^{\ell-1} n})$. Then, considering different cases whether or not components are embeddable or not, solitary or not, and according to their diameter, we show that $\mathbb{P}(\tilde{K}_\ell - K'_{\varepsilon,\ell} > 0) = O(\frac{1}{\log^{\ell} n})$. Finally, the proof is completed by showing that he second factorial moment $\mathbb{E}(K'_{\varepsilon,\ell})_2$ satisfies $\mathbb{E}(K'_{\varepsilon,\ell})_2 = \Theta(\frac{1}{\log^{2\ell-2} n})$. Moreover, using similar arguments as in the proof of Theorem 3.5.1, we obtain a corollary of a well known result: given a random set $\mathscr{X}$ of n points in $[0,1)^2$, let $(\mathscr{G}(n,r))_{r\in\mathbb{R}^+}$ be the continuous random graph process describing the evolution of $\mathscr{G}(n,r)$ for r between 0 and $+\infty$ ($\mathscr{X}$ remains fixed for the whole process). Observe that the graph process starts at $r = 0$ with all n vertices being isolated, then edges are progressively added, and finally at $r \geq \sqrt{2}/2$ we have the complete graph on n vertices. In this context, consider the random variables $r_c = r_c(n) = \inf\{r \in \mathbb{R}^+ : \mathscr{G}(n,r) \text{ is connected}\}$ and $r_i = r_i(n) = \inf\{r \in \mathbb{R}^+ : \mathscr{G}(n,r) \text{ has no isolated vertex}\}$. As a corollary of Theorem 3.5.1 we obtain an alternative proof of the following well known result (see Theorem 1 of [141]), which says that a.a.s. $(\mathscr{G}(n,r))_{r\in\mathbb{R}^+}$ becomes connected exactly at the same moment when the last isolated vertex disappears:

Corollary 3.5.2. *A.a.s., we have $r_c = r_i$.*

3.6 Analysis of connectivity in dynamic random geometric graphs

This section is based on [53]. In contrast to the previous sections, here we introduce a dynamic random graph model motivated by its applications to MANETS (mobile ad hoc networks). Several "practical" models of mobility have been proposed in the literature; for a survey of these models we refer to [96]. In all these models, the connections in the network are created and destroyed as the vertices move closer together or further apart. Many *empirical* results have been obtained for connectivity issues and routing performance and the different MANET models (see for example [153]). The paper [81] also deals with the problem of maintaining connectivity of mobile vertices communicating by radio, but from an orthogonal perspective to the one in our paper: it describes a *kinetic data structure* to maintain the connected components of the union of unit-radius disks moving in the plane.

We introduce a variation of the *Random Walk* model introduced by Guerin [79]. Before giving a formal description, roughly speaking, vertices can move from their current positions, changing the graph therefore at each time. We will use the parameters s to be the distance travelled by each vertex before possibly changing its direction, and the parameter m which is such that at every step each vertex updates its direction independently and with probability $1/m$.

As in the previous section, we consider the unit torus $[0,1)^2$ in order to be able to ignore boundary issues. For the case of static random geometric graphs, the connectivity thresholds for the torus $[0,1)^2$ and for the unit square $[0,1]^2$ are asymptotically the same (see for instance [140]). When talking about generic models of MANETS, most authors consider the unit square setting, where the vertices that touch the boundary of $[0,1]^2$, bounce back as a ball banging against a wall. From the experimental point of view, when doing simulations on large areas, the torus $[0,1)^2$ seems to behave similarly as $[0,1]^2$ (see for example [34]). However, when using a rigorous analytic approach, the model on $[0,1]^2$ adds a greater degree of difficulty (the main problem is that at each step where one or more vertices touch the boundary, the probability space changes).

Our main result below provides precise asymptotic results for the expected number of steps that the dynamic graph remains connected once it becomes connected, and the expected number of steps the graph remains disconnected once it becomes disconnected. Our results are expressed in terms of n, s and m. Surprisingly, the final expression on the length of connectivity periods (asymptotically) does not depend on the expected number m of steps between consecutive change of angles of a vertex (as long as the angles do eventually change, no matter how large the value of m is). It is worth to note here that the evolution of connectivity of this model is *not* Markovian, in the sense that staying connected for a large number of steps does have an impact on the probability of being connected at the next step. However, one key and rather counterintuitive fact is that, despite of this absence of the Markovian property, the argument to prove our result is mainly based on the analysis of the connectivity changes in two consecutive steps.

We describe our model now more formally: given positive reals $s = s(n)$ and $m = m(n)$, consider the following random process $(\mathcal{X}_t)_{t\in\mathbb{Z}} = (\mathcal{X}_t(n,s,m))_{t\in\mathbb{Z}}$: at step $t = 0$, n vertices are scattered independently and u.a.r. over $[0,1)^2$, as in the static model. Moreover, at any

time step t, each vertex i jumps a distance s in some direction $\alpha_{i,t} \in [0, 2\pi)$. The initial angle $\alpha_{i,0}$ is chosen independently and uniformly at random for each vertex i, and then at every step each vertex changes its angle independently with probability $1/m$. New angles are also selected independently and uniformly at random in $[0, 2\pi)$. Observe that the number of steps that each vertex must wait between two consecutive changes of angle has a geometric distribution with expectation m. Since the dynamic process is time-reversible, it also makes sense to consider negative steps. The dynamic random geometric graph is then defined as a sequence $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{Z}}$, where for each particular value of t, $G(\mathcal{X}_t; r)$ is the random geometric graph with vertex set $\mathcal{X}_t$.

The case when s tends to 0 very fast is of special interest. In fact, given any $d \in \mathbb{R}^+$ (d might depend on n), we can choose s arbitrarily small and m arbitrarily large such that $d = sm$, and the distance travelled by each vertex between two consecutive changes of angles is approximately exponentially distributed with mean $d = sm$. As a result, our model can be regarded as a discrete-time approximation of the following natural continuous-time counterpart, which we denote by $\widetilde{\mathscr{G}}_{R,u,v}(n, r)$: the vertices move continuously at speed 1 around the torus rather than performing jumps at discrete steps, and each vertex changes direction according to an independent Poisson process of intensity $1/d$ (thus the waiting time between two consecutive changes is exponential with mean d).

To state our main theorem precisely, we need a few definitions. Let $\mu = e^{-r^2\pi n}$ be defined as in the previous section, and also as before let K_1 denote the random variable counting the isolated vertices. We suppose $\mu = \Theta(1)$ or equivalently $r = r_c = \sqrt{\frac{\log n \pm O(1)}{\pi n}}$. Let us denote by $\mathcal{C}$ and $\mathcal{D}$ the events that $\mathscr{G}(n, r)$ is connected and disconnected, respectively. Observe that, when $\mu = \Theta(1)$, the probability that $\mathscr{G}(n, r)$ is (dis)connected can be easily obtained:

$$\mathbb{P}(\mathcal{C}) \sim \mathbb{P}(K_1 = 0) \sim e^{-\mu} \quad \text{and} \quad \mathbb{P}(\mathcal{D}) \sim \mathbb{P}(K_1 > 0) \sim 1 - e^{-\mu}. \tag{3.12}$$

We denote by $\mathcal{C}_t$ ($\mathcal{D}_t$) the event that $\mathcal{C}$ ($\mathcal{D}$) holds at step t. In $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{Z}}$, define $L_t(\mathcal{C})$ to be the number of consecutive steps that $\mathcal{C}$ holds starting at step t (possibly ∞ and also 0 if C_t does not hold). The distribution of $L_t(\mathcal{C})$ does not depend on t, and we often omit t when it is understood. $L_t(\mathcal{D})$ is defined analogously by interchanging $\mathcal{C}$ and $\mathcal{D}$ (it is shown that $L_t(\mathcal{C})$ and $L_t(\mathcal{D})$ are indeed random variables).

