Random graphs from minor-closed classes

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Notes for talks in Nice, on Wednesday 16 and Thursday 17 July 2019.
There are 5 exercises for Wednesday afternoon on pages 3-7.

1 Introduction

We will be interested in minor-closed classes and related structured classes of graphs. The development will be set in a natural general context. Let $\mathcal{G}$ be a class of graphs (closed under isomorphism), for example the class $\mathcal{P}$ of planar graphs. We use $\mathcal{G}_n$ to denote the set of graphs in $\mathcal{G}$ on vertex set $[n] := \{1, \ldots, n\}$. The notation $R_n \in_u \mathcal{G}$ means that R_n is a random graph sampled uniformly from the set $\mathcal{G}_n$ (assumed non-empty). We are interested in typical properties of R_n. For example does R_n usually have a giant component? How likely is R_n to be connected?

For a class $\mathcal{G}$ of graphs, the exponential generating function (*egf*) is

$$G(x) = \sum_n |\mathcal{G}_n| x^n / n!.$$

We use ρ_G, $\rho_{\mathcal{G}}$ (and $\rho(G)$, $\rho(\mathcal{G})$) to denote the radius of convergence. For suitable classes $\mathcal{G}$, we can relate the egfs (or two variable versions) of all graphs, connected graphs, 2-connected graphs and 3-connected graphs. **If** we know enough about the 3-connected graphs (as we do for planar graphs, thanks to Tutte and others) then we **may** be able to extend to all graphs. We aim to proceed in greater generality.

A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by edge-contractions, see for example [18]. A class $\mathcal{G}$ of graphs is *minor-closed* if

$$G \in \mathcal{G},\ H \text{ a minor of } G \quad \Rightarrow\ H \in \mathcal{G}.$$

Here are some examples.

1. forests, series-parallel graphs, and more generally graphs of treewidth $\leq k$;

2. outerplanar graphs, planar graphs, and more generally graphs embeddable in a given surface;

3. graphs with at most k vertex disjoint cycles.

We use $\mathrm{Ex}(H)$ to denote the class of graphs with no minor (isomorphic to) H. Similarly, $\mathrm{Ex}(\mathcal{H})$ denotes the class of graphs with no minor a graph in $\mathcal{H}$. For example: forests = $\mathrm{Ex}(C_3)$, series-parallel graphs = $\mathrm{Ex}(K_4)$, planar graphs = $\mathrm{Ex}(\{K_5, K_{3,3}\})$, graphs with no two disjoint cycles = $\mathrm{Ex}(2C_3)$.

It is easy to see that: $\mathcal{G}$ is minor-closed iff $\mathcal{G} = \mathrm{Ex}(\mathcal{H})$ for some class $\mathcal{H}$. Robertson and Seymour's *graph minors theorem* (once Wagner's conjecture) says that if $\mathcal{G}$ is minor-closed then $\mathcal{G} = \mathrm{Ex}(\mathcal{H})$ for some **finite** class $\mathcal{H}$. The unique minimal such $\mathcal{H}$ consists of the *excluded minors* for $\mathcal{G}$.

For any fixed graph H there is a polynomial time algorithm to test if an input graph G has a minor H. Thus for any minor-closed class there exists a polynomial time algorithm to test if an input graph G is in the class.

Mostly we shall assume that $\mathcal{G}$ is minor-closed and *proper* (that is, not empty and not all graphs). For such $\mathcal{G}$, a result of Mader says that there is a $c = c(\mathcal{G})$ such that the average degree of each graph in $\mathcal{G}$ is at most c. Thus our graphs are *sparse*. For $\mathrm{Ex}(K_t)$ the maximum average degree is of order $t\sqrt{\log t}$ (Kostochka, Thomason).

Recall that $\rho(\mathcal{G})$ denotes the radius of convergence of the egf of $\mathcal{G}$. We call $\mathcal{G}$ *small* if $\rho(\mathcal{G}) > 0$; that is, there exists a constant c such that $|\mathcal{G}_n| \leq c^n n!$ for all n (sufficiently large). We shall be interested here only in small graph classes. Norine, Seymour, Thomas and Wollan [28] and then Dvorák and Norine [10] showed the following result, see also Lemma 12 below.

Lemma 1. *Each proper minor-closed graph class is small.*

2 Connectivity and bridge-addability

In the subsections below we introduce some useful basic properties that a graph class $\mathcal{G}$ may possess, and present corresponding results on connectivity and components for random graphs $R_n \in_u \mathcal{G}$.

2.1 Decomposable, bridge-addable and addable classes

When a graph is in $\mathcal{G}$ if and only if each component is, we call $\mathcal{G}$ *decomposable*. For example the class of planar graphs is decomposable but the class of graphs embeddable on the torus is not.

Exercise 1 (a) *A minor-closed class is decomposable iff each excluded minor is connected.*

Following [25] we say a graph class $\mathcal{G}$ is *bridge-addable* if whenever $G \in \mathcal{G}$ and u and v are in different components of G then $G + uv \in \mathcal{G}$. (Here $G + uv$ denotes the graph obtained by adding the edge uv to G.) The class $\mathcal{G}$ is *addable* if it is both decomposable and bridge-addable. The class $\mathcal{P}$ of planar graphs is addable, and as we noted $\mathcal{P} = \mathrm{Ex}(K_5, K_{3,3})$. The class $\mathcal{G}^S$ of graphs embeddable on the surface S is bridge-addable, but it is **not** decomposable and so not addable, except in the planar case.

Exercise 1 (b) *A minor-closed class $\mathcal{G}$ is addable iff each excluded minor is 2-connected.*

2.2 Connectedness and number of components

It is perhaps natural to expect that a typical graph in a bridge-addable class does not have many components, as there are many possible ways to join components together. In fact we have the following non-asymptotic bounds, from 2005. Let $\kappa(G)$ denote the number of components of G.

Theorem 2. *([25]) If the class $\mathcal{G}$ of graphs is bridge-addable and $R_n \in_u \mathcal{G}$ then*

$$\mathbb{P}(R_n \textit{ is connected}) > 1/e,\ \mathbb{E}[\kappa(R_n)] < 2 \ \ \textit{and} \ \ \kappa(R_n) \leq_s 1 + \mathrm{Po}(1).$$

The last part above means that for each k, $\mathbb{P}(\kappa(R_n) \geq k+1) \leq \mathbb{P}(X \geq k)$, where X has the Poisson distribution with mean 1. Since for $k = 1, 2, ..$ we have

$$\mathbb{P}(\mathrm{Po}(1) \geq k) = e^{-1} \sum_{j \geq k} \frac{1}{j!} = \frac{1}{ek!}(1 + \frac{1}{k+1} + \cdots) < \frac{1}{k!}$$

it follows from Theorem 2 that

$$\mathbb{P}(\kappa(R_n) \geq k+1) < \frac{1}{k!} \quad \text{for each } k = 1, 2, \ldots. \tag{1}$$

Exercise 2 is to prove the first part of the above theorem. Let the class $\mathcal{G}$ of graphs be bridge-addable and let $R_n \in_u \mathcal{G}$. Prove that $\mathbb{P}(R_n \text{ is connected}) > 1/e$, after showing a preliminary result in part (a).
(a) Given a graph G, let $\text{Bridge}(G)$ denote the set of bridges, and let $\text{Cross}(G)$ denote the set of 'non-edges' or 'possible edges' between components. Show that if the graph G has n vertices, then $|\text{Bridge}(G)| \leq n - \kappa(G)$; and if $\kappa(G) = k+1$ for some positive integer k then $|\text{Cross}(G)| \geq k(n-k)$.
(b) Complete the proof, as follows (or otherwise). Let $\mathcal{G}_n^k$ be the set of graphs $G \in \mathcal{G}_n$ with k components. Show that $|\mathcal{G}_n^{k+1}| \leq \frac{1}{k}|\mathcal{G}_n^k|$, by comparing the set of pairs (G, e) such that $G \in \mathcal{G}_n^{k+1}$ and $e \in \text{Cross}(G)$ with the set of pairs (G, e) such that $G \in \mathcal{G}_n^k$ and $e \in \text{Bridge}(G)$.

