Giant Components in $G_{\mathcal{D}}$

Recall:

$0<d_1 \leq d_2 \leq \ \ldots \ \leq d_n$

$j_{\mathcal{D}}$ =min{n,min{j | $\sum_{i=1}^{j_D} d_i(d_i$-2) >0}}

$R_{\mathcal{D}} = \sum_{i=j_D}^{n} d_i$

We assume ∄ i s.t. d_i=2.

So M= $\sum_{i=1}^{n} d_i$.

Theorem 1: For all sufficiently small $\omega > 0$, if $R_{\mathcal{D}} < \omega \mathrm{M}$ then:

a.s every component of $G_{\mathcal{D}}$ has less than $\omega^{\frac{1}{9}} M$ edges.

Theorem 2: $\forall\ \varepsilon$ >0 $\exists \delta$>0 s.t. if

$R_{\mathcal{D}}$ >ϵM then a.s $G_{\mathcal{D}}$ has a component with more than δn edges.

Handling The High Degree Vertices

Threshold for H(igh) degree is $d_i > \sqrt{M}/\log M$.

Subcritical Case($R_{\mathcal{D}} = o(M)$) : We start our exploration from a set of vertices including all the high degree vertices, eliminating the problems they cause.

Supercritical Case($R_{\mathcal{D}} > \varepsilon M$ for some $\varepsilon > 0$) : We need:

Lemma: If the union of the components containing the vertices in H contains $\varepsilon M/100$ vertices then it is one component and we are done.

Otherwise deleting the vertices in the component we are still in the supercritical case, and no longer have any high degree vertices.
Proof: Switching.

A Sample Lemma

Lemma: If u and v are in H then the probability that u is in a component of size $M^{2/3}$ not containing v is $o(M^{1/4})$

Proof: Let A be the graphs with degree distribution $\mathcal{D}$ where u and v are in the same component. Let B be the graphs with degree distribution $\mathcal{D}$ where u is in a component of size $M^{2/3}$ not containing v.

We consider swaps from B to A where we swap an edge vw wih an edge xy such that . is at least as close to u as y. There are at least $(\frac{\sqrt{M}}{\log M})M^{2/3}$ such swaps. xv and wv are the only possibilities for the edges leaving v in the resultant graphs which are on a shortest uv path.. So each graph in A arises via at most 2M such switches.

The Random Object We Explore/Construct

$G_{\mathcal{D}}$ and a uniformly random permutation of each adjacency list.

The Exploration Process

Start with a set S_0 of vertices. In iteration t, we add a vertex w_t to S_t-1 to obtain S_t. We expose the edges from w_t to S_{t-1} and their position on the adjacency lists of their endpoints.

In iteration t,

(i) if there are no edges from S_{t-1} to V-S_{t-1} then P(w_t=w)=$\frac{d(w)}{\sum_{v \notin S_{t-1}} d(v)}$

(ii) Otherwise, v_t is the lowest indexed vertex of S_{t-1} with edges to V-S_{t-1} and we expose the vertex w_t of V-S_{t-1} appearing first on its adjacency list.

X_t is |E(S_t,V-S_t)|. X_t' =($\sum_{v\ in\ S_t} d(v)$)-2t.

The Subcritical Case

Lemma 3: For every sufficiently small $\omega > 0$, and every degree sequence $\mathcal{D}$ s.t. no d_i=2 & $R_{\mathcal{D}} \leq \omega M$, the probability a vertex v lies in a component of G($\mathcal{D}$) with more than $\omega^{\frac{1}{9}} M$ edges is o($\frac{1}{M}$).

Proof: Let S be the smallest set of vertices $\{v_i, v_{i+1}, \ldots, v_n\}$ such that $\sum_{j=i}^{n} d_j > 5\omega^{\frac{1}{4}} M$. Note that S contains v_j for j$\geq j_D$. Set S_0=S∪ $\{v\}$.

Lemma 4:

P($\exists t \leq \omega^{1/9}$M s.t X_t=0 and $X_t' \leq \omega^{\frac{1}{5}} M$) =1-o($\frac{1}{M}$).

=> Lemma 3 => Theorem 1.

Lemma 4: For $S_0 = S \cup \{v\}$,
Prob $(\exists t \leq \omega^{1/9} M$ s.t $X_t = 0$ & $X'_t \leq \omega^{\frac{1}{5}} M)$=1-o($\frac{1}{M}$).

Claim 1: $\exists\, u\ in\ S\ s.t\ d(u) \leq \omega^{-1/4}$, hence ∀ v in V-$S_0$ d(v)$\leq \omega^{-\frac{1}{4}}$.

Further, $X'_0 \leq 7\omega^{1/4}$M and$\sum_{v\ not\ in\ S_0} d(v)(d(v)-2) \leq$ -4$\omega^{1/4}$M

We let Y_t =d(w_t)-2-E[d(w_t)-2]

Claim 2: P(∃ t such that $\sum_{t' \leq t} Y_{t'} \geq \mathrm{M}^{2/3}$) =o($\frac{1}{M}$)

We let τ= min {t | X_t=0 or $\sum_{t' \leq t} Y_{t'} \geq \mathrm{M}^{\frac{2}{3}}$ or t $= \left\lfloor \frac{\omega^{1/9} M}{2} \right\rfloor$

Claim 3: for any t ≤ τ, E(d(w_t)-2)≤ $\frac{-t}{M}$+19$\omega^{1/5}$

τ= min {t | X_t=0 or $\sum_{t' \le t} Y_{t'} \ \geq M^{\frac{2}{3}}$ or $t = \left\lfloor \frac{\omega^{1/9} M}{2} \right\rfloor$

Claim 3: for any t $\leq \tau$, E(d(w_t)-2)$\leq \frac{-t}{M}$+19$\omega^{1/5}$

Proof:

Claim 6: If t $\leq \omega^{1/9}$M***,*** $X'_{\boldsymbol{t}} \leq \ \omega^{1/5}$M and $X_{t'}$>0 for all t'<t, then:

(a) If w in V-S_{t-1} and d(w)=1. then P[w_t=w] $\geq \frac{1-9\omega^{1/5}}{M}$

(b) If w in V-S_{t-1}, then P[w_t=w] $\leq \frac{1+9\omega^{1/5}}{M}$

Claim 7: For any sequence $a_1,\ldots,a_j$ of positive integers none of which are two, for any nonnegative l s.t $\sum_{i=1}^{j} a_i \geq 2j - l$ we have: $\sum_{i=1}^{j} a_i \ (a_i$ -2)$\geq$ j $-$ 2l.

Future Directions

Determining size of components of $G_{\mathcal{D}}$

Determining the conditions ensuring P($G_{\mathcal{D}}$ is (not) connected)=1-o(1)

Determining the mixing time of (the giant component of) $G_{\mathcal{D}}$.