We are interested in the length of the periods in which $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{Z}}$ stays connected (disconnected). More precisely, we consider the expected number of steps that $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{Z}}$ stays connected (disconnected) starting at step t conditional upon the fact that it becomes connected (disconnected) precisely at step t:

$$\lambda_{\mathcal{C}} = \mathbb{E}(L_t(\mathcal{C}) \mid \mathcal{D}_{t-1} \wedge \mathcal{C}_t) \qquad \text{and} \qquad \lambda_{\mathcal{D}} = \mathbb{E}(L_t(\mathcal{D}) \mid \mathcal{C}_{t-1} \wedge \mathcal{D}_t).$$

Our main theorem then reads as follows:

Theorem 3.6.1. *Let $r = r_c$. The expected lengths of the connectivity and disconnectivity periods in $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{Z}}$ are as follows:*
If $srn = \Theta(1)$, then

$$\lambda_{\mathcal{C}} \sim \frac{1}{1 - e^{-\mu(1-e^{-4srn/\pi})}} \quad \textit{and} \quad \lambda_{\mathcal{D}} \sim \frac{e^{\mu} - 1}{1 - e^{-\mu(1-e^{-4srn/\pi})}}.$$

Otherwise, we have

$$\lambda_{\mathcal{C}} \sim \begin{cases} \frac{\pi}{4\mu srn} & \text{if } srn = o(1), \\ \frac{1}{1-e^{-\mu}} & \text{if } srn = \omega(1), \end{cases}$$

and

$$\lambda_{\mathcal{D}} \sim \begin{cases} \frac{\pi(e^{\mu}-1)}{4\mu srn} & \text{if } srn = o(1), \\ e^{\mu} & \text{if } srn = \omega(1). \end{cases}$$

Note that the results of $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{D}}$ of both cases $srn = o(1)$ and $srn = \omega(1)$ correspond to the respective limits of the case where $srn = \Theta(1)$.

Intuitively speaking, the consequences of the result are the following. First observe that, asymptotically, the expected number of steps in a period of connectivity (disconnectivity) does not depend on how often the vertices of $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{Z}}$ change their direction, since the expressions we obtained for $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{D}}$ do not contain m. Moreover, $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{D}}$ are non-increasing with respect to s, which corroborates the intuitive fact that having a big jump of the vertices at each step reduces the positive correlation existing between consecutive time steps for state $\mathcal{C}$ (or state $\mathcal{D}$). In particular, for $srn = \omega(1)$, $\lambda_{\mathcal{C}}$ and $\lambda_{\mathcal{D}}$ do not depend on s, since for such a large s the events of being (dis)connected at consecutive time steps are roughly independent. The case $srn = o(1)$ deserves some extra attention. Let us denote the expected total distance covered by each vertex during a connectivity period and a disconnectivity period by $\tau_{\mathcal{C}} = s \cdot \lambda_{\mathcal{C}}$ and $\tau_{\mathcal{D}} = s \cdot \lambda_{\mathcal{D}}$, respectively. In this case we have

$$\tau_{\mathcal{C}} \sim \frac{\pi}{4\mu rn} \sim \frac{\pi\sqrt{\pi}}{4\mu\sqrt{n \ln n}},$$

and

$$\tau_{\mathcal{D}} \sim \frac{\pi(e^{\mu}-1)}{4\mu rn} \sim \frac{\pi\sqrt{\pi}(e^{\mu}-1)}{4\mu\sqrt{n \ln n}},$$

which asymptotically do not depend on s. Note that these asymptotic relations still hold if s tends to 0 arbitrarily fast, as long as $s = o(1/(rn))$. In particular, this suggests that the related continuous-time model $\widetilde{\mathscr{G}}_{R,u,v}(n, r)$ has a similar behavior, and thus in that model the travelled distance during the periods of (dis)connectivity does not presumably depend either on the average distance $d = sm$ between changes of angle.

Overview of the proof. The main ingredient of the proof of Theorem 3.6.1 is the fact that the probabilities of the events of being connected and disconnected can be expressed in terms of the probabilities of events involving only two consecutive steps. This is remarkable, since as mentioned, the sequence of connected/disconnected states of $G(\mathcal{X}_t; r)$ is not Markovian; staying connected for a long period of time makes it more likely to remain connected for one more step. We show that it suffices to compute the probabilities of the events:

$$(\mathcal{C}_t \wedge \mathcal{D}_{t+1}), \quad (\mathcal{D}_t \wedge \mathcal{C}_{t+1}), \quad \mathcal{C} \quad \text{and} \quad \mathcal{D}. \tag{3.13}$$

In order to apply the previous result, it is also required that the expectations $\mathbb{E}(L_t(\mathcal{C}))$ and $\mathbb{E}(L_t(\mathcal{C}))$ are finite, which is proven using the Monotone Convergence Theorem. To obtain

the probabilities of the events in (3.13), we start from (3.12) and then characterize the connectivity of $\big(G(\mathcal{X}_t; r)\big)_{t\in\mathbb{Z}}$ at two consecutive steps. It turns out that the existence/non-existence of isolated vertices is asymptotically equivalent to the disconnectivity/connectivity of the graph, both in the static case $\mathscr{G}(n, r)$ and for two consecutive steps of $G(\mathcal{X}_t; r)$. We characterize the changes of the number of isolated vertices between two consecutive steps. The proof is based on the computation of the joint factorial moments of the variables accounting for these changes and using a well known theorem in probability (Theorem 1.23 in [27]), to characterize the fact that the random variables are Poisson. At first sight, it is not obvious that the probability of existence of components of larger sizes is negligible compared to the probability of sudden appearance of isolated vertices, but using the results of the previous section about the static $\mathscr{G}(n, r)$ this is indeed shown to be the case.

3.7 Future work

In the field of $\mathscr{G}(n, r)$, the arguably most important open problem to be solved is the determination of the constant of the threshold radius of the appearance of the giant component. Whereas this might be a hard problem probably requiring new ideas, I would also like to attack the problem of calculating the metric dimension of these graphs. Another open problem is the determination of the spy number of the game of revolutionaries and spies whose results for $\mathcal{G}(n, p)$ were mentioned in the previous chapter. A new approach might be the analysis of certain parameters in a slightly more general model, recently introduced by Penrose (see [138]) as "soft geometric graphs". Starting from $\mathscr{G}(n, r)$, each edge is retained independently of other edges with probability p. This model somehow interpolates between $\mathscr{G}(n, r)$, corresponding to $p = 1$, and between $\mathcal{G}(n, p)$, corresponding to a radius r so that $\mathscr{G}(n, r)$ is a clique.

Chapter 4

Random hyperbolic graphs

In this chapter we consider the model of random hyperbolic graphs $\mathcal{G}_{\alpha,C}(n)$ as defined in the introduction. The restriction $\alpha > 1/2$ and the role of R, guarantee that the resulting graph has a bounded average degree (depending on α and C only). If $\alpha < 1/2$, then the degree sequence is so heavy tailed that this is impossible. Research in random hyperbolic graphs is in a sense in its infancy. Besides the results mentioned above, very little else is known. Notable exceptions are the emergence and evolution of giant components (see [23] and [24] for an extended version), connectedness [25], results on the global clustering coefficient of the so called binomial model of random hyperbolic graphs [38], and on the evolution of graphs on more general spaces with negative curvature [68]. In this chapter we analyze the diameter of the giant component for the range $\frac{1}{2} < \alpha < 1$ and we bound the size of the second largest component in this range. The chapter is based on [103].

4.1 The diameter of the giant component and the size of the second largest component

The main problem we address in this work is the natural question, explicitly stated in [80, page 6], that asks to determine the expected diameter of the giant component of a random hyperbolic graph G chosen according to $\mathcal{G}_{\alpha,C}(n)$ for $\frac{1}{2} < \alpha < 1$. We look at this range, since for $\alpha < \frac{1}{2}$ a.a.s. a very small central configuration yielding a diameter of at most 3 exists (see [25]): consider a ball of sufficiently small radius around the origin and partition it into 3 sectors. It can be shown that in each of the sectors a.a.s. there will be at least one vertex, and also that every other vertex is connected to at least one of the three vertices. For $\alpha = \frac{1}{2}$ the probability of this configuration to exist depends on C (see [25]). For $\alpha > 1$, there exists no giant component (see [23]); the case $\alpha = 1$ is a matter requiring further study. For a disconnected graph G, we define the diameter of G, $\mathrm{diam}(G)$ as the maximal length of a shortest path between any two vertices of the (same) largest component (ties broken arbitrarily). Our main result is the following:

Theorem 4.1.1. *Let $G \in \mathcal{G}_{\alpha,C}(n)$ for some $C > 0$ and some $\frac{1}{2} < \alpha < 1$. There exists a constant $C_0 = C_0(\alpha) > 0$ such that a.a.s.*

$$diam(G) = O(\log^{C_0} n).$$

By a result of [23] a.a.s. all vertices belonging to the (hyperbolic) ball of radius $R/2$ around the center are part of a giant component. We also show that there is no other giant component, and that in fact the size of the second largest component is at most polylogarithmic: denoting by $|L_i|$ be size of the i-th largest component, we obtain:

Theorem 4.1.2. *Let $G \in \mathcal{G}_{\alpha,C}(n)$ for some $C > 0$ and some $\frac{1}{2} < \alpha < 1$. There exists a constant $D_0 = D_0(\alpha) > 0$ such that a.a.s.*

$$|L_2| = O(\log^{D_0} n).$$

Remark 4.1.3. *In our proof we choose $C_0 = 2/(\frac{1}{2} - \frac{3}{4}\alpha + \frac{\alpha^2}{4})$ and $D_0 = 2C_0 + o(1)$.*