For trees $\mathcal{T}$ and forests $\mathcal{F}$, $|\mathcal{T}_n| = n^{n-2}$ and $|\mathcal{F}_n| \sim e^{\frac{1}{2}} n^{n-2}$, see [30]. Thus for $F_n \in_u \mathcal{F}$,

$$\mathbb{P}(F_n \text{ is connected}) \sim e^{-\frac{1}{2}}. \tag{2}$$

Forests have the fewest edges of all bridge-addable classes: perhaps they are the 'least likely to be connected'? It was conjectured in 2006 [26] that, if the class $\mathcal{G}$ of graphs is bridge-addable and $R_n \in_u \mathcal{G}$, then

$$\mathbb{P}(R_n \text{ is connected}) \geq e^{-\frac{1}{2}+o(1)}. \tag{3}$$

Balister, Bollobás and Gerke [2, 3] improved on the lower bound e^{-1}; and Addario-Berry, McDiarmid and Reed [1], and Kang and Panagiotou [17] independently proved the conjectured inequality (3) in the special case when we assume that $\mathcal{G}$ is also closed under deleting bridges. Recently Chapuy and Perarnau (to appear in JCTB) established the full conjecture.

Theorem 3. *[9] If the class $\mathcal{G}$ of graphs is bridge-addable and $R_n \in_u \mathcal{G}$, then*

$$\mathbb{P}(R_n \textit{ is connected}) \geq e^{-\frac{1}{2}+o(1)}.$$

Here is a natural possible strengthening of this result.

Conjecture 1. *[3] If the class $\mathcal{G}$ of graphs is bridge-addable, $R_n \in_u \mathcal{G}$ and $F_n \in_u \mathcal{F}$ (where $\mathcal{F}$ is the class of forests), then*

$$\mathbb{P}(R_n \textit{ is connected}) \geq \mathbb{P}(F_n \textit{ is connected}).$$

2.3 The fragment

The *big component* $\mathrm{Big}(G)$ of a graph G is the component with most vertices, with ties broken in some way (for example, we take the lex first component with the maximum number of vertices). The *fragment* 'left over', $\mathrm{Frag}(G)$, is the subgraph induced on the vertices not in the big component (which could be the empty graph). We think of $\mathrm{Frag}(G)$ as an unlabelled graph. Write $\mathrm{frag}(G)$ for $v(\mathrm{Frag}(G))$. The next result shows that, for a bridge-addable class, $\mathrm{Big}(R_n)$ is giant!

Theorem 4 ([21]). *If the class $\mathcal{G}$ of graphs is bridge-addable and $R_n \in_u \mathcal{G}$ then $\mathbb{E}[\mathrm{frag}(R_n)] < 2$.*

Exercise 3 is to prove this theorem, after a preliminary result.
(a) Show that, if the graph G has n vertices, then $|\mathrm{Cross}(G)| \geq (n/2) \cdot \mathrm{frag}(G)$.
(b) Now prove Theorem 4 .

3 Growth constant

In order to deduce results about a class $\mathcal{G}$ of graphs, we need to know that the numbers $|\mathcal{G}_1|, |\mathcal{G}_2|, |\mathcal{G}_3|, \ldots$ do not behave too erratically. Recall that $\rho = \rho(\mathcal{G})$ denotes the radius of convergence of the corresponding egf: thus $0 \leq \rho \leq \infty$ and $\rho^{-1} = \limsup_{n\to\infty}(|\mathcal{G}_n|/n!)^{1/n}$. We say that $\mathcal{G}$ has *growth constant* γ if $0 < \gamma < \infty$, and

$$(|\mathcal{G}_n|/n!)^{1/n} \to \gamma \quad \text{as } n \to \infty$$

that is

$$|\mathcal{G}_n| = (\gamma + o(1))^n \; n!.$$

In this case clearly $\rho^{-1} = \gamma$.

Many of the graph classes in which we are interested are decomposable, but not all. For example, consider the class $\mathcal{G}^S$ of graphs embeddable on a given surface S, where S is not the sphere, say S is the torus. For each graph $G \in \mathcal{G}^S$, each component certainly must be in $\mathcal{G}^S$. But given two disjoint graphs in $\mathcal{G}^S$ it does not follow that their union is: for example, we cannot embed two disjoint copies of K_5 in the torus.

Exercise 4 Let us call a class $\mathcal{G}$ of graphs *down-decomposable* if each component of each graph in $\mathcal{G}$ is also in $\mathcal{G}$. Of course, any hereditary family has this property, and in particular any minor-closed family.

Suppose that $\mathcal{G}$ is down-decomposable, let $\mathcal{C}$ be the class of connected graphs in $\mathcal{G}$, and suppose that $\mathcal{C}$ has growth constant γ. Show that $\mathcal{G}$ also has growth constant γ. (Hint: you may want to use the 'exponential formula'.)

Exercise 5 A graph is called *apex* if by deleting a vertex we may obtain a planar graph. More generally, given a class $\mathcal{G}$ of graphs, the corresponding *apex class*, $\operatorname{Apex}\mathcal{G}$, consists of all graphs G such that we may obtain a graph in $\mathcal{G}$ by deleting at most one 'apex' vertex from G. It is easy to check that if $\mathcal{G}$ is minor-closed then so is $\operatorname{Apex}\mathcal{G}$. Show that, if $\mathcal{G}$ has growth constant γ, then $\operatorname{Apex}\mathcal{G}$ has growth constant 2γ.

Observe that if $\mathcal{G}$ contains all paths then $|\mathcal{G}_n| \geq \frac{1}{2}n!$ and so $\rho(\mathcal{G}) \leq 1$. Bernardi, Noy and Welsh [6] showed in 2010 that, if $\mathcal{G}$ is monotone (closed under deleting edges) and does not contain all paths then $\rho(\mathcal{G}) = \infty$.

3.1 When is there a growth constant?

We need 'Fekete's lemma', which is widely useful in combinatorics, in its supermultiplicative form: see for example [32].

Lemma 5. *Let $f : \mathbb{N} \to \mathbb{R}_{\geq 0}$ be a function satisfying $f(m+n) \geq f(m)f(n)$ for all $m, n \in \mathbb{N}$, and such that $f(n) > 0$ for n sufficiently large. Then*

$$f(n)^{1/n} \to \sup_m f(m)^{1/m} \leq \infty \quad \text{as } n \to \infty.$$

Now we can prove the following lemma, which is a crucial first step in investigating suitable graph classes.

