1. The Existence of Giant Components in $G_{\mathcal{D}}$(continued)

We continue with the proof of the following two theorems.

Theorem 1.1. *For any sufficiently small $\omega > 0$, if M is sufficiently large in terms of ω and if $R_D < \omega D$ then*

$$P(F > \omega^{1/9} M) = o(1).$$

Theorem 1.2. *For every $\epsilon > 0$ there is a $\delta > 0$ such that if M is sufficently large in terms of epsilon, $d_n \leqslant \frac{\sqrt{M}}{log\ M}$, and $R > \epsilon M$ then:*

$$P(F > \delta M) = 1 - o(1).$$

We have stated three claims which imply Theorem 1.1. Before proving them we sketch the proof of Theorem 1.2.

1.1. **A proof sketch for Theorem 1.2.**

In this case, we cannot be content with tracking X'_t we must focus directly on X_t. Letting $d'(w_t)$ be the number of edges between w_t and S_{t-1} we have that X_t is at least $X_{t-1} + d(w_t) - 2 - 2d'(w_t)$ (we have at least here because when $X_{t-1} = 0$ we do not subtract 2). We will focus on the contribution from $d(w_t) - 2$ and $d'(w_t)$ separately. We let $A_t = d(w_t) - E[d(w_t)]$ and $B_t = d'(w_t) - E[d'(w_t)]$ so $X_t - E(X_t) = A_t + B_t$. We let $\mathcal{F}_{bad}$ be the event that for some t either $|\sum_{t' \leqslant t} A_t| \geqslant \frac{M}{log\ log\ M}$ or $|\sum_{t' \leqslant t} B_t| \geqslant \frac{M}{log\ log\ M}$.

We note that if the theorem is true for any ϵ it is true for all larger ϵ so we can and do assume ϵ is as small as we like..

The following has the same flavour as Claim 2.

Claim 4: $Prob(\mathcal{F}_{bad}) = o(1)$.

We next claim that the expected drift is positive until we have explored a constant fraction of the edges

Claim 5: Provided $X'_t + 2t \leqslant 10^{-6}\epsilon^2 M$, we have: $E[d(w_t) - 2] \geqslant \frac{\epsilon}{4}$ and $E[d'(w_t)] \leqslant \frac{E[d(w_t)-2]}{3}$. Hence $E[X_t - X_{t-1}] \geqslant \frac{E[d(w_t)-2]}{3} \geqslant \frac{\epsilon}{12}$.

We consider the first t for which $X'_t + 2t$ exceeds $10^{-6}\epsilon^2 M$. We note that by Claim 5,

$$E[\sum_{t' \leqslant t} X_t - X_{t-1}] \geqslant min(\frac{\sum_{t' \leqslant t} E[d(w_t) - 2]}{3}, \frac{\epsilon t}{12})$$

I.e.

$$X_t - \Sigma_{t' \leqslant t} A_{t'} - \Sigma_{t' \leqslant t} B_t \geqslant min(\frac{X'_t - X'_0 - \Sigma_{t' \leqslant t} B_t}{3}, \frac{\epsilon t}{12})$$

So we have that for large M and sufficiently small ϵ,, unless $\mathcal{F}_{bad}$ holds:

$$X_t \geqslant min(10^{-7}\epsilon^2 M - t, \frac{\epsilon t}{12} - \frac{M}{loglogM}) - \frac{2M}{log\ log\ M} \geqslant 10^{-9}\epsilon^3 M.$$

Since $P(\mathcal{F}_{bad}) = o(1)$, Theorem 1.2 follows.

1.2. **Some Details.**

We give now the details of the proof of Theorem 1.1. We let M_t be the sum of the degrees of the vertices in $V - S_t$. I.e. $M_t = M - X'_t - 2t$. We let n_1 be the number of vertices of degree 1 in $G_{\mathcal{D}}$.

Proof of Claim 1: We note that $d_i(d_i - 2)$ is only negative if d_i is 1, so the sum of all such negative terms is at most $-M$. It follows that for all $j < j_{\mathcal{D}}$, $d_j \leqslant \sqrt{M} + 2$. Thus, letting $S' = S - \{v_j | j \geqslant j_{\mathcal{D}} - 1\}$, we have that $\sum_{v \in S'} d(v) \geqslant 5\omega^{1/4}M - R_{\mathcal{D}} - \sqrt{M} - 2 \geqslant 4\omega^{1/4}M$. If every vertex of S has degree exceeding $\omega^{-1/4}$ then

$$\sum_{v \in S'} d(v)(d(v) - 2) \geqslant (\omega^{-1/4} - 2) \sum_{v \in S'} d(v) \geqslant (\frac{\omega^{-1/4}}{2}) 4\omega^{1/4} M = 2M.$$

But this contradicts the definition of $j_{\mathcal{D}}$. So (i) holds.