Remark 4.1.4. *The result of Theorem 4.1.1 was shortly after our result improved by [69] to show that a.a.s. diam$(G) = \Theta(\log n)$.*

Overview of the proof. We show that for $\frac{1}{2} < \alpha < 1$, a.a.s., for any two vertices of the same component, their shortest path length is $O(\log^{C_0+1+o(1)} n)$: to establish this result we rely on the known, and easily established fact, that for the range of α we are concerned with, the graph $G_{\alpha,C}(n)$ has a "center" clique whose size is, with probability at least $1-\exp(-\omega \log n)$, equal to $\Theta(n^{1-\alpha})$. Then, we show that, depending on how "far away" from the center clique a vertex Q is, there is either a very high or at least non-negligible probability that the vertex connects to the center clique through a path of polylogarithmic (in n) length, or otherwise all paths starting from Q have at most polylogarithmic length. It immediately follows that two vertices in the same connected component, a.a.s., either connect to the center clique through paths of polylogarithmic length, or belong to paths of size at most polylogarithmic. Either way, a bound on the diameter of $G_{\alpha,C}(n)$ follows. Rigorously developing the preceding argument requires overcoming significant obstacles, not only technical but also in terms of developing the insight to appropriately define the relevant typical events which are also amenable to a rigorous study of their probabilities of occurrence. Our main result's proof argument also yields that the size of the second largest component is $O(\log^{2C_0+1+o(1)} n)$, thus answering a question of Bode, Fountoulakis and Müller [23].

Another contribution of our work is that it proposes at least two refinements and variants of the *breadth exploration process* introduced by Bode, Fountoulakis and Müller [23]. Specifically, we strengthen the method by identifying more involved strategies for exploring hyperbolic space, not solely dependent on the angular coordinates of its points, and not necessarily contiguous regions of space. We hope these refinements will be useful in tackling other problems concerning the newly minted (and captivating) hyperbolic random graph model.

4.2 Future work

In the field of random hyperbolic graphs, many parameters are unknown yet. Together with Marcos Kiwi (Univ. de Chile) we have preliminary results about the conductance and the second eigenvalue of the normalized Laplacian of such graphs. We plan to extend these to obtain more details on the spectrum of the normalized Laplacian, and also we plan to use the results on the conductance for an analysis of certain rumor spreading protocols. Other

parameters of interest we plan to attack are the hyperbolicity, the acquaintance time, and the metric dimension of such graphs. It would be clearly also interesting to strengthen the known bounds on the size of the second largest component.

Chapter 5

Uniform random graph models

In this chapter we consider two parameters: the metric dimension (see [125]) of a tree (resp. forest) chosen uniformly at random from all labelled trees (resp. forests) on the vertex set $\{1, \ldots, n\}$, and the maximum degree (see [76]) of a graph chosen uniformly at random from all labelled graphs on the vertex set $\{1, \ldots, n\}$ that do not contain a cycle of length k or more, for k being any fixed constant $k \geq 3$. The main motivation of the study of the first parameter was a comparison with the values of the metric dimension of the sparse case of $\mathcal{G}(n, p)$ mentioned above, and the main motivation of the study of the second parameter was to find a class of graphs which are very tree-like but which have a slightly bigger maximum degree. From the mathematical point of view, the proof techniques used in this chapter (analytic combinatorics in the case of [125] and a certain kind of double counting together with a characterization of biconnected components in the case of [76]) are very different to the ones in the other chapters.

5.1 Metric dimension on uniform random trees and uniform random forests

In [125], to complement the analysis of sparse $\mathcal{G}(n, p)$ graphs, we study the metric dimension also in a different family. We consider the metric dimension for a random tree, chosen uniformly at random among all labelled trees with n vertices $\{1, \ldots, n\}$, and the same result applies for random forests. These models are a reminiscent of the random planar graph model introduced by Denise, Vanconcellos and Welsh [50] (see also [118]). Our main result is the following one:

Theorem 5.1.1. *Let T_n, F_n be a random tree (respectively random forest) chosen uniformly at random among all trees (respectively forests) with n vertices. Then, each of the sequences of random variables*

$$\frac{\beta(T_n) - \mathbb{E}\left[\beta(T_n)\right]}{\sqrt{\mathbb{V}ar\beta(T_n)}}, \ \frac{\beta(F_n) - \mathbb{E}\left[\beta(F_n)\right]}{\sqrt{\mathbb{V}ar\beta(F_n)}}$$

converges in distribution to a standard normal distribution as $n \to \infty$. Additionally, $\mathbb{E}\left[\beta(T_n)\right] = \mathbb{E}\left[\beta(F_n)\right] = \mu n(1 + o(1))$ and $\mathbb{V}ar\beta(T_n) = \mathbb{V}ar\beta(F_n) = \sigma^2 n(1 + o(1))$, and $\mu \simeq 0.14076941$, $\sigma^2 \simeq 0.063748151$.

Comparison of the model with the results in $\mathcal{G}(n,p)$ with $p = c/n$. In order to compare this result with the sparse $\mathcal{G}(n,p)$ case, recall Theorem 2.3.2 and Figure 2.3: by looking at $\mathcal{G}(n,p)$ with $p = c/n$ and values of c approaching 1, one can see that the constant term in the expectation of $\mathcal{G}(n,p)$ obtained in Theorem 2.3.2 is much bigger than the expectation obtained here. This shows that these two models are qualitatively different. A possible explanation for this is the following: the $\mathcal{G}(n,p)$ model for $p = c/n$ with $c < 1$ generates many small trees, for which, relatively to the number of vertices in the whole graph, a bigger subset is needed to distinguish all vertices (for example, for isolated vertices all of them except one has to be taken, for trees of size 2 and 3 one vertex has to be taken, and in general, the bigger the number of vertices of a tree, the smaller the proportion of vertices that has to be chosen). Unfortunately, we are not able to calculate the leading constant of the variance in $\mathcal{G}(n,p)$ with $p = c/n$, and thus we cannot compare the two variances.

Overview of the proof. The proof of Theorem 5.1.1 is based on methods from analytic combinatorics. The framework of analytic combinatorics is also powerful to handle probabilities in a combinatorial class. Consider a certain parameter $\chi : \mathcal{A} \to \mathbb{N}$ on $\mathcal{A}$. For $n, m \in \mathbb{N}$, denote by $a_{n,m}$ the number of objects of $\mathcal{A}$ of size n and parameter χ being equal to m. Define the bivariate generating function

$$A(x,y) = \sum_{n,m\in\mathbb{N}} \frac{1}{n!} a_{n,m}\, x^n\, y^m,$$

where y marks the parameter χ. Observe that $A(x,1) = A(x)$. For each value of n, the parameter χ defines a random variable X_n over elements of $\mathcal{A}$ of size n with the according discrete probability distribution $\mathbb{P}(X_n = m) = a_{m,n}/a_n$. Hence, this discrete probability distribution can be encapsulated by means of the following expression:

$$p_n(y) = \frac{[x^n]A(x,y)}{[x^n]A(x,1)},$$

where $[x^n]$ denotes the extraction of the coefficient of $A(x,y)$ of x^n (which is a function of y). The result of uniform trees is based on the analysis of such probability distributions. These relations remain valid when dealing with multivariate counting formulas. By means of complex analytic techniques, we perform a singularity analysis on bivariate counting formulas to extract asymptotic estimates of the coefficients. We consider expressions of the form

$$T(x,y) = F(x,y,T(x,y)),$$

for certain analytic functions $F(x,y,z)$. Under natural conditions on F, we can obtain the singular expansion of $T(x,y)$ around its smallest singularity. We rephrase Theorem 2.21 of [58] (based on the earlier works [56, 57, 59]) in a simplified version:

Theorem 5.1.2 (Square-root singularity for implicit equations)**.** *Let $F(x,y,z)$ an analytic function around the origin, such that all Taylor coefficients are non-negative, $F(0,y,z)$ is identically equal to the zero function and $F(x,y,0) \neq 0$. Assume that in the region of convergence of $F(x,y,z)$ the system of equations*

$$z = F(x,1,z),\ 1 = \frac{\partial}{\partial z} F(x,1,z) \tag{5.1}$$

has a non-negative solution $(x, z) = (\rho, \tau)$ *such that* $\frac{\partial}{\partial x}F(\rho, 1, \tau) \neq 0$ *and* $\frac{\partial^2}{\partial y^2}F(\rho, 1, \tau) \neq 0$. *Assume that the counting formula* $T(x, y)$ *is defined by the implicit scheme* $T(x, y) = F(x, y, T(x, y))$. *Then,* $T(x, y)$ *is an analytic function around the origin, with non-negative Taylor coefficients. Additionally, there exist functions* $f(y)$, $g(y)$, $h(y)$, $q(y)$ *and* $\rho(y)$ *which are analytic around* $x = \rho = \rho(1)$, $y = 1$ *such that* $T(x, y)$ *is analytic for* $|x| < \rho$ *and* $|y - 1| < \varepsilon$ *(for some* $\varepsilon > 0$*), and has an expansion of the form*

$$T(x, y) = f(y) + g(y)\left(1 - \frac{x}{\rho(y)}\right)^{1/2} +$$
$$h(y)\left(1 - \frac{x}{\rho(y)}\right) + q(y)\left(1 - \frac{x}{\rho(y)}\right)^{3/2} + O\left(\left(1 - \frac{x}{\rho(y)}\right)^{2}\right),$$

locally around $x = \rho(y)$.