Lemma 6 ([25])**.** *Let the non-empty class $\mathcal{G}$ of graphs be small and addable. Then $\mathcal{G}$ has a growth constant.*

Proof. Since $\mathcal{G}$ is bridge-addable, by Theorem 2

$$\mathbb{P}(R_n \text{ is connected}) = \frac{|\mathcal{C}_n|}{|\mathcal{G}_n|} \geq \frac{1}{e}.$$

Also, since $\mathcal{G}$ is decomposable, the disjoint union G of a graph in $\mathcal{C}_a$ and one in $\mathcal{C}_b$ is in $\mathcal{G}_{a+b}$. If $a = b$, as well as G consider G plus an arbitrary edge e between the components. Then

$$|\mathcal{G}_{a+b}| \geq \binom{a+b}{a} |\mathcal{C}_a||\mathcal{C}_b| \geq \frac{(a+b)!}{a!\, b!} \frac{|\mathcal{G}_a|}{e} \frac{|\mathcal{G}_b|}{e}.$$

It follows that $f(n) = \frac{|\mathcal{G}_n|}{e^2 n!}$ satisfies $f(a+b) \geq f(a) \cdot f(b)$ for all positive integers a, b (that is, f is supermultiplicative). Now we can use 'Fekete's lemma', Lemma 5, gives

$$f(n)^{1/n} \to \gamma := \sup_k f(k)^{1/k} > 0,$$

where $\gamma < \infty$ since $\mathcal{G}$ is small. Thus $\mathcal{G}$ has growth constant γ. □

By Lemmas 1 and 6 we have

Theorem 7. *Every proper addable minor-closed class $\mathcal{G}$ of graphs has a growth constant.*

In particular the class $\mathcal{P}$ of planar graphs has a growth constant. For any surface S other than the plane, the class $\mathcal{G}^S$ of graphs embeddable in S is bridge-addable but not addable. However, we can still see [19] that $\mathcal{G}^S$ has a growth constant, since it is 'not much bigger' than $\mathcal{P}$, and in fact has the same growth constant as $\mathcal{P}$. We now know *much* more, as we will see later.

By a result of Bernardi, Noy and Welsh (2010)) mentioned earlier, a minor-closed class $\mathcal{G}$ of graphs has $\rho(\mathcal{G}) < \infty$ iff it contains all paths.

Conjecture 2 ([6]). *Every proper minor-closed class $\mathcal{G}$ of graphs with $\rho(\mathcal{G}) < \infty$ has a growth constant; and indeed the class $\mathcal{C}$ of connected graphs in $\mathcal{G}$ has the same growth constant.*

3.2 Pendant Appearances Theorem

Perhaps the most striking result we can deduce when a class has a growth constant is the 'Pendant Appearances Theorem', from [25, 26]. We shall want a slight extension of this result.

Given a connected graph H, let us say that a graph G has a *pendant copy* of H if 'G has a bridge with H at one end'; that is, if G has a bridge e such that one component of $G - e$ is a copy of H.

It is convenient to ask for more. Let H be a connected h-vertex graph with vertex set $V(H) \subset \mathbb{N}$, and let G be a graph on vertex set $\{1, \ldots, n\}$ where $n > h$. Let W be an h-set of vertices of G, and let the 'root' r_W be the least element in W. We say that H has a *pendant appearance* at W in G if (a) the increasing bijection from $V(H)$ to W gives an isomorphism between H and the induced subgraph $G[W]$ of G; and (b) there is exactly one edge in G between W and the rest of G, and this edge is incident with the root r_W.

Call a connected rooted graph H *attachable* to $\mathcal{G}$ if whenever we have a graph G in $\mathcal{G}$ and a disjoint copy of H, and we add an edge between a vertex in G and the root of H, then the resulting graph (which has a pendant copy of H) must be in $\mathcal{G}$. For an addable minor-closed class $\mathcal{G}$, the class of attachable graphs is the class of all connected rooted graphs in $\mathcal{G}$. For $\mathcal{G}^S$, the class of attachable graphs is the class of connected rooted planar graphs.

In the standard version of the Pendant Appearances Theorem, we assume that $\mathcal{G}$ has a growth constant, and can then often deduce that $R_n \in_u \mathcal{G}$ has a linear numbers of vertices of each degree, has exponentially many automorphisms, and so on. We give a slight generalisation of the theorem, where we do not assume the existence of a growth constant.

Theorem 8. *Let $\mathcal{G}$ be a class of graphs with $0 < \rho(\mathcal{G}) < \infty$, and let the connected rooted graph H be attachable to $\mathcal{G}$. Then there exists $\alpha > 0$ such that the following holds. Let $\mathcal{H}$ denote the class of graphs G in $\mathcal{G}$ with at most $\alpha\, v(G)$ pendant appearances of H. Then $\rho(\mathcal{H}) > \rho(\mathcal{G})$.*

If $\mathcal{G}$ has a growth constant we quickly obtain the usual version.

Corollary 9. *(Pendant Appearances Theorem [25, 26]) Let the class $\mathcal{G}$ of graphs have a growth constant, and let the connected rooted graph H be attachable to $\mathcal{G}$. Then there exists $\alpha > 0$ such that for $R_n \in_u \mathcal{G}$*

$$\mathbb{P}(R_n \textit{ has } \leq \alpha n \textit{ pendant appearances of } H) = e^{-\Omega(n)}.$$

Proof of Corollary 9. Let $\mathcal{H}$ be as in Theorem 8. Then the above probability equals

$$\frac{|\mathcal{H}_n|}{|\mathcal{G}_n|} \leq \frac{(\rho(\mathcal{H})^{-1} + o(1))^n}{(\rho(\mathcal{G})^{-1} + o(1))^n} = e^{-\Omega(n)},$$

as required. □

Sketch of proof of Theorem 8. Of course $\rho(\mathcal{H}) \geq \rho(\mathcal{G})$. The idea of the proof is to show that, if $\rho(\mathcal{H}) = \rho(\mathcal{G})$, and $|\mathcal{H}_n|$ is close to $\rho(\mathcal{H})^{-n}n!$ for some large n, then we can construct "too many" graphs in $\mathcal{G}$ on $(1+\delta)n$ vertices. Essentially this is done by attaching linearly many copies of H to each graph G in $\mathcal{H}_n$: the fact that G has 'few' appearances of H limits the amount of double-counting involved.

□

3.3 Applications of the Pendant Appearances Theorem

In this section we present a few applications of Corollary 9. We shall meet more applications later.

Vertex degrees

By applying Corollary 9 to particular graphs H, for example to a star on k vertices, we can learn about vertex degrees.

Proposition 10 ([25])**.** *Let the class $\mathcal{G}$ of graphs have a growth constant, and let $R_n \in_u \mathcal{G}$. Let $k \geq 1$ and suppose that the star with $k-1$ edges, rooted at its centre, is attachable to $\mathcal{G}$. Then there exists a constant $\alpha > 0$ such that*

$$\mathbb{P}(R_n \text{ has } < \alpha n \text{ vertices of degree } k) = e^{-\Omega(n)}.$$

Symmetries

The next result follows immediately from the fact that a graph with linearly many vertices that are connected to at least two leaves has exponentially many automorphisms.

Proposition 11. *Let the class $\mathcal{G}$ of graphs have a growth constant, and suppose that the 3-vertex path, rooted at its centre, can be attached to $\mathcal{G}$. Then there is a constant $\alpha > 0$ such that, for $R_n \in_u \mathcal{G}$,*

$$\mathbb{P}(R_n \text{ has } < 2^{\alpha n} \text{ automorphisms }) = e^{-\Omega(n)}.$$

This result is in contrast to what happens with classical random graphs: for example, the Erdős-Rényi random graph $G_{n,\frac{1}{2}}$ has whp only the trivial automorphism; and indeed the probability that $G_{n,\frac{1}{2}}$ has a non-trivial automorphism is $2^{-(1+o(1))n}$. We will use Proposition 11 in the next section on unlabelled graphs.

Distinct growth constants

Consider two distinct proper addable minor-closed classes of graphs $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A} \subset \mathcal{B}$. By Theorem 7, they have growth constant $\gamma_{\mathcal{A}}$ and $\gamma_{\mathcal{B}}$ respectively. Let us see that $\gamma_{\mathcal{A}} < \gamma_{\mathcal{B}}$ (as was shown in [6]).

