1. The Existence of Giant Components in $G_{\mathcal{D}}$

We consider a degree sequence $\mathcal{D} = \{d_1, d_2, ..., d_n\}$ such that $d_1 \leqslant d_2 \leqslant ... \leqslant d_n$. We let $M = M_{\mathcal{D}} = \sum_{i=1}^{n} d_i$ and insist: $d_1 > 0$ and no $d_i = 2$. We let $j_{\mathcal{D}} = min(n \cup \{j | \sum_{i=1}^{j} d_i(d_i - 2) > 0\})$ and $R = R_{\mathcal{D}} = \sum_{i=j_{\mathcal{D}}}^{n} d_i(d_i - 2)$. We let $F = F_{\mathcal{D}}$ be the number of edges in the largest component of $G = G_{\mathcal{D}}$. We prove:

Theorem 1.1. *For any sufficiently small $\omega > 0$, if M is sufficiently large in terms of ω and if $R_D < \omega D$ then*

$$P(F > \omega^{1/9} M) = o(1).$$

Theorem 1.2. *For every $\epsilon > 0$ there is a $\delta > 0$ such that if M is sufficently large in terms of epsilon, $d_n \leqslant \frac{\sqrt{M}}{log\ M}$, and $R > \epsilon M$ then:*

$$P(F > \delta M) = 1 - o(1).$$

In proving these theorems we explore a random object consisting of G and a uniformly chosen permutation of the adjacency list of v for each vertex of G.

We start with some set S_0 of vertices, and expose the edges in $G[S_0]$ and where they appear on the adjacency lists of their endpoints. In each iteration t, we add a vertex w_t to S_{t-1} to obtain S_t and expose the edges from w_t to S_{t-1} and where they appear on the adjacency lists of their endpoints. If there are no edges of G between S_t and $V - S_t$, w_t is a random vertex of $V - S_t$ chosen proportional to its degree. Otherwise we let v_t be the lowest indexed vertex of S_t which has a neighbour in V_t and make a random choice of w_t, its neighbour in $V - S_t$ appearing first on its adjacency list, and the edges from w_t to S_{t-1} and where they appear on the adjacency lists of their endpoints. Our choice is conditioned by the part of the random object we have already exposed.

We will track X_t, the number of edges between S_t and $V - S_t$. We will also track $X'_t = \Sigma_{x \in S_t} d(x) - 2t$. Now, $X_0 \leqslant X'_0$, $X'_t = X'_{t-1} + d(w_t) - 2$ and unless $X_{t-1} = 0$, we have $X_t = X_{t-1} + d(w_t) - 2$. So, until $X_t = 0$ we have $X_t \leqslant X'_t$, and if $X'_t = 0$ then for some $t' \leqslant t$ we have $X_{t'} = 0$.

1.1. **The Proof of Theorem 1.1.**

To prove Theorem 1.1, we actually show that under its hypotheses, for any vertex v of G we have:

$$(*)\ P(v \textit{ is in a component with more than } \omega^{-1/9} M \textit{ edges}) = o(\frac{1}{m}).$$

Summing over all v yields Theorem 1.1.

To prove (*) for a specific vertex v we actually analyze our exploration process starting with a set S_0 which contains v and prove:

$$(**)\ P(\exists t \leqslant \frac{\omega^{1/9}M}{2}\ s.t.\ X_t = 0\ \&\ X'_t \leqslant \omega^{1/5}M) = 1 - o(\frac{1}{m}).$$

Now if $\exists t \leqslant \frac{\omega^{1/9}M}{2}$ such that $X_t = 0$ and $X'_t \leqslant \omega^{1/5}M$. then the sum of the degrees of the vertices in the component containing v is less than the sum of the degrees of the vertices in S_t which is $X'_t + 2t \leqslant \omega^{1/5}M + \frac{\omega^{1/9}M}{2}$. So this implies that the number of edges in the component containing v is less than $\omega^{1/9}M$.

So to prove Theorem 1.1 it remains to prove (**) for a set S_0 which contains v.

We choose the minimum j such that $\Sigma_{i=j}^n d_i \geqslant 5\omega^{-1/4}M$. We set $S = \{v_j, ..., v_n\}$ and set $S_0 = S \cup \{v\}$.

It is straightforward to prove, as we do below, the following results about the initial situation:

Claim 1:

(i) There is a $u \in S$ such that $d(u) \leqslant \omega^{-1/4}$ and hence for all $u \in V - S_0$, $d(u) \leqslant \omega^{-1/4}$.
(ii) $\Sigma_{u \in V-S} d(u)(d(u) - 2) \leqslant -4\omega^{1/4}M$.
(iii) $X'_0 \leqslant 7\omega^{1/4}M$.

We will need to show that as time goes on, X'_t tracks its expectation. The difference between X'_t and X'_{t-1} is $d(w) - 2$ so we will focus on the following variable: $Y_t = d(w) - 2 - E(d(w) - 2)$. Now, the expectation of each Y_t is 0 and by (i), each Y_t is at most $\omega^{-1/4}$. This allows us to apply Azuma's Inequality to prove:

Claim 2: The probability that there is a t for which $\sum_{t' \leqslant t} Y_t \geqslant M^{2/3}$ is $o(\frac{1}{m})$.

Finally we show that provided X'_t does track its expectation, if the process does not die out then the expected step size becomes more and more negative.

Claim 3: For any t with $t \leqslant \tau = min(\{t | X_t = 0\} \cup \{t | \Sigma_{t' \leqslant t} Y_t \geqslant M^{2/3}\} \cup \{\lfloor \frac{\omega^{1/9}M}{2} \rfloor\})$, we have:

$$E(d(w_t) - 2) \leqslant \frac{-t}{M} + 19\omega^{1/5}.$$

With these three claims in hand, we can prove (**)as follows.

$$X'_\tau = X'_0 + \sum_{t=1}^{\tau}(E(d(w_t) - 2)) + \sum_{t<\tau} Y_t \leqslant 7\omega^{1/4}M - \frac{1}{M}\sum_{t=1}^{t^*} t + 19\omega^{1/5}t^* + M^{2/3}$$

$$\leqslant 7\omega^{1/4}M - \frac{\tau^2}{3} + 19\omega^{14/45}M + M^{2/3} < \omega^{1/5}M$$

. Furthermore, if $\tau = \lfloor \frac{\omega^{1/9}M}{2} \rfloor$ then $X'_\tau < 0$ so $X'_t = 0$ for some $t < \tau$. Hence $X_{t'} = 0$ for some $t' < \tau$. This contradicts the definition of τ. So it always holds that $\tau < \lfloor \frac{\omega^{1/9}M}{2} \rfloor$. Since the probability there is a t such that $\sum_{t' \leqslant t} Y_t \geqslant M^{2/3}$ is $o(\frac{1}{m})$, we see that with probability $1 - o(\frac{1}{m})$ there is a $t = \tau < \lfloor \frac{\omega^{1/9} M}{2} \rfloor$ with $X_t = 0$ and $X'_t \leqslant \omega^{1/5} M$.