Once we know the singular behavior of a bivariate generating function, we can study, by means of general results, the limiting distribution of the parameter we are codifying. In this context, the *Quasi-powers Theorem* [90] gives sufficient conditions to assure normal limiting distributions. In the following simplified version we adapt the hypothesis to the expansions we will find in the analysis:

Theorem 5.1.3 (Quasi-Powers Theorem [90])**.** *Let* $F(x, y)$ *be a bivariate analytic function on a neighborhood of* $(0, 0)$*, with non-negative coefficients. Assume that the function* $F(x, y)$ *admits, in a region*

$$\mathcal{R} = \{|y - 1| < \varepsilon\} \times \{|x| \leq r\}$$

for some $r, \varepsilon > 0$*, a representation of the form*

$$F(x, y) = A(x, y) + B(x, y)\, C(x, y)^{-\alpha},$$

where $A(x, y)$, $B(x, y)$ *and* $C(x, y)$ *are analytic in* $\mathcal{R}$*, and such that*

- $C(x, y) = 0$ *has a unique simple root* $\rho < r$ *in* $|x| \leq r$,
- $B(\rho, y) \neq 0$,
- *neither* $\partial_x C(\rho, y)$ *nor* $\partial_y C(\rho, y)$ *vanish, so there exists a non-constant function* $\rho(y)$ *analytic at* $y = 1$ *such that* $\rho(1) = \rho$ *and* $C(\rho(y), y) = 0$,
- *and finally,*

  $$\sigma^2 = -\frac{\rho''(1)}{\rho(1)} - \frac{\rho'(1)}{\rho(1)} + \left(\frac{\rho'(1)}{\rho(1)}\right)^2$$

  is different from 0.

Then the sequence of random variables with density probability function

$$p_n(y) = \frac{[x^n]F(x, y)}{[x^n]F(x, 1)}$$

converges in distribution to a normal distribution. The corresponding expectation μ_n *and variance* σ_n^2 *converge asymptotically to* $-\frac{\rho'(1)}{\rho(1)}\, n$ *and* $\sigma^2 n$*, respectively.*

5.2 Maximum degree in minor-closed classes of graphs

This section is based on [76]. We consider a graph chosen uniformly at random from the set of all labelled graphs with vertex set $\{1, \ldots, n\}$ without containing a certain *minor*: a graph H is a minor of a graph G, if H can be formed from G by deleting edges and vertices and by contracting edges, that is, removing an edge from a graph while simultaneously merging the two vertices that it previously joined. A class of labelled graphs $\mathcal{G}$ is minor-closed if whenever a graph G is in $\mathcal{G}$ and H is a minor of G, then H is also in $\mathcal{G}$, and the set of all such graphs with vertices $\{1, \ldots, n\}$ is denoted by $\mathcal{G}_n$. Denote in general throughout this section $\mathrm{Ex}(H)$ the class of graphs not containing H as a minor. By a random graph from $\mathrm{Ex}(H)$ of size n we mean a graph drawn with uniform probability among all graphs of size n in $\mathrm{Ex}(H)$, and we are interested in the maximum degree Δ_n of such a graph.

The motivation of our work comes from the analysis of graphs that are close to trees: a classical result says that for labelled trees, Δ_n is of order $\log n / \log\log n$ (see [127]). In fact, much more precise results are known in this case, in particular that (see [39])

$$\frac{\Delta_n}{\log n / \log\log n} \to 1 \qquad \text{in probability.}$$

Many more results about the distribution of maximum degree, its concentration, and several different models of randomly generated trees can be found in the survey of [92]. McDiarmid and Reed [117] showed that for the class of planar graphs there exist constants $0 < c_1 < c_2$ such that

$$c_1 \log n < \Delta_n < c_2 \log n \qquad \text{a.a.s.}$$

More recently this result has been strengthened using subtle analytic and probabilistic methods [61], by showing the existence of a computable constant c such that

$$\frac{\Delta_n}{\log n} \to c \qquad \text{in probability.}$$

For planar maps (planar graphs with a given embedding), more precise results on the distribution of Δ_n can be found in [73]. Analogous results have been proved for series-parallel and outerplanar graphs [60], with suitable constants. Using the framework of Boltzmann samplers, results about the degree distribution of subcritical graph classes such as outerplanar graphs, series-parallel graphs, cactus graphs and clique graphs can also be found in [22]. The latter paper also contains conjectures of the exact values of c_{OP} (c_{SP}, respectively) saying that the maximum degree in outerplanar graphs (series-parallel graphs, respectively) will be roughly $c_{OP} \log n$ ($c_{SP} \log n$, respectively).

The goal in our paper [76] is to analyze the maximum degree in additional minor-closed classes of graphs. Let C_n be the cycle on n vertices. Let F_n be the fan graph, that is, the graph consisting of a path with $n-1$ vertices plus a vertex adjacent to all the vertices in the path. A graph G is 2-connected, if at least 2 vertices have to be removed so that G becomes disconnected. The following is our main result:

Theorem 5.2.1. *Let c, c_1, c_2 be suitable positive constants. A.a.s., we have:*

- *In* $\mathrm{Ex}(C_4)$ *we have, for all* $\epsilon > 0$,

$$(2-\epsilon)\frac{\log n}{\log\log n} \le \Delta_n \le (2+\epsilon)\frac{\log n}{\log\log n}.$$

- *In* $\mathrm{Ex}(C_5)$ *we have, for all* $\epsilon > 0$,

$$(1-\epsilon)\frac{\log n}{\log\log\log n} \le \Delta_n \le (1+\epsilon)\frac{\log n}{\log\log\log n}.$$

- *In* $\mathrm{Ex}(C_6)$ *we have*

$$c_1\frac{\log n}{\log\log\log n} \le \Delta_n \le c_2\frac{\log n}{\log\log\log n}.$$

- *In* $\mathrm{Ex}(C_7)$ *we have*

$$c_1\frac{\log n}{\log\log\log\log n} \le \Delta_n \le c_2\frac{\log n}{\log\log\log\log n}.$$

- *If* H *is 2-connected and contains* $C_{2\ell+1}$ *as a minor, then in* $\mathrm{Ex}(H)$ *we have*

$$\Delta_n \ge c\frac{\log n}{\log^{(\ell+1)} n},$$

  where $\log^{(\ell+1)} n = \log\cdots\log n$, *iterated* $\ell+1$ *times.*

- *If* H *is 2-connected and is not a minor of* F_n *for any* n, *then in* $\mathrm{Ex}(H)$ *we have*

$$\Delta_n \ge c\log n.$$

Idea of the proof. The main proof idea is inspired by a double counting technique developed by McDiarmid and Reed [117], and we explain this technique now. Let $\mathcal{G}$ be a class of graphs and suppose we want to show that a property P holds in $\mathcal{G}$ a.a.s. Let $\mathcal{B}_n$ be the graphs in $\mathcal{G}_n$ that do not satisfy P (the 'bad' graphs). Suppose that for a constant fraction $\alpha > 0$ of graphs in $\mathcal{B}_n$ we have a rule producing at least $C(n)$ graphs in $\mathcal{G}_n$ (the 'construction' function). A graph in $\mathcal{G}_n$ can be produced more than once, but assume every graph in $\mathcal{G}_n$ is produced at most $R(n)$ times (the 'repetition' function). By double counting we have

$$\alpha|\mathcal{B}_n|C(n) \le |\mathcal{G}_n|R(n),$$

hence

$$\alpha\frac{|\mathcal{B}_n|}{|\mathcal{G}_n|} \le \frac{R(n)}{C(n)}.$$

If the procedure is such that $C(n)$ grows faster than $R(n)$, that is $R(n) = o(C(n))$, then we conclude that $|\mathcal{B}_n| = o(|\mathcal{G}_n|)$, that is, the proportion of bad graphs goes to 0. Equivalently, property P holds a.a.s.