For let G be a connected graph in $\mathcal{B}$ and not in $\mathcal{A}$. Then by the Pendant Appearances Theorem, there is a constant $\delta > 0$ such that the proportion of graphs in $\mathcal{B}_n$ with no pendant appearances of G is $O(e^{-\delta n})$. Hence $|\mathcal{A}_n|/|\mathcal{B}_n| = O(e^{-\delta n})$, and so $\gamma_{\mathcal{A}} \leq \gamma_{\mathcal{B}}\, e^{-\delta}$.

Colouring

There is a fast expected-time colouring algorithm for random planar graphs, as follows. Let $R_n \in_u \mathcal{P}$, where $\mathcal{P}$ is the class of planar graphs. Since with very high probability R_n has linearly many pendant copies of K_4, we may check if R_n has a pendant copy of K_4 in constant expected time. If there is one, then we apply the quadratic time algorithm to four-colour planar graphs, which follows from the proof of the four-colour theorem, see [31]. If there is no pendant copy of K_4, which happen with probability $e^{-\Omega(n)}$, we colour the graph optimally in subexponential time $O(c^{\sqrt{n}})$ by using the $\sqrt{n}$-separator theorem. It follows that we can colour a random planar graph R_n optimally in quadratic expected time. This observation is due to Anusch Taraz and Michael Krivelevich, see [25]. Further, we see that we can determine $\chi(R_n)$ in constant expected time.

Hadwiger's Conjecture is one of the major conjectures of graph theory. It says that, for each positive integer k, if $\chi(G) \geq k$ then G has a minor K_k; and this is known to be true for $k \leq 6$. Hadwiger's Conjecture being false says that for some k, there is a graph $G \in \mathrm{Ex}(K_k)$ with $\chi(G) \geq k$. But in this case, Corollary 9 then implies that all but an exponentially small proportion of graphs G in $(\mathrm{Ex}(K_k))_n$ have $\chi(G) \geq k$ and so are counterexamples.

3.4 Unlabelled graphs

For unlabelled graphs we can follow some of the steps which worked for labelled graphs. We think of an unlabelled graph as an equivalence class of labelled graphs. Thus the set $\mathcal{G}_n$ of (labelled) graphs is partitioned into a set of unlabelled graphs, which we denote by $\tilde{\mathcal{G}}_n$; and similarly we write $\tilde{\mathcal{G}}$ for the set of unlabelled graphs in $\mathcal{G}$, and write $\tilde{\rho}_{\mathcal{G}}$ for the radius of convergence of the generating function $\tilde{G}(x) = \sum_{n \geq 0} |\tilde{\mathcal{G}}_n| x^n$.

Let $\mathcal{G}$ be a class of graphs such that $\mathcal{G}_n$ is non-empty for n sufficiently large. Observe that $\tilde{\rho}_{\mathcal{G}} \leq 1$. We say that $\mathcal{G}$ (and the corresponding set $\widetilde{\mathcal{G}}$ of unlabelled graphs) has *unlabelled growth constant* $\widetilde{\gamma} = \widetilde{\gamma}_{\mathcal{G}}$ if $|\widetilde{\mathcal{G}}_n|^{1/n} \to \widetilde{\gamma}$ as $n \to \infty$. In this case $\widetilde{\gamma}$ must $\tilde{\rho}_{\mathcal{G}}^{-1}$. We shall see that the class $\mathcal{P}$ of planar graphs has an unlabelled growth constant, and indeed this holds more generally. Lemma 1 was stated in terms of labelled graph classes, but in fact the 'smallness' result of Dvorák and Norine [10] is for unlabelled graph classes.

Lemma 12 ([10])**.** *For each proper minor-closed class $\mathcal{G}$ of graphs there is a constant c such that $|\widetilde{\mathcal{G}}_n| \leq c^n$ for each n.*

This result immediately implies Lemma 1, since $|\mathcal{G}_n| \leq n! \cdot |\tilde{\mathcal{G}}_n|$. We may now prove the main result of this section.

Theorem 13. *Let $\mathcal{G}$ be a proper addable minor-closed class of graphs, and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{G}$. Then $\mathcal{G}$ and $\mathcal{C}$ have an unlabelled growth constant, and $\widetilde{\gamma}_{\mathcal{G}} = \widetilde{\gamma}_{\mathcal{C}} > \gamma_{\mathcal{G}}$ $(= \gamma_{\mathcal{C}})$.*

For example, the class $\mathcal{F}$ of forests has unlabelled growth constant $\widetilde{\gamma}_{\mathcal{F}} \approx 2.956 > \gamma_{\mathcal{F}} = e \approx 2.718$, see Otter [29]. Thus we must have $\widetilde{\gamma}_{\mathcal{G}} \geq \widetilde{\gamma}_{\mathcal{F}} \approx 2.956$ for each class $\mathcal{G}$ as above.

Proof. Let $\widetilde{\mathcal{C}}^{\bullet}$ denote the set of (vertex-) rooted graphs in $\widetilde{\mathcal{C}}$. Then $f(n) = |\widetilde{\mathcal{C}}_n^{\bullet}|$ is supermultiplicative; that is, for positive integers a and b

$$|\widetilde{\mathcal{C}}_{a+b}^{\bullet}| \geq |\widetilde{\mathcal{C}}_a^{\bullet}| \cdot |\widetilde{\mathcal{C}}_b^{\bullet}|. \tag{4}$$

To see this, note first that we may assume that $a \leq b$. We may form a graph H in $\widetilde{\mathcal{C}}_{a+b}^{\bullet}$ from disjoint copies of graphs $H_a \in \widetilde{\mathcal{C}}_a^{\bullet}$ with root r_a and $H_b \in \widetilde{\mathcal{C}}_b^{\bullet}$ with root r_b, by adding the edge $r_a r_b$ and making r_a the new root r^*. There is no double counting, since there is a unique bridge e in H incident with r^* such that at least half the vertices are in the component of $H \setminus e$ not containing r^*. Thus (4) holds, as required.

Since $f(n)$ is supermultiplicative, by Fekete's Lemma (Lemma 5), as $n \to \infty$

$$|\widetilde{\mathcal{C}}_n^{\bullet}|^{1/n} \to \widetilde{\gamma} := \sup_k |\widetilde{\mathcal{C}}_k^{\bullet}|^{1/k} > 1;$$

and $\widetilde{\gamma} < \infty$ by Lemma 12.

We have now seen that $\widetilde{\mathcal{C}}^{\bullet}$ has unlabelled growth constant $\widetilde{\gamma}$, and so this holds also for $\widetilde{\mathcal{C}}$, since $|\widetilde{\mathcal{C}}_n| \leq |\widetilde{\mathcal{C}}_n^{\bullet}| \leq n\,|\widetilde{\mathcal{C}}_n|$. It follows that $\widetilde{\mathcal{G}}$ also has unlabelled growth constant $\widetilde{\gamma}$, for example by noting that $|\widetilde{\mathcal{C}}_n| \leq |\widetilde{\mathcal{G}}_n| \leq |\widetilde{\mathcal{C}}^{\bullet}_{n+1}|$.