Since the lowest degree vertex in S has degree at most $\omega^{-1/4}$, as does v if it is not in S, Further, by the definition of S the sum of the degrees of the vertices in S_0 is at most $5\omega^{1/4}M + 2\omega^{-1/4} \leqslant 7\omega^{1/4}M$. So (iii) holds. Hence, the sum of the degrees of the vertices in $V - S_0$ is at least $M - 7\omega^{1/4}M > \frac{M}{2}$. So, if these vertices all have degree one, (ii) holds. Otherwise every vertex in S has degree at least 3, so $\sum_{v \in S'} d(v)(d(v) - 2) \geqslant \sum_{v \in S'} d(v) \geqslant 4\omega^{1/4}M$. Hence by the definition of $J_{\mathcal{D}}$, (ii) holds.

Proof of Claim 2: We need to apply Azuma's Inequality which is obtained by deleting independent in the statement of the Simplified Azuma's Inequality. We apply it to $X = \sum_{t' \leqslant t} Y_t$. This is determined by t trials each of which is a choice of a $w_{t'}$. Now, given the first i trials, $E(X)$ is simply $\sum_{j=1}^{i} Y_j$ because the expectation of Y_j for $j > i$ is by definition 0. Since $d(w_{t'}) - 2$ and its expectation both lie between -1 and $\omega^{-1/4} - 2$, it follows that the conditions of the inequality hold with $c_i = \omega^{-1/4}$. Since t is at most M, and $E(X) = 0$ it follows that:

$$P(|X| > M^{2/3}) \leqslant P(|X - E(X)| > M^{2/3}) \leqslant e^{\frac{-M^{4/3}}{\omega^{-1/2}M}} = o(\frac{1}{M^2}).$$

Since this holds for all of the at most M values of t, the claim follows.

Proof of Claim 3: We will need

Claim 6: if $t \leqslant \omega^{1/9}M, X'_{t-1} \leqslant \omega^{1/5}M$, and $X_{t'} > 0$ for all $t' < t$ then the following hold:

(a) if $w \in V - S_{t-1}$ and $d(w) = 1$ then $P[w_t = w] \geqslant (1 - 9\omega^{1/5})\frac{1}{M_{t-1}}$,

(b) if $w \in V - S_{t-1}$ and $d(w) = d$ then $P[w_t = w] \leqslant (1 + 9\omega^{1/5})\frac{1}{M_{t-1}}$.

and

Claim 7: For any sequence $a_1, .., a_j$ of positive integers none of which are 2 and a nonnegative integer l such that $\sum_{i=1}^{j} a_i \geqslant 2j - l$, we have $\sum_{i=1}^{j} a_i(a_i - 2) \geqslant j - 2l$.

The easy proof of Claim 7 is omitted it can be found on page 18 of Joos et al.

We note that there are least $\frac{M}{4}$ vertices of degree 1 in G as otherwise

$$\sum_{v \in V-S} d(v)(d(v) - 2) \geqslant \frac{-M}{4} + (3 - 2)(M - \frac{M}{4} - 7\omega^{1/4}M) > 0$$

contradicting the definition of $j_{\mathcal{D}}$.

We will prove Claim 3 by induction on t. For $t = 1$, Claim 1(iii) tells that $X'_0 \leqslant \frac{\omega^{1/5}M}{2}$. Hence Claim 6 implies:

$$E[d(w_t) - 2] \leqslant \frac{(1 + 9\omega^{1/5}) \sum_{w \in V-S_0} d(w)(d(w) - 2) + 18\omega^{1/5}n_1}{M_0}$$

. So, applying Claim (1)(ii) and then Claim 1(iii), we have:

$$E[(d(w_t) - 2] \leqslant \frac{18\omega^{1/5}M}{M_0} \leqslant 18\omega^{1/5}(1 - 7\omega^{1/4})^{-1} \leqslant \frac{-1}{M} + 19\omega^{1/5}.$$

For $2 \leqslant t \leqslant \tau$, by induction we obtain

$$X'_{t-1} = X'_0 + \Sigma_{i=1}^{t-1} E[d(w_t) - 2] + \Sigma_{t'<t} Y_t \leqslant \frac{\omega^{1/5}M}{2} + 19\omega^{1/5}t + M^{2/3} \leqslant \omega^{1/5}M.$$

Since $X_{t'} > 0$ for all $t' \leqslant t - 1$, $X'_{t-1} > 0$ and hence;

$$\Sigma_{i=1}^{t-1} d(w_i) = 2(t - 1) + \Sigma_{i=1}^{t-1}(d(w_i) - 2) = 2(t - 1) + (X'_{t-1} - X'_0) \geqslant 2(t - 1) - X'_0$$

.