We will apply this principle in order to obtain lower and upper bounds on the maximum degree for several classes. Roughly speaking, we start from a random graph of a given

class and suppose that its maximum degree does not satisfy the given bounds. Then, by modifying the graph, we can construct many other graphs, still belonging to this class, and still allowing reconstruction of the original graph; if the new family is much bigger, and the maximum degree for each graph of the new family is different, then we arrive at the desired contradiction. In this context, lower bounds are easier to obtain, and only in some cases we are able to prove matching upper bounds. The proof of the upper bound for planar graphs in [117] depends very strongly on planarity, and it seems difficult to adapt it to general situations; however we obtain such a proof for outerplanar graphs. On the other hand, we develop new tools for proving upper bounds based on the decomposition of a connected graph into 2-connected components.

5.3 Future work

Needless to say, the results in [76] leave open the question whether the lower bound of $\Delta_n \geq c\frac{\log n}{\log^{(\ell+1)} n}$ in the case of $\mathrm{Ex}(H)$ with H being 2-connected and containing $C_{2\ell+1}$ as a minor, is tight. The current techniques based on a characterization of biconnected components unfortunately are not generalizable to all ℓ, and some new idea is required. Also, in the same framework, it would be interesting to show an upper bound of $\Delta_n = O(\log n)$ in the case of $\mathrm{Ex}(H)$ with H being 2-connected. When forbidding a 2-connected graph, it would also be interesting to determine all possible orders of Δ_n; perhaps no other values than $\log n$ and $\frac{\log n}{\log^{(\ell+1)} n}$ can be obtained. Finally, when omitting the restriction of H being 2-connected, the example of H being two disjoint triangles shows that Δ_n can be linear in such a case; a natural question to be asked again is to determine all possible orders of Δ_n.

Chapter 6

Random directed acyclic graphs

In hierarchical social networks such as Twitter, information flows downwards from the source to sinks. Disrupting the flow of information may correspond to halting the spread of news or gossip in on-line social networks, or intercepting a message sent in a terrorist network. How do we disrupt this flow of information while minimizing the resources used? Motivated by this question, we study a vertex-pursuit game, called Seepage, in this chapter. This chapter is based on [32].

6.1 Seepage on random directed acyclic graphs

We consider a simple model of Seepage introduced in [46]. Seepage is motivated by the 1973 eruption of the Eldfell volcano in Iceland. In order to protect the harbor, the inhabitants poured water on the lava in order to solidify it and thus, halt its progress. The game has two players, the *sludge* and a set of *greens* (note that one player controls all the greens), a directed acyclic graph (DAG) with one source (corresponding to the top of the volcano) and many sinks (representing the lake). The players take turns, with the sludge going first by contaminating the top node (source). Then it is the greens' turn, and they choose some non-protected, non-contaminated nodes to protect. On subsequent moves the sludge moves to a non-protected node that is adjacent (that is, downhill) to the node the sludge is currently located at and contaminates it; note that the sludge is located at a single node in each turn. The greens, on their turn, as before, choose some non-protected, non-contaminated nodes to protect. Once protected or contaminated, a node stays in that state to the end of the game. The sludge wins if some sink is contaminated; the greens win otherwise, that is, if they erect a cutset of nodes which separates the contaminated nodes from the sinks. The name "Seepage" is used because the rate of contamination is slow. The game is related to vertex-pursuit games such as Cops and Robbers (for an introduction and further reading on such games, see [33]), although the greens in our case need not move to neighboring nodes. An interesting feature that comes out from our analysis is the different winning strategies that are employed by the two players. Whereas in some cases one of the two players can play arbitrarily (at least to some point), in other cases the optimal strategy for the greens is a "greedy" strategy protecting neighbors as close as possible to the current position of the sludge, and in other cases "wise" strategies have to be applied by the sludge.

To date the only analysis of Seepage was in [46], which presented results for DAGs. Seepage may be extended to certain directed graphs with cycles, although we do not consider this variation here. In [46], a characterization was given of directed trees where one green has a winning strategy, and bounds were given on the number of greens needed to win in truncated products of paths. See also Chapter 9 of [33].

Seepage displays some interesting similarities to an approach used in mathematical counterterrorism, where cut sets in partially ordered sets (which are just a special kind of DAG) are used to model the disruption of terrorist cells. As described in Farley [65], the maximal elements of the poset are viewed as the leaders of the terrorist organization, who submit plans down via the edges to the nodes at the bottom (the foot soldiers or minimal nodes). Only one messenger needs to receive the message for the plan to be executed. Farley considered finding minimum-order sets of elements in the poset, which when deleted, disconnect the minimal elements from the maximal one (that is, find a *minimum cut*). We were struck by the similarities in the underlying approaches in [46] and [65]; for example, in Seepage the greens are trying to prevent the sludge from moving to the sinks by blocking nodes. The main difference is that Seepage is "dynamic" (that is, the greens can move, or choose new sets of nodes each time-step), while the min-cut-set approach is "static" (that is, find a cutset in one time-step). Seepage is perhaps a more realistic model of counterterrorism, as the agents do not necessarily act all at once but over time. However, in both approaches deterministic graphs are used.

Our goal is to analyze Seepage and the green number when played on a random DAG as a model of disrupting a given hierarchical social network. Our model includes as a parameter the total degree distribution of nodes in the DAG. This has some similarities to the $G(\mathbf{w})$ model of random graphs with expected degree sequences (see [45]) or the pairing model (see [166]). We study two cases: regular DAGs (where we would expect each level of the DAG to have nodes with about the same out-degree), and power law DAGs (where the degree distribution is heavy tailed, with many more low degree nodes but a few which have a high degree).

We now give a formal definition of our vertex-pursuit game. Fix $v \in V(G)$ a node of G. We will call v the *source*. For $i \in \mathbb{N} \cup \{0\}$ let

$$L_i = L_i(G, v) = \{u \in V(G) : d_G(u, v) = i\},$$

where $d_G(u, v)$ is the graph distance between u and v in G. In particular, $L_0 = \{v\}$. For a given $j \in \mathbb{N}$ and $c \in \mathbb{R}^+$, let $\mathcal{G}(G, v, j, c)$ be the game played on graph G with the source v and the *sinks* L_j. The game proceeds over a sequence of discrete time-steps. Exactly

$$c_t = \lfloor ct \rfloor - \lfloor c(t-1) \rfloor$$

new nodes are protected at time-step t. (In particular, at most ct nodes are protected by time t.) Note that if c is an integer, then exactly c nodes are protected at each time-step, so this is a natural generalization of Seepage. To avoid trivialities, we assume that $L_j \neq \emptyset$.

The *sludge* starts the game on the node $v_1 = v$. The second player, the *greens*, can protect $c_1 = \lfloor c \rfloor$ nodes of $G \setminus \{v\}$. Once nodes are protected they will stay protected to the end of the game. At time $t \geq 2$, the sludge makes the first move by sliding along a directed edge from v_{t-1} to v_t, which is an out-neighbor of v_{t-1}. After that the greens have a chance

to protect another c_t nodes. Since the graph is finite and acyclic, the sludge will be forced to stop moving, and so the game will eventually terminate. If he reaches any node of L_j, then the sludge wins; otherwise, the greens win.

If $c = \Delta(G)$ (the maximum out-degree of G), then the game $\mathcal{G}(G, v, j, c)$ can be easily won by the greens by protecting of all neighbors of the source. Therefore, the following graph parameter, *the green number*, is well defined:

$$g_j(G, v) = \inf\{c \in \mathbb{R}^+ : \mathcal{G}(G, v, j, c) \text{ is won by the greens}\}.$$

It is clear that for any $j \in \mathbb{N}$ we have $g_{j+1}(G, v) \leq g_j(G, v)$.

In the case of random regular DAGs we obtain the following result:

Theorem 6.1.1. *Let ω be any function that grows (arbitrarily slowly) as n tends to infinity. For the random d-regular DAGs, we have the following.*

(i) *A.a.s. $g_1 = d$.*

(ii) *If $2 \leq j = O(1)$, then a.a.s.*
$$g_j = d - 2 + \frac{1}{j}.$$

(iii) *If $\omega \leq j \leq \log_{d-1} n - \omega \log\log n$, then a.a.s.*
$$g_j = d - 2.$$

(iv) *If $\log_{d-1} n - \omega \log\log n \leq j \leq \log_{d-1} n - \frac{5}{2} s \log_2 \log n + \log_{d-1} \log n - O(1)$ for some $s \in \mathbb{N}$, then a.a.s.*
$$d - 2 - \frac{1}{s} \leq g_j \leq d - 2.$$

(v) *Let $s \in \mathbb{N}$, $s \geq 4$. There exists a constant $C_s > 0$ such that if $j \geq \log_{d-1} n + C_s$, then a.a.s.*
$$g_j \leq d - 2 - \frac{1}{s}.$$

Theorem 6.1.1 tells us that the green number is slightly bigger than $d-2$ if the sinks are located near the source, and then it is $d-2$ for a large interval of j. Later, it might decrease slightly since an increasing number of vertices have already in-degree 2 or more, but only for large j (part (v)) we can prove better upper bounds than $d-2$. One interpretation of this fact is that the resources needed to disrupt the flow of information in a typical regular DAG is (almost) independent of j, and relatively low (as a function of j).