It remains to show that $\widetilde{\gamma}_{\mathcal{G}} > \gamma_{\mathcal{G}}$. Recall that the isomorphism class of a graph G in $\mathcal{G}_n$ has size $n!/\mathrm{aut}(G)$. Thus by Proposition 11, with the constant $\alpha > 0$ introduced there, the number of graphs in $\mathcal{G}_n$ which are in isomorphism classes of size $> 2^{-\alpha n} n!$ is at most $2^{-\Omega(n)}|\mathcal{G}_n|$, which is at most $\frac{1}{2}|\mathcal{G}_n|$ for n sufficiently large. But then

$$|\widetilde{\mathcal{G}}_n| \geq \frac{1}{2}|\mathcal{G}_n|/(2^{-\alpha n} n!),$$

that is

$$|\mathcal{G}_n|/n! \leq 2^{1-\alpha n}|\widetilde{\mathcal{G}}_n|;$$

and it follows that $\gamma_{\mathcal{G}} \leq 2^{-\alpha}\widetilde{\gamma}_{\mathcal{G}}$. □

By the last theorem, the class $\mathcal{P}$ of planar graphs has unlabelled growth constant $\widetilde{\gamma}_{\mathcal{P}} > \gamma_{\mathcal{P}}$. (This was first shown in [25].) We know $\gamma_{\mathcal{P}}$ precisely – from Giménez and Noy [13, 14] in 2009 we have $\gamma \approx 27.226878$ – but this is not the case for $\widetilde{\gamma}_{\mathcal{P}}$. The best known bounds are $\gamma_{\mathcal{P}} < \widetilde{\gamma}_{\mathcal{P}} \leq 30.061$, see [7, 25]. (We do not know an asymptotic counting formula or even smoothness for $\widetilde{\mathcal{P}}$.) We noted that, for each fixed surface S, the class $\mathcal{G}^S$ of graphs embeddable in S has growth constant the planar growth constant $\gamma_{\mathcal{P}}$. Similarly, $\mathcal{G}^S$ has unlabelled growth constant $\widetilde{\gamma}_{\mathcal{P}}$.

There is a natural conjecture corresponding to Conjecture 2 for the labelled case.

Conjecture 3. *Each proper minor-closed class $\mathcal{G}$ has an unlabelled growth constant $\widetilde{\gamma}$.*

Connectivity

We could adapt the first part of the proof of Theorem 13 above to prove the key result Lemma 6 on the existence of growth constants for labelled graphs. The proof we gave for Lemma 6 relied on the connectivity lower bound for a (labelled) bridge-addable class given in Theorem 2. We do not know any corresponding result for a bridge-addable class of unlabelled graphs.

Conjecture 4. *There is a $\delta > 0$ such that, if the graph class $\mathcal{A}$ is bridge-addable, $\mathcal{A}_n \neq \emptyset$, and $\tilde{R}_n \in_u \tilde{\mathcal{A}}$, then*

$$\mathbb{P}(\tilde{R}_n \textit{ is connected}) \geq \delta \quad \textit{for each } n. \tag{5}$$

For trees and forests we have

$$|\tilde{\mathcal{T}}_n|/|\tilde{\mathcal{F}}_n| \to \tau \approx e^{-0.5226} \approx 0.5930 \quad \text{as } n \to \infty,$$

4 Smoothness and the fragment

In this section we introduce the concept of smoothness for a graph class, and we see that each proper addable minor-closed proper graph class $\mathcal{G}$ is smooth (we saw in Theorem 7 that $\mathcal{G}$ has a growth constant); and each surface class $\mathcal{G}^S$ is smooth (we saw that it has growth constant $\gamma_{\mathcal{P}}$). Next we introduce the Boltzmann Poisson random graph $BP(\mathcal{G}, \rho)$, and see that, under suitable conditions, the unlabelled fragment of the random graph R_n converges in distribution to a Boltzmann Poisson random graph. These results tell us for example about the limiting probability that R_n is connected. Finally we sketch a fuller picture for the addable case.

Let us define smoothness. Let $\mathcal{G}$ be any class of graphs with $0 < \rho = \rho(\mathcal{G}) < \infty$. Let $g_n = |\mathcal{G}_n|$ and let $r_n = ng_{n-1}/g_n$. As an aside, observe that if isolated vertices can be freely added to or removed from graphs in $\mathcal{G}$, then r_n is the expected number of isolated vertices in $R_n \in_u \mathcal{G}$. It is easy to see that

$$\liminf_{n\to\infty} r_n \le \rho \le \limsup_{n\to\infty} r_n. \tag{6}$$

Thus if r_n tends to a limit as $n \to \infty$ then that limit must be ρ. We call $\mathcal{G}$ *smooth* if $r_n \to \rho$ as $n \to \infty$.

In this case, $\mathcal{G}$ must have growth constant ρ^{-1}. For suppose that $\mathcal{G}$ is smooth and let $0 < \epsilon < 1$. Then there is a k such that $g_n > 0$ for $n \ge k$ and $\rho/(1+\epsilon) < r_n < \rho/(1-\epsilon)$ for $n \ge k+1$. Thus

$$(1-\epsilon)\rho^{-1} < \frac{g_n}{ng_{n-1}} < (1+\epsilon)\rho^{-1}$$

and so for $n > k$

$$((1-\epsilon)\rho^{-1})^{n-k}\frac{g_k}{k!} < \frac{g_n}{n!} < ((1+\epsilon)\rho^{-1})^{n-k}\frac{g_k}{k!}.$$

But now

$$((1-\epsilon)^{-k}\rho^k\frac{g_k}{k!})^{\frac{1}{n}}\,(1-\epsilon)\rho^{-1} < \left(\frac{g_n}{n!}\right)^{\frac{1}{n}} < ((1+\epsilon)^{-k}\rho^k\frac{g_k}{k!})^{\frac{1}{n}}\,(1+\epsilon)\rho^{-1}$$

and we see that $(g_n/n!)^{1/n} \to \rho^{-1}$ as $n \to \infty$, as required.

For example, for the class $\mathcal{F}$ of forests,

$$r_n = n|\mathcal{F}_{n-1}|/|\mathcal{F}_n| \to e^{-1} \text{ as } n \to \infty$$

so $\mathcal{F}$ is smooth. Indeed all the classes for which we know an asymptotic counting formula are smooth, including (as well as the forests) series-parallel graphs, $\mathcal{P}$, $\mathcal{G}^S$. Showing smoothness is often a crucial step in proving results about $R_n \in_u \mathcal{G}$.

Smoothness for addable classes and for $\mathcal{G}^S$

We know that $\mathcal{G}^S$ has growth constant γ, where $\gamma = \gamma_{\mathcal{P}}$ is the planar graph growth constant. Further, the asymptotic counting formula for $|\mathcal{P}_n|$ given by Giménez and Noy [13] shows that $\mathcal{P}$ is smooth. Bender, Canfield and Richmond [4] in 2008 show that the surface class $\mathcal{G}^S$ is smooth for any surface S. Their proof does not involve an asymptotic counting formula (and indeed none was then known). Such a formula was given in 2011 by Chapuy, Fusy, Giménez, Mohar and Noy [8], and by Bender and Gao [5].

The composition method used by Bender, Canfield and Richmond will tell us more. The key idea in the proof is to consider the core. The *core* of G, core(G), is the unique maximal subgraph H such that the minimum degree $\delta(H) \geq 2$ (and is empty if G is a forest). Let $\mathcal{G}^{\delta \geq 2}$ denote the class of graphs in $\mathcal{G}$ with minimum degree $\delta \geq 2$. The idea is that if $\mathcal{G}^{\delta \geq 2}$ grows reasonably smoothly then rooting trees at vertices in the core leads to the class $\mathcal{G}$ being smooth.

Theorem 14. *([4, 20]) Let $\mathcal{G}$ either be a proper addable minor-closed class of graphs, or be the class $\mathcal{G}^S$ of graphs embeddable in a given surface S; and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{G}$. Then $\mathcal{G}$ and $\mathcal{C}$ are both smooth.*

Perhaps more is true?

Conjecture 5. *Every proper minor-closed class is smooth.*

Sketch proof of smoothness for addable case

Let $\mathcal{G}$ be an addable minor-closed class, not $\mathcal{F}$. Let us sketch why $\mathcal{G}$ is smooth. Note that $\mathcal{F} \subseteq \mathcal{G}$, and C_3 is attachable to $\mathcal{G}$. Hence, by the Pendant Appearances Theorem, $\rho(\mathcal{G}) < \rho(\mathcal{F})$ $(= 1/e)$.

