The Structure of Typical H-free and $\mathcal{H}$-*free* Graphs I

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Two Observations And A Definition

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wpn(H)- max t such that for some s+c=t, H cannot be partitioned into s stable sets and c cliques

Some Examples of The Structure of Typical H-free and $\mathcal{H}$-*free* graphs I

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- Almost every triangle free graph is bipartite.

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- For k>3, a.e. C_{2k+1}-*free* graph is partitionable into k cliques. Balogh+Butterfield, 2011

Some Examples of The Structure of Typical H-free and *H-free* graphs II

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- The vertex set of almost every C_6-free graph can be partitioned into a stable set and a graph which induces the complement of a graph of girth 5. R. & Scott 201?

A Conjecture

H-freeness witnessing partition *OF V(G):* into wpn(H) parts X_1 to $X_{wpn(H)}$ all of size (1+o(1)) $\frac{n}{wpn(H)}$ s.t. for any partition of V(H) into wpn(H) parts S_1 to $S_{wpn(h)}$ there is some i s. t. $H[S_i]$ is not an induced subgraph of $G[X_i]$.

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Similar statement for H-free shown false by Balogh, Bollobas, and Simonovitz 2013.

The Structure of $\mathcal{H}$-*free* Graphs For arbitrary H.

Theorem: ∀H, large k & $\delta > 0$ ∃α>0 s.t. A.e G in $IForb_n^H$ contains a subset Z with. |Z|$\leq n^{1-\alpha}$ such that G-Z has an H-freeness witnessing partition.

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Definition: U(k) is bipartite with vertex set $\{a_1,\ldots,a_k\} \cup B=\{b_1,\ldots,b_{2^k}\}$**.**
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Theorem: ∀H, large k & $\delta > 0$ ∃α>0 s.t. A.e G in $IForb_b^H$ has a partition into $X_1,\ldots,X_{wpn(H)}$,Z s.t. |Z|≤ $n^{1-\alpha}$, each S_i is U[k]-free & has (1+o(1))$\frac{n}{wpn(H)}$ vertices

Alon, Balogh,Bollobas,Morris 2009

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If $X_1,\ldots,X_{wpn(H)}$ is an H-freeness witnessing partition of G, then for some i, there are four induced subgraphs F_1, F_2, F_3, F_4 of H which are not induced subgraphs of $G[X_i]$ such that:

$V(F_1)$ can be partitioned into a clique and an edge,

$V(F_2)$ can be partitioned into a stable set & an edge

$V(F_3)$ (resp. $V(F_4)$) can be partitioned into a clique (stable set) and a stable set of size two

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=> There is a k s.t. G is U_k-free.

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Alon, Balogh,Bollobas,Morris 2009

The Structure of H-free Graphs

Let M(H) be the set of graphs F such that H is a subgraph of the join of (i) the disjoint union of F and a stable set with (ii) a χ(H)-2 partite graph.

Theorem: $\forall H$, $\exists$ B s.t. almost every H-free graph G contains a set Z of at most B vertices s.t. G-Z can be partitioned into χ(H)-1 parts each of which contains no subgraph in M(H).

Balogh, Bollobas, & Simonovits 2009.

An Example

Let