In the case of random power law DAGs we obtain the following result:

Theorem 6.1.2. *Let*

$$\gamma = d^{\beta-1} \left(\frac{\beta-2}{\beta-1}\right)^{\beta-2} \left(\left(1 + \left(d\frac{\beta-2}{\beta-1}\right)^{1-\beta}\right)^{\frac{\beta-2}{\beta-1}} - 1\right)$$

if $\frac{1}{\alpha} - \beta + 3 \in \mathbb{N} \setminus \{1, 2\}$, *and* $\gamma = 1$ *otherwise. Let* j_1 *be the largest integer satisfying* $j_1 \leq \max\{\frac{1}{\alpha} - \beta + 3, 2\}$. *Let* $j_2 = O(\log\log n)$ *be the largest integer such that*

$$d^{\beta-1}\left(\frac{\gamma}{d^{\beta-1}} n^{\alpha(j_1-1)-1}\right)^{\left(\frac{\beta-2}{\beta-1}\right)^{j_2-j_1}} \leq (\omega \log\log n)^{-\max\{2,(\beta-1)^2\}}.$$

Finally, let

$$\xi = \left(\frac{\beta-2}{\beta-1}\right) d \left(\left(\frac{d(\beta-2)}{\beta-1}\right)^{\beta-1} + 1\right)^{-\frac{1}{\beta-1}}.$$

Then, for $1 \leq j \leq j_2 - 1$ *we have that a.a.s.*

$$(1 + o(1))\bar{w}_j \leq g_j \leq (1 + o(1))\bar{w}_{j-1}, \tag{6.1}$$

where $\bar{w}_0 = \bar{w}_1 = M$, *for* $2 \leq j < \frac{1}{\alpha} - \beta + 3$,

$$\bar{w}_j = \begin{cases} n^\alpha & \text{if } 2 \leq j < \frac{1}{\alpha} - \beta + 2 \\ \xi n^\alpha & \text{if } 2 \leq j = \frac{1}{\alpha} - \beta + 2 \\ \left(\frac{\beta-2}{\beta-1}\right) d n^{\frac{1-\alpha(j-1)}{\beta-1}} & \text{if } \frac{1}{\alpha} - \beta + 2 < j < \frac{1}{\alpha} - \beta + 3 \text{ and } j \geq 2, \end{cases}$$

and for $j_1 \leq j \leq j_2 - 1$,

$$\bar{w}_j = \left(\frac{\beta-2}{\beta-1}\right)\left(\frac{\gamma}{d^{\beta-1}} n^{\alpha(j_1-1)-1}\right)^{-\left(\frac{\beta-2}{\beta-1}\right)^{j-j_1}/(\beta-1)}.$$

In the power law case, Theorem 6.1.2 tells us that the green number is smaller for large j. This reinforces the view that intercepting a message in a hierarchical social network following a power law is more difficult close to levels near the source.

Idea of the proofs. In both theorems we give recursive bounds on the sizes $|L_i|$ that hold a.a.s. In the case of Theorem 6.1.2 in addition to this, we calculate bounds on the minimum weight and maximum weight of each layer. In the case of Theorem 6.1.1 the desired bounds then follow from showing that a.a.s. the subgraph of each depth pending from a given vertex is very close to a tree, and there are few *bad* vertices, that is, vertices with 2 or more incoming edges. In the case of Theorem 6.1.2 the results follow by bounding the number of paths from any vertex v, possibly occupied by the sludge, and any vertex u two layers below, possibly occupied by greens, by an absolute constant K.

6.2 Future work

We considered Seepage on regular DAGs and on power law DAGs. It would be interesting to analyze the game on random DAGs with other degree sequences; for example, where the degree distribution remains the same at each level, or there are the same number of vertices at each level. Also, hierarchical social networks are usually not strictly acyclic; for example, on Twitter, directed cycles of followers may occur. Seepage naturally extends to settings with directed cycles (the directed graphs considered must have at least one source and a set of sinks; the game is then played analogously as before). A next step would then be to extend our results to a setting where such cycles occur and analyze the green number on the strongly connected components of such directed graphs.

Chapter 7

Bootstrap percolation

This chapter is slightly different from the previous ones: here, in contrast to the above results, the family of graphs under consideration is deterministic (with the exception of a randomly chosen matching added at the end), and the randomness is only part of the percolation. Since however the proof techniques in our model are very similar to both random geometric graphs as well as random graphs with a given degree sequence, we decided to introduce this chapter as well. We consider here bootstrap percolation with strong majority. This chapter is based on the paper [122].

Let us first observe the following obvious monotonicity properties for the bootstrap percolation $\mathbb{B}_j(G;A)$ as defined in the introduction: for any $A' \subseteq A'' \subseteq V$, if $\mathbb{B}_j(G;A')$ disseminates, then $\mathbb{B}_j(G;A'')$ disseminates as well; similarly, if $i \leq j$ and $\mathbb{B}_j(G;A)$ disseminates, then $\mathbb{B}_i(G;A)$ must also disseminate. Therefore, the probability that $\mathbb{B}_j(G;q)$ disseminates is non-increasing in j and non-decreasing in q. In view of this, one may expect that, for some sequences of graphs G_n, there may be a sharp probability threshold $\widehat{q}_n$ such that for every $\varepsilon > 0$, $\mathbb{B}_j(G_n;q_n)$ disseminates, if $q_n \geq (1+\varepsilon)\widehat{q}_n$; and a.a.s. it does not disseminate, if $q_n \leq (1-\varepsilon)\widehat{q}_n$. If such a value $\widehat{q}_n$ exists, we call it a *dissemination threshold* of $\mathbb{B}_j(G_n;q_n)$. Moreover, if $\lim_{n\to\infty} \widehat{q}_n = \widehat{q} \in [0,1]$ exists, we call this limit $\widehat{q}$ the *critical probability* for dissemination, which is *non-trivial* if $0 < \widehat{q} < 1$. A lot of work has been done to establish dissemination thresholds or related properties of this process for different graph classes. In particular, as in the chapter on hyperbolic graphs, letting $[n] = \{1, 2, \ldots, n\}$, the case of G being the m-dimensional grid $[n]^m$ has been extensively studied: starting with the work of Holroyd [88] analyzing the 2-dimensional grid, the results then culminated in [12], where Balogh et al. gave precise and sharp thresholds for the dissemination of $\mathbb{B}_j([n]^m;q)$ for any constant dimension $m \geq 2$ and every $2 \leq j \leq m$. Other graph classes that have been studied are trees, hypercubes and hyperbolic lattices (see e.g. [15, 11, 14, 155]). Bootstrap percolation was also considered in the context of random graphs: Janson et al. [95] considered the model $\mathbb{B}_j(G;A)$ with $j \geq 2$, $G = \mathcal{G}(n,p)$ and A being a set of vertices chosen at random from all sets of size $a(n)$. They showed a sharp threshold with respect to the parameter $a(n)$ that separates two regimes in which the final set of active vertices has a.a.s. size $o(n)$ or $n - o(n)$ (i.e. 'almost' dissemination), respectively. Moreover, there is full dissemination in the supercritical regime provided that $\mathcal{G}(n,p)$ has minimum degree at least j. Balogh and Pittel [16] analyzed the bootstrap percolation process on random d-regular graphs, and established non-trivial critical probabilities for dissemination for all $2 \leq j \leq d-1$, and Amini [8] considered

random graphs with more general degree sequences. Finally, extensions to inhomogeneous random graphs were considered by Amini, Fountoulakis and Panagiotou in [9].