We shall use the product and the composition of graph classes from the *symbolic method* – see [11]. Let $\mathcal{G}^-$ be the class of graphs $G \in \mathcal{G}$ with no tree components. We may think of $\mathcal{G}$ as the product of $\mathcal{G}^-$ and $\mathcal{F}$. But $\rho(\mathcal{G}^-) < \rho(\mathcal{F})$, so it suffices to show that $\mathcal{G}^-$ is smooth.

$\mathcal{G}^{\delta\geq 2}$ is addable and small, so by Lemma 6 it has a growth constant. Call this class $\mathcal{B}$. Graphs G in $\mathcal{G}^-$ are obtained by starting with a graph in $\mathcal{B}$ and rooting a tree at each vertex. Then $\mathcal{G}^-$ is the composition of $\mathcal{B}$ with the class $\mathcal{T}^\bullet$ of rooted trees, so

$$G^-(x) = B(\mathcal{T}^\bullet(x)).$$

Now

$$\begin{aligned} |\mathcal{G}_n^-|/n! = [x^n]G^-(x) &= [x^n] \sum_{k=0}^{n} |\mathcal{B}_k| \, (T^\bullet(x))^k/k! \\ &\sim [x^n] \sum_{k:|k-\alpha n|<\epsilon n} |\mathcal{B}_k|/k! \, (T^\bullet(x))^k, \end{aligned}$$

where $\alpha = 1 - \rho(\mathcal{G}^{\delta\geq 2})$, $0 < \alpha < 1$. This last 'concentration' result is a key step in the proof, see [4], and see also [22]. We have

$$\frac{|\mathcal{G}_n^-|}{n!} \sim \sum_{k:|k-\alpha n|<\epsilon n} |\mathcal{B}_k|/k! \, [x^n]T^\bullet(x)^k$$

and

$$\frac{|\mathcal{G}_{n+1}^-|}{(n+1)!} \sim \sum_{k:|k-\alpha n|<\epsilon n} |\mathcal{B}_k|/k! \, [x^{n+1}]T^\bullet(x)^k.$$

Each individual ratio

$$[x^{n+1}]T^\bullet(x)^k / \, [x^n]T^\bullet(x)^k$$

is close to $\rho_{\mathcal{G}}^{-1}$. Hence

$$\frac{|\mathcal{G}_{n+1}^-|}{(n+1)|\mathcal{G}_n^-|} \sim \rho_{\mathcal{G}}^{-1}.$$

This completes our sketch proof of smoothness, but what will smoothness yield?

4.1 Boltzmann Poisson random graph

Let $\mathcal{G}$ be a non-empty class of graphs, and let $\widetilde{\mathcal{G}}$ denote the set of unlabelled graphs in $\mathcal{G}$. Fix $\rho > 0$ such that $G(\rho)$ is finite; and let

$$\mu(H) = \frac{\rho^{v(H)}}{\mathrm{aut}(H)} \quad \text{for each } H \in \widetilde{\mathcal{G}} \tag{7}$$

(where $\mathrm{aut}(\phi) = 1$ and so $\mu(\phi) = 1$, if the empty graph ϕ is in $\widetilde{\mathcal{G}}$). We will normalise these quantities to give probabilities. Since each graph $H \in \widetilde{\mathcal{G}}$ consists of $\frac{v(H)!}{\mathrm{aut}(H)}$ isomorphic graphs $G \in \mathcal{G}$, we have

$$\frac{z^{v(H)}}{\mathrm{aut}(H)} = \sum_{G \in H} \frac{\mathrm{aut}(H)}{v(H)!} \frac{z^{v(H)}}{\mathrm{aut}(H)} = \sum_{G \in H} \frac{z^{v(G)}}{v(G)!}.$$

Therefore

$$G(z) = \sum_{H \in \widetilde{\mathcal{G}}} \sum_{G \in H} \frac{z^{v(G)}}{v(G)!} = \sum_{H \in \widetilde{\mathcal{G}}} \frac{z^{v(H)}}{\mathrm{aut}(H)}.$$

Thus

$$G(\rho) = \sum_{H \in \widetilde{\mathcal{G}}} \mu(H). \tag{8}$$

Now assume that $\mathcal{G}$ is decomposable. By convention, the empty graph ϕ is in $\mathcal{G}$. The *Boltzmann Poisson random graph* $R = \mathrm{BP}(\mathcal{G}, \rho)$ takes values in $\widetilde{\mathcal{G}}$, with

$$\mathbb{P}(R = H) = \frac{\mu(H)}{G(\rho)} \quad \text{for each } H \in \widetilde{\mathcal{G}}.$$

Let $\mathcal{C}$ denote the class of connected graphs in $\mathcal{G}$. Observe that $\mathbb{P}(R = \emptyset) = G(\rho)^{-1} = e^{-C(\rho)}$. Also, if the one-vertex graph K_1 is in $\mathcal{G}$, then $\mathbb{P}(R = K_1) = \rho e^{-C(\rho)}$. For each $H \in \widetilde{\mathcal{C}}$ let $\kappa(G, H)$ denote the number of components of G isomorphic to H.

Proposition 15. *The random variables $\kappa(R, H)$ for $H \in \widetilde{\mathcal{C}}$ are independent, with $\kappa(R, H) \sim \mathrm{Po}(\mu(H))$.*

Proof. Each sum and product below is over all H in $\widetilde{\mathcal{C}}$. Let the unlabelled graph G consist of n_H components isomorphic to H for each $H \in \widetilde{\mathcal{G}}$, where $0 \leq \sum_H n_H < \infty$. Then

$$\rho^{v(G)} = \prod_H \rho^{v(H) n_H}$$

and

$$\mathrm{aut}(G) = \prod_H \mathrm{aut}(H)^{n_H} n_H!.$$

Also since $\sum_H \mu(H) = C(\rho)$ by (8) applied to $\mathcal{C}$,

$$\frac{1}{G(\rho)} = e^{-C(\rho)} = \prod_H e^{-\mu(H)}.$$

Hence

$$\begin{aligned}
\mathbb{P}(R = G) &= e^{-C(\rho)} \frac{\rho^{v(G)}}{\mathrm{aut}(G)} \\
&= \prod_H e^{-\mu(H)} \frac{\mu(H)^{n_H}}{n_H!} \\
&= \prod_H \mathbb{P}(\mathrm{Po}(\mu(H)) = n_H).
\end{aligned}$$

Thus the probability factors appropriately, and the random variables $\kappa(R, H)$ for $H \in \widetilde{\mathcal{C}}$ satisfy

$$\mathbb{P}(\kappa(R,H) = n_H \ \ \forall H \in \widetilde{\mathcal{C}}) = \prod_H \mathbb{P}(\kappa(R,H) = n_H).$$

This completes the proof. □

4.2 Fragments for addable classes and surface classes

Recall that the fragment $\mathrm{Frag}(G)$ of a graph G is the graph remaining when we discard the largest component (breaking ties in some way): we view it as an unlabelled graph. Recall also that for a decomposable graph class $\mathcal{G}$, the Boltzmann Poisson random graph $BP(\mathcal{G}, \rho)$ is well defined when $0 < G(\rho) < \infty$, and takes values in $\widetilde{\mathcal{G}}$. Here is the Fragments Theorem for addable minor-closed classes and surface classes $\mathcal{G}$ (there are more general versions).

Theorem 16. *(Fragments Theorem)*

(a) Let $\mathcal{G}$ be a proper addable minor-closed class and let $\rho = \rho(\mathcal{G})$. Then $0 < G(\rho) < \infty$; and for $R_n \in_u \mathcal{G}$, the fragment converges in distribution to $BP(\mathcal{G}, \rho)$.