So, Claim 7 implies $\Sigma_{i=1}^{t-1} d(w_i)(d(w_i) - 2) \geqslant (t - 1) - 2X'_0$. Since $V - S_{t-1} = V - S_0 - \{w_1, ..., w_{t-1}\}$ Claim 1(ii) yields:

$$\Sigma_{w \in V-S_{t-1}} d(w)(d(w) - 2) \leqslant 2X'_0 - (t - 1).$$

Combining this bound with Claim 6 yields:

$$E[d(w_t) - 2] \leqslant \frac{(1 + 9\omega^{1/5})(2X'_0 - (t - 1)) + 18\omega^{1/5}n_1}{M_{t-1}}.$$

Now, $n_1 \leqslant M, X'_0 \leqslant \frac{\omega^{1/5}M}{2}$, and $M_{t-1} = M - X'_{t-1} - 2t$, so we obtain:

$$E[d(w) - 2] \leqslant \frac{1 - t}{M} + \frac{9\omega^{1/5}\omega^{1/5}M + 18\omega^{1/5}M}{(1 - \omega^{1/5} - 2\omega^{1/9})M} \leqslant \frac{-t}{M} + 19\omega^{1/5}.$$

This completes the proof of Claim 3, it remains to prove Claim 6.

Proof of Claim 6: We can and do assume $X_{t-1} > 0$ as otherwise $Prob(w_t = w) = \frac{d(w)}{M_{t-1}}$. For any vertex w of $V - S_{t-1}$, we let A_w be the set of extensions of the choices

we have made for the random object we have made so far for which $w = w_t$ and B_w be those for which $w \neq w_t$. We switch between our random objects much as we switch between graphs, with all non switched edges maintaining their positions in the ordered adjacency lists and the new edges taking the positions of the old edges.

Proof of (a): To switch from A_w to B_w we must swap the edge $v_t w$ with some oriented edge xy with $x \in V - S_{t-1}$ to obtain $v_t x$ and wy. There are at most $|M_{t-1}|$ such switchings. To swap from B_w to A_w we must swap $v_t w_t$ with the oriented edge wy. We can only do so, if w is not already a neighbour of v_t and y is not a neigbour of w_t. Now, we know $n_1 \geqslant \frac{M}{4}$ so since $t \leqslant \frac{\omega^{1/9}M}{2}$ and $X_0' \leqslant 7\omega^{1/4}M$ we know that the number n_1' of vertices of degree 1 in $V - S_{t-1}$ is at least $\frac{M}{4} - \frac{\omega^{1/9}M}{2} - 7\omega^{1/4}M > \frac{M}{5}$.

We can split the random objects extending what we have exposed so far into equivalence classes where two objects are in the same equivalence classes if they can be obtained from each other via a permutation of the labels on the vertices of degree 1 in $V - S_{t-1}$. it is not hard to see that there are $n_1'!$ elements in each equivalence class. So, we can thing of selecting an element of B_w uniformly by first choosing an equivalence class uniformly and then assigning labels to the vertices of degree one.

Now, v_t has at most $R_{\mathcal{D}} \leqslant \omega M$ neighbours and w_t has at most $\omega^{-1/4}$ neighbours each with at most ωM neighbours. So, for any equivalence class, there are at most $2\omega^{3/4}M$ vertices which are a neighbour of v_t or a neighbour of a neighbour of w_t. it follows that for any equivalence class, the proportion of elements for which we cannot switch the edge wy with $v_t w_t$ is less than $10\omega^{3/4}$. Thus we have:

$$M_{t-1}|A_w| \geqslant (1 - \omega^{1/5})|B_w|.$$

I.e. $|B_w| \leqslant |A_w| \frac{M_{t-1}}{1-\omega^{1/5}}$. So,

$$P[w = w_t] = \frac{|A_w|}{|A_W| + |B_w|} \geqslant \frac{1}{1 + M_{t-1}(1 - \omega^{1/5})^{-1}}.$$

Now since $M_{t-1} = M - X_{t-1}' \geqslant (1 - \omega^{1/5} - 2\omega^{1/9})M$, it is large and (a) follows.

Proof of (b): For a switching from B_w to A_w, we have to switch $v_t w_t$ with wy for some neighbour y of w. Therefore, there are at most d such switches. For a switching from A_w to B_w, we must switch the edge $v_t w$ with some (oriented) edge yz such that $y \in V - S_{t-1} - N(v_t)$ and z is not a neighbour of w. Now there are at most ωM neighbours of v_t in $V - S_{t-1}$ each incident to at most $\omega^{-1/4}$ edges and w has at most $\omega^{-1/4}$ neighbours each incident to at most ωM edges. So, there are at least $M_{t-1} - 2\omega^{3/4}M$ such swaps. Since $M_{t-1} > (1 - 7\omega^{1/4} - 2\omega^{1/9})M$ it follows that $|A_W| \leqslant \frac{d}{(1-\omega^{1/5})M_{t-1}}|B_W|$. So,

$$P[w = w_t] = \frac{|A_w|}{|A_W| + |B_w|} \leqslant \frac{|A_W|}{|B_W|} \leqslant \frac{(1 + 9\omega^{1/5})d}{M_{t-1}}.$$