7.1 Bootstrap percolation with strong majority

We introduce a natural variant of the bootstrap percolation process. Given a graph $G = (V, E)$, an initially active set $A \subseteq V$, and $r \in \mathbb{Z}$, the *r-majority bootstrap percolation* process $\mathbb{M}_r(G; A)$ is defined as follows: starting with an initial set of active vertices A, at each round, each inactive vertex becomes active if the number of its active neighbors minus the number of its inactive neighbors is at least r. In other words, the activation rule for an inactive vertex v of degree $\deg(v)$ is that v has at least $\lceil(\deg(v) + r)/2\rceil$ active neighbors. As in ordinary bootstrap percolation, we are mainly interested in characterizing the set of inactive vertices in the final state of and determining whether it is empty (i.e. the process disseminates) or not. Note that for a d-regular graph G, $\mathbb{M}_r(G; A)$ is exactly the same process as $\mathbb{B}_{\lceil(d+r)/2\rceil}(G; A)$, and therefore the final set of inactive vertices of $\mathbb{M}_r(G; A)$ is precisely the vertex set of the $\lfloor(d - r)/2 + 1\rfloor$-core of the graph induced by the initial set of inactive vertices. If G is not regular, the two models are not comparable. The process $\mathbb{M}_r(G; q)$ is defined analogously for a random initial set A of active vertices, where each vertex belongs to A (i.e. is initially active) with probability q and independently of all other vertices. Note that $\mathbb{M}_r(G; A)$ and $\mathbb{M}_r(G; q)$ satisfy the same monotonicity properties with respect to A, to r, and to q that we described above for ordinary bootstrap percolation, and thus we define the *critical probability* $\widehat{q}$ for dissemination (if it exists) analogously as before. Additionally, for any (random or deterministic) sequence of graphs G_n, define

$$\widehat{q}^{\,+} = \inf\{q \in [0, 1] : \text{a.a.s. } \mathbb{M}_r(G_n; q) \text{ disseminates}\} \qquad \text{and}$$
$$\widehat{q}^{\,-} = \sup\{q \in [0, 1] : \text{a.a.s. } \mathbb{M}_r(G_n; q) \text{ does not disseminate}\}.$$

Trivially, $0 \le \widehat{q}^{\,-} \le \widehat{q}^{\,+} \le 1$; and, in case of equality, the critical probability $\widehat{q}$ must exist and satisfy $\widehat{q} = \widehat{q}^{\,-} = \widehat{q}^{\,+}$. The r-majority bootstrap percolation process is a generalization of the *non-strict majority* and *strict majority* bootstrap percolation models, which correspond to the cases $r = 0$ and $r = 1$, respectively. The study of these two particular cases has received a lot of attention recently. For instance, Balogh, Bollobás and Morris [14] obtained the critical probability $\widehat{q} = 1/2$ for the non-strict majority bootstrap percolation process $\mathbb{M}_0(G; q)$ on the hypercube $[2]^n$, and extended their results to the m-dimensional grid $[n]^m$ for $m \ge (\log\log n)^2(\log\log\log n)$. Also, Stefánsson and Vallier [158] studied the non-strict majority model for the random graph $\mathscr{G}(n, p)$. (Note that, since $\mathcal{G}(n, p)$ is not a regular graph, this process cannot be formulated in terms of ordinary bootstrap percolation). For the strict majority case, we first state a consequence of the work of Balogh and Pittel [16] on random d-regular graphs mentioned earlier. Let $G \in \mathcal{G}_{n,d}$ denote a graph chosen u.a.r. from the set of all d-regular graphs on n vertices (note that n is even if d is odd). Then, for any constant $d \ge 3$, the critical probability for dissemination of the process $\mathbb{M}_1(\mathcal{G}_{n,d}; q)$ is equal to

$$\widehat{q}(d) = 1 - \inf_{y \in (0,1)} \frac{y}{F(d - 1, 1 - y)}, \tag{7.1}$$

where $F(d, y)$ is the probability of obtaining at most $d/2$ successes in d independent trials with success probability equal to y. Moreover,

$$\widehat{q}(3) = 1/2, \qquad \min\{\widehat{q}(d) : d \geq 3\} = \widehat{q}(7) \approx 0.269 \qquad \text{and} \lim_{d\to\infty} \widehat{q}(d) = 1/2. \tag{7.2}$$

The case of strict majority was studied by Rapaport, Suchan, Todinca and Verstraëte [147] for various families of graphs. They showed that, for the wheel graph W_n (a cycle of length n augmented with a single universal vertex), $\widehat{q}^{\,+}$ is the unique solution in the interval $[0, 1]$ to the equation $\widehat{q}^{\,+} + (\widehat{q}^{\,+})^2 - (\widehat{q}^{\,+})^3 = \frac{1}{2}$ (that is, $\widehat{q}^{\,+} \approx 0.4030$); and they also gave bounds on $\widehat{q}^{\,+}$ for the toroidal grid augmented with a universal vertex. Moreover, they proved that, for every sequence G_n of 3-regular graphs of increasing order (that is, $|V(G_n)| < |V(G_{n+1})|$ for all $n \in \mathbb{N}$) and every $q < 1/2$, a.a.s. the process $\mathbb{M}_1(G_n; q)$ does not disseminate (so $\widehat{q}^{\,-} \geq 1/2$). Together with the result from (7.2) that $\widehat{q}(3) = 1/2$, their result implies, roughly speaking, that, for every sequence of 3-regular graphs, dissemination is at least as 'hard' as for random 3-regular graphs. In view of this, they conjectured the following:

Conjecture 7.1.1 ([147])**.** *Fix any constant $d \geq 3$, and let G_n be any arbitrary sequence of d-regular graphs of increasing order. Then, for the strict majority bootstrap percolation process on G_n, we have $\widehat{q}^{\,-} \geq \widehat{q}(d)$. That is, for any constant $0 \leq q < \widehat{q}(d)$, a.a.s. the process $\mathbb{M}_1(G_n; q)$ does not disseminate.*

Observe that, if the conjecture were true, then for every sequence of d-regular graphs of growing order, $\widehat{q}^{\,-} \geq \widehat{q}(d) \geq \widehat{q}(7) \approx 0.269$. This motivated the following question:

Question 7.1.2 ([147])**.** *Is there any sequence of graphs G_n such that their critical probability of dissemination (for strict majority bootstrap percolation) is $\widehat{q} = 0$?*

Further results for strict majority bootstrap percolation on augmented wheels were given in [104], and some experimental results for augmented tori and augmented random regular graphs were presented in [126]. The underlying motivation in both papers (in view of Question 7.1.2) was the attempt to construct sequences of graphs G_n such that a.a.s. $\mathbb{M}_1(G_n; q)$ disseminates for small values of q (i.e., sequences G_n with a small value of $\widehat{q}^{\,+}$). However, to the best of our knowledge, for all graph classes investigated before our paper, the values of $\widehat{q}^{\,+}$ obtained were strictly positive. We disprove Conjecture 7.1.1 by constructing a sequence of d-regular graphs such that $\widehat{q}^{\,+}$ can be made arbitrarily small by choosing d large enough (see Theorem 7.1.3 and Corollaries 7.1.6 and 7.1.5 below). Moreover, by allowing $d \to \infty$, we achieve $\widehat{q}^{\,+} = 0$, and thus we answer Question 7.1.2 in the affirmative. It is worth noting that, if one considers the non-strict majority model ($r = 0$) instead of the strict majority model ($r = 1$), then Question 7.1.2 has a trivial answer as a result of the work of Balogh et al. [12] on the m-dimensional grid $[n]^m$. Indeed, their results imply that the process $\mathbb{M}_0([n]^m; q)$ has critical probability $\widehat{q} = 0$. (In fact, they establish a sharp threshold for dissemination at $\widehat{q} = \lambda/\log n \to 0$, for a certain constant $\lambda > 0$). However, the aforementioned results do not extend to the strict majority model. As a matter of fact, it is easy to show that the process $\mathbb{M}_1([n]^m; q)$ has critical probability $\widehat{q} = 1$. In order to prove this, observe that if all the vertices in the cube $\{1, 2\}^m$ or any of its translates in the grid $[n]^m$ are initially inactive, then they remain inactive at the final state. If $q < 1$, then each of these cubes is initially inactive with positive probability, so a.a.s. there exists an initially inactive cube and we do not get dissemination.

Our sequence of regular graphs. To state our results precisely, we first need to define a sequence of regular graphs that disseminates 'easily'. For each $n \in \mathbb{N}$ and $k = k(n) \in \mathbb{N}$, consider the following graph $\mathscr{L}(n,k)$: the vertices are the n^2 points of the toroidal grid $[n]^2$ with coordinates taken modulo n; each vertex $v = (x,y)$ is connected to the vertices $v + w$, where $w \in K = \{-k,\dots,-1,0,1,\dots,k\} \times \{-1,1\}$. Assuming that $2k+1 \le n$ (so that the neighborhood of a vertex does not wrap around the torus), we have that $|K| = 2(2k+1) = 4k+2$, and thus our graph $\mathscr{L}(n,k)$ is $(4k+2)$-regular. Therefore, in the process $\mathbb{M}_{2r}(\mathscr{L}(n,k),q)$, an inactive vertex needs at least $2k+r+1$ active neighbors to become active. Next, for even n and $r = r(n) \in \mathbb{N}$, we also consider the (random) graph $\mathscr{L}^*(n,k,r)$, consisting of adding r random perfect matchings to $\mathscr{L}(n,k)$. These matchings are chosen u.a.r. from the set of perfect matchings of $[n]^2$ conditional upon not creating multiple edges (i.e. the perfect matchings are pairwise disjoint and do not use any edge from $\mathscr{L}(n,k)$). Note that $\mathscr{L}^*(n,k,r)$ is $(4k+r+2)$-regular. Moreover, the process $\mathbb{M}_r(\mathscr{L}^*(n,k,r);q)$ has the same activation rule as $\mathbb{M}_{2r}(\mathscr{L}(n,k);q)$: namely, an inactive vertex becomes active at some round of the process if it has at least $2k+r+1$ active neighbors. In view of this and since $\mathscr{L}(n,k)$ is a subgraph of $\mathscr{L}^*(n,k,r)$, we can couple the two processes in a way that the set of active vertices of $\mathbb{M}_{2r}(\mathscr{L}(n,k);q)$ is always a subset of that of $\mathbb{M}_r(\mathscr{L}^*(n,k,r);q)$. We will show that for every $q > 0$ (and even $q \to 0$ not too fast as $n \to \infty$) and every not too large $r \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that a.a.s. $\mathbb{M}_r(\mathscr{L}^*(n,k,r);q)$ disseminates. On a high level, our analysis comprises two phases: in phase 1, we will consider $\mathbb{M}_{2r}(\mathscr{L}(n,k);q)$ and show that most vertices become active in this phase. In phase 2, we incorporate the effect of the r perfect matchings and consider then $\mathbb{M}_r(\mathscr{L}^*(n,k,r);q)$ to show that all remaining inactive vertices become active. This 2-phase analysis is motivated by the fact that the final set of inactive vertices of $\mathbb{M}_r(\mathscr{L}^*(n,k,r);q)$ is a subset of the final set of inactive vertices of $\mathbb{M}_{2r}(\mathscr{L}(n,k);q)$, in view of the aforementioned coupling between the two processes.

Our main result is the following:

Theorem 7.1.3. *Let $q_0 > 0$ be a sufficiently small constant. Given any $q \in [0,1]$, $k = k(n) \in \mathbb{N}$ and $r = r(n) \in \mathbb{N}$ satisfying (eventually for all large enough even $n \in \mathbb{N}$),*

$$200\frac{(\log\log n)^{2/3}}{(\log n)^{1/3}} \le q \le q_0, \tag{7.3}$$

$$\frac{1000}{q}\log(1/q) \le k \le \frac{q^2 \log n}{3000\log(1/q)}, \quad \text{and} \quad 1 \le r \le \frac{qk}{20}, \tag{7.4}$$

consider the r-majority bootstrap percolation process $\mathbb{M}_r(\mathscr{L}^*(n,k,r);q)$ on the $(4k+r+2)$-regular graph $\mathscr{L}^*(n,k,r)$, where each vertex is initially active with probability q. Then, $\mathbb{M}_r(\mathscr{L}^*(n,k,r);q)$ disseminates a.a.s.

Remark 7.1.4.

1. *By our assumptions on q, it is easy to verify that $\lceil \frac{1000}{q}\log(1/q)\rceil < \lfloor \frac{q^2\log n}{3000\log(1/q)}\rfloor$, and so the range for k is non-empty, and the statement is not vacuously true. In particular, $k = \lceil \frac{1000}{q}\log(1/q)\rceil$ satisfies the assumptions of the theorem.*

2. *Note that the lower bound required for k in terms of q is almost optimal: in Theorem 2 of [147], the authors showed (for the 1-majority model) that for any sequence of d-regular graphs (of increasing order) with $d < 1/q$ (in the case of odd d) or $d < 2/q$ (in the case of even d), a.a.s. dissemination does not occur. (For the r-majority model with $r \geq 2$, dissemination is even harder.) Hence, setting $k = \lceil \frac{1000}{q} \log(1/q) \rceil$, our sequence of $\Theta(k)$-regular graphs $\mathscr{L}^*(n,k,r)$ has the smallest possible degree up to an additional $\Theta(\log(1/q))$ factor for achieving dissemination.*

As a consequence of Theorem 7.1.3, we get the following two corollaries. The first one follows from an immediate application of Theorem 7.1.3 with

$$q = 200\frac{(\log\log n)^{2/3}}{(\log n)^{1/3}}, \qquad k = \lfloor \tfrac{q^2 \log n}{3000 \log(1/q)} \rfloor, \qquad r = \lfloor 400 \log\log n \rfloor,$$

together with the monotonicity of the process $\mathbb{M}_r(\mathscr{L}^*(n,k,r);q)$ with respect to q and r.

Corollary 7.1.5. *There is $d = \Theta\left((\log n \cdot \log\log n)^{1/3}\right)$, and a sequence G_n of d-regular graphs of increasing order such that, for every*

$$200\frac{(\log\log n)^{2/3}}{(\log n)^{1/3}} \leq q \leq 1 \qquad \text{and} \qquad 1 \leq r \leq 400\log\log n,$$

the process $\mathbb{M}_r(G_n;q)$ disseminates a.a.s.

Setting $r = 1$, this corollary answers Question 7.1.2 in the affirmative. The second corollary concerns the case in which all the parameters are constant.

Corollary 7.1.6. *For any constants $0 < q \leq 1$ and $r \in \mathbb{N}$, there exists $d_0 \in \mathbb{N}$ satisfying the following. For every natural $d \geq d_0$, there is a sequence G_n of d-regular graphs of increasing order such that the r-majority bootstrap percolation process $\mathbb{M}_r(G_n;q)$ a.a.s. disseminates.*

In particular, since $\lim_{d\to\infty} \widehat{q}(d) = 1/2$ (cf. (7.2)), Corollary 7.1.6 implies that, for every sufficiently large constant d, there is a sequence of d-regular graphs of increasing order such that (for the 1-majority model) $\widehat{q}^{\,+} < \widehat{q}(d)$, which disproves Conjecture 7.1.1.

Idea of the proofs. The proof of Theorem 7.1.3 is as follows: first one shows that there is a small set of vertices forming a certain parabolic shape of points that is completely active. Then deterministically this structure will grow provided that in the neighborhood of this vertex set at least a small number of vertices is active as well. Comparing this process with 2-dependent site percolation one can then deduce that there will be a unique big component containing a linear fraction of active vertices, and all other components are small. Finally, by adding a random matching these small components become active as well.

7.2 Future work

We would like to consider the following generalization: instead of having a fixed probability q for each vertex to become initially active, the probability of each vertex could be chosen from a given distribution, again independently for each vertex. We would like to characterize the conditions for such a distribution so that there is still dissemination in the model of bootstrap percolation with strong majority (on either our family of graphs or on a different family), knowing only that the expected value of this distribution tends to 0.

Further future work

Here we mention some further future work that does not fit into the previous chapters. On the one hand, it deals with problems on different random graph models, on the other hand it also deals with problems in the broader field of probability and combinatorics, outside the field of random graphs.

1. In the model of random inhomogeneous graphs introduced recently by [29], together with Nicolas Fraiman (UPenn Philadelphia), we recently obtained preliminary results on the diameter of the case of a graph having a (slightly) superlinear number of edges. We also plan to determine the spectrum of such graphs.

2. Together with Abbas Mehrabian (Carnegie Mellon Univ.) and Nick Wormald (Monash Univ.) we started to calculate diameters in a variant of the preferential attachment model, originally introduced by [17], in which a new vertex is attached to $d \geq 1$ previous vertices with a probability that is a function of the degree of the previous vertex; our goal is to classify the functions for which the diameter of the resulting graph is a.a.s. still logarithmic, which is the case when the vertices are added uniformly at random to one vertex ($d = 1$). As a first step, we want to consider functions that are monotone with respect to the depth of a vertex, i.e., the deeper a vertex, the more likely it is to be the chosen neighbor of the new vertex to be attached.

3. Outside the field of random graphs, in the more general field of probabilistic combinatorics, together with Miloš Stojaković (Univ. Novi Sad) we would like to attack Maker-Breaker type combinatorial games with geometric flavor, say these games played on the plane, where the goal is to achieve a certain property of the claimed point set.

4. Once more, outside of the field of random graphs, together with Balázs Patkós (Rényi Institute) we would like to determine the acquaintance time of Kneser graphs, of complements of Kneser graphs, and of hypercubes.

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