(b) Let S be a given surface, let $\mathcal{G}$ be $\mathcal{G}^S$, and let $\rho = \rho(\mathcal{P})$ (where $\mathcal{P}$ is the class of planar graphs). Then $0 < G(\rho) < \infty$; and for $R_n \in_u \mathcal{G}$, the fragment converges in distribution to $BP(\mathcal{P}, \rho)$.

In the addable case (a) above, let $\mathcal{C}$ be the class of connected graphs in $\mathcal{G}$, and in the surface case (b), let $\mathcal{C}$ be the class of connected graphs in $\mathcal{P}$. Then

$$\mathbb{P}(R_n \text{ is connected }) \to \mathbb{P}(R \text{ is connected }) = e^{-C(\rho)}.$$

Consider the classes $\mathcal{T}$ of trees and $\mathcal{F}$ of forests, which have radius of convergence $\rho = e^{-1}$. For $R_n \in_u \mathcal{F}$, since $T(\rho) = \frac{1}{2}$ we have

$$\mathbb{P}(R_n \text{ is connected }) = \frac{|\mathcal{T}_n|}{|\mathcal{F}_n|} \to e^{-T(\rho)} = e^{-\frac{1}{2}} \quad \text{as} \quad n \to \infty,$$

as we already saw in (2).

Fragments theorem - proof idea (assuming smoothness)

Call a graph H *freely addable* to $\mathcal{G}$ when a graph G is in $\mathcal{G}$ if and only if the disjoint union $G \cup H$ is in $\mathcal{G}$. Observe that $\mathcal{G}$ is decomposable if and only if each graph $H \in \mathcal{G}$ is freely addable to $\mathcal{G}$. Also, for a surface class $\mathcal{G}^S$, the freely addable graphs are precisely the planar graphs. Recall that $\mu(H) = \rho^{v(H)}/\mathrm{aut}(H)$, and $r_n = n|\mathcal{G}_{n-1}|/|\mathcal{G}_n|$. The following lemma is essentially Lemma 5.1 in [19].

Lemma 17. *Let $\mathcal{G}$ be any class of graphs and let $\rho > 0$. Let $H_1, \ldots, H_m$ be pairwise non-isomorphic connected graphs, each freely addable to $\mathcal{G}$. Let $k_1, \ldots, k_m$ be non-negative integers, and let $K = \sum_{i=1}^m k_i v(H_i)$. Then for $R_n \in_u \mathcal{G}$,*

$$\mathbb{E}\left[\prod_{i=1}^m (\kappa(R_n, H_i))_{k_i}\right] = \prod_{i=1}^m \mu(H_i)^{k_i} \prod_{j=1}^K (r_{n-j+1}/\rho).$$

Proof. Let $v_i = v(H_i)$ for $i = 1, \ldots, m$. We may construct a graph G in $\mathcal{G}_n$ with at least k_i components isomorphic to H_i as follows: choose the vertex sets for the different components (listing the vertex sets in order); then insert appropriate copies of H_i on the vertices of each component; and finally choose any graph in $\mathcal{G}$ of order $n-K$ on the remaining $n-K$ vertices. Since the number of ways of putting a copy of H_i on a given set of $v(H_i)$ vertices is $v(H_i)!/\mathrm{aut}(H_i)$, the total number of constructions is

$$\begin{aligned} &\prod_{i=1}^m \prod_{j=1}^{k_i} \left(\binom{n - \sum_{s=1}^{i-1} k_s v_s - (j-1)v_i}{v_i} \cdot \frac{v_i!}{aut(H_i)} \right) \cdot |\mathcal{G}_{n-K}| \\ = \quad &(n)_K \prod_{i=1}^m (\mathrm{aut}(H_i))^{-k_i} \cdot |\mathcal{G}_{n-K}|. \end{aligned}$$

Now observe that each graph $G \in \mathcal{G}_n$ is constructed exactly $\prod_{i=1}^{m}(\kappa(G, H_i))_{k_i}$ times; and so the last expression above equals

$$\sum_{G \in \mathcal{G}_n} \prod_{i=1}^{m} (\kappa(G, H_i))_{k_i}.$$

But by definition $\mathbb{E}\left[\prod_{i=1}^{m}(\kappa(R_n, H_i))_{k_i}\right]$ is this last quantity divided by $|\mathcal{G}_n|$. Hence

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^{m}(\kappa(R_n, H_i))_{k_i}\right] &= (n)_K \prod_{i=1}^{h} (\mathrm{aut}(H_i))^{-k_i} \cdot |\mathcal{G}_{n-K}|/|\mathcal{G}_n| \\
&= \prod_{i=1}^{h} \mu(H_i)^{k_i} \cdot \prod_{j=1}^{K} \left(\rho^{-1}(n-j+1)\frac{|\mathcal{G}_{n-j}|}{|\mathcal{G}_{n-j+1}|}\right) \\
&= \prod_{i=1}^{h} \mu(H_i)^{k_i} \cdot \prod_{j=1}^{K} (r_{n-j+1}/\rho)
\end{aligned}$$

as required. □

When we add the assumption that $\mathcal{G}$ is smooth, we find convergence of distributions.

Lemma 18. *Let the graph class $\mathcal{G}$ be smooth, and let $\rho = \rho(\mathcal{G})$. Let $H_1, \ldots, H_m$ be a fixed family of pairwise non-isomorphic connected graphs, each freely addable to $\mathcal{G}$. Then as $n \to \infty$ the joint distribution of the random variables $\kappa(R_n, H_1)$, …, $\kappa(R_n, H_m)$ converges in total variation distance to the product distribution $\mathrm{Po}(\mu(H_1)) \otimes \cdots \otimes \mathrm{Po}(\mu(H_m))$.*

Proof. Since $r_n \to \rho$ as $n \to \infty$, by the last lemma

$$\mathbb{E}\left[\prod_{i=1}^{m} (\kappa(R_n, H_i))_{k_i}\right] \to \prod_{i=1}^{m} \mu(H_i)^{k_i}$$

as $n \to \infty$, for all non-negative integers $k_1, \ldots, k_m$. A standard result on the Poisson distribution now shows that the joint distribution of the random variables $\kappa(R_n, H_1), \ldots, \kappa(R_n, H_m)$ tends to that of independent random variables $\mathrm{Po}(\mu(H_1)), \ldots, \mathrm{Po}(\mu(H_m))$, see for example Lemma 5.4 of McDiarmid, Steger and Welsh [25] or see Janson, Łuczak and Ruciński [16]. Thus for each m-tuple of non-negative integers $(t_1, \ldots, t_m)$

$$\mathbb{P}[\kappa(R_n, H_i) = t_i \ \forall i] \to \prod_i \mathbb{P}[\kappa(R_n, H_i) = t_i] \quad \text{as } n \to \infty;$$

and so we have pointwise convergence of probabilities, which is equivalent to convergence in total variation. $\square$

Putting the last lemma together with

$$\mathbb{E}[v(\text{Frag}(R_n))] < 2$$

(which shows that we need not worry about large components H) will yield the Fragments Theorem.

4.3 Fuller story for addable minor-closed classes

Recall that the core (or 2-core) core(G) of a graph G is the unique maximal subgraph with minimum degree at least 2, and it is empty if G is a forest. Thus core(G) is the graph obtained by repeatedly trimming off (deleting) leaves until none remain, and then deleting any isolated vertices. The core is at the heart of all our proofs, and should appear naturally in the theorems. Given a class $\mathcal{G}$ of graphs, we let $\mathcal{G}^{\delta \geq 2}$ denote the class of graphs in $\mathcal{G}$ with minimum degree at least two. The first two parts of the next result collect together results mentioned above, in Theorems 14 and 16 (a), for the addable case. Part (d) corresponds to the concentration result in the sketch proof of smoothness for the addable case.

Theorem 19. *Let $\mathcal{G}$ be a proper addable minor-closed class of graphs, and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{G}$. Then the following statements hold.*

(a) Both $\mathcal{G}$ and $\mathcal{C}$ are smooth, with the same radius of convergence ρ_0.

(b) $G(\rho_0) < \infty$; and for $R_n \in_u \mathcal{G}$, the fragment of R_n converges in distribution to $BP(\mathcal{G}, \rho_0)$. In particular,

$$\mathbb{P}(R_n \textit{ is connected }) \to e^{-C(\rho_0)} \quad \textit{as } n \to \infty.$$

Now suppose that $\mathcal{G}$ is not just the class of forests. Then $\rho_0 < 1/e$, and the following statements hold.

(c) Both $\mathcal{G}^{\delta \geq 2}$ and $\mathcal{C}^{\delta \geq 2}$ are smooth, with radius of convergence ρ_2, where ρ_2 is the unique root x in $(0, 1)$ to $xe^{-x} = \rho_0$ (and $\rho_2 = T^\bullet(\rho_0)$).

(d) Given $\epsilon > 0$, for $R_n \in_u \mathcal{G}$ or $R_n \in_u \mathcal{C}$,

$$\mathbb{P}(|v(\mathrm{core}(R_n)) - (1-\rho_2)n| > \epsilon n) = e^{-\Omega(n)}. \tag{9}$$

(e) $G^{\delta\geq 2}(\rho_2) < \infty$; and both the core of the unlabelled fragment of $R_n \in_u \mathcal{G}$, and the unlabelled fragment of $R_n \in_u \mathcal{G}^{\delta\geq 2}$, converge in distribution to $BP(\mathcal{G}^{\delta\geq 2}, \rho_2)$.

We may see easily that, when $\mathcal{G}$ is not just the class of forests, with high probability the number of vertices in the core of $R_n \in_u \mathcal{G}$ tends to infinity. Also, given that the core has vertex set W, the core is uniformly distributed on the graphs in $\mathcal{G}^{\delta\geq 2}$ on W. Hence the result that the core of the fragment of $R_n \in_u \mathcal{G}$ has limiting distribution $BP(\mathcal{G}^{\delta\geq 2}, \rho_2)$ is implied easily by the corresponding result for the fragment of $R_n \in_u \mathcal{G}^{\delta\geq 2}$.

It is possible to extend the above picture to encompass also graphs on surfaces – but not here!

References

[1] L. Addario-Berry, C. McDiarmid and B. Reed. Connectivity for bridge-addable monotone graph classes. *Combinatorics, Probability and Computing* **21** (2012) 803 – 815.

[2] P. Balister, B. Bollobás and S. Gerke. Connectivity of addable graph classes. *J. Combin. Theory Ser. B* 98 (2008), 577–584.

[3] P. Balister, B. Bollobás and S. Gerke, Connectivity of random addable graphs, *Proc. ICDM 2008* No 13 (2010) 127 – 134.

[4] E. A. Bender, E. R. Canfield, L. B. Richmond. Coefficients of functional compositions often grow smoothly. *Electron. J. Combin.* **15** #R21.

[5] E. A. Bender, Z. Gao. Asymptotic enumeration of labelled graphs by genus. *Electron. J. Combin.* 18 (2011), no. 1, Paper 13, 28 pp.

[6] O. Bernardi, M. Noy, D. Welsh. Growth constants of minor-closed classes of graphs. *J. Combin. Theory Ser. B* 100 (2010), 468–484.

[7] N. Bonichon, C. Gavoille, N. Hanusse, D. Poulalhon, G. Schaeffer. Planar Graphs, via Well-Orderly Maps and Trees. *Graphs Combin.* **22** (2006), 185–202.

[8] G. Chapuy, E. Fusy, O. Giménez, B. Mohar, M. Noy. Asymptotic enumeration and limit laws for graphs of fixed genus. *J. Combin. Theory Ser. A* 118 (2011), 748–777.

[9] G. Chapuy, G. Perarnau. Connectivity in bridge-addable graph classes: the McDiarmid-Steger-Welsh conjecture. arXiv: 1504.06344 (2015).

[10] Z. Dvorák and S. Norine. Small graph classes and bounded expansion. *J. Combinatorial Theory B* **100** (2010) 171 – 175.

[11] P. Flajolet, R. Sedgewick. *Analytic Combinatorics.* Cambridge University Press, Cambridge, 2009.

[12] O. Giménez, M. Noy. The number of planar graphs and properties of random planar graphs. *DMTCS Proc.* AD (2005), 147–156.

[13] O. Giménez, M. Noy. Asymptotic enumeration and limit laws of planar graphs. *J. Amer. Math. Soc.* 22 (2009), 309–329.

[14] O. Giménez, M. Noy. Counting planar graphs and related families of graphs. In *Surveys in Combinatorics 2009*, 169–210, Cambridge Univ. Press, Cambridge, 2009.

[15] O. Giménez, M. Noy, J. Rué. Graph classes with given 3-connected components: asymptotic enumeration and random graphs. *Random Structures Algorithms* **42** (2013), 438 – 479.

[16] S. Jansen, T. Łuczak and A. Ruciński. *Random Graphs*, Wiley, 2001.

[17] M. Kang and K. Panagiotou, On the connectivity of random graphs from addable classes, *J. Combinatorial Theory B* **103** (2013), 306 – 312.

[18] L. Lovász. Graph Minor Theory. *Bull. Amer. Math. Soc.*, 43, (2006), 75–86.

[19] C. McDiarmid. Random graphs on surfaces. *J. Combin. Theory Ser. B* 98 (2008), 778–797.

[20] C. McDiarmid. Random graphs from a minor-closed class. *Combin. Probab. Comput.* 18 (2009), 583–599.

[21] C. McDiarmid. Connectivity for random graphs from a weighted bridge-addable class, *Electronic J Combinatorics* **19**(4) (2012) P53.

[22] C. McDiarmid. Random graphs from a weighted minor-closed class, *Electronic J Combinatorics* **20** (2) (2013) P52, 39 pages.

[23] C. McDiarmid. Connectivity for bridge-alterable graph classes, *European J. Comb.* **56** (2016) 33 – 39.

[24] C. McDiarmid, B. Reed. On the maximum degree of a random planar graph. *Combin. Probab. Comput.* **17** (2008) 591– 601.

[25] C. McDiarmid, A. Steger, D. J. A. Welsh. Random planar graphs. *J. Combin. Theory Ser. B* **93** (2005), 187–205.

[26] C. McDiarmid, A. Steger, D. J. A. Welsh. Random graphs from planar and other addable classes. Topics in discrete mathematics, 231–246, Algorithms Combin., 26, Springer, Berlin, 2006.

[27] C. McDiarmid, K. Weller. Bridge-addability, edge-expansion and connectivity. *Combin. Probab. Comput.* to appear.

[28] S. Norine, P. Seymour, R. Thomas, P. Wollan, Proper minor-closed families are small, *J. Combin. Theory Ser. B* **96** (2006), 754–757.

[29] R. Otter. The number of trees. *Annals Math.* 49 (1948), 583–599.

[30] A. Rényi. Some remarks on the theory of trees. *Publications of the Mathematical Institute of the Hungarian Academy of Sciences* **4** (1959) 73 – 85.

[31] N. Roberton, D.P. Sanders, P.D. Seymour, R. Thomas. A new proof of the four-colour theorem. *Electronic Research Announcements of the AMS* **2** (1), August 1996.

[32] J. H. van Lint, R. M. Wilson. *A Course in Combinatorics*, Second edition. Cambridge University Press, Cambridge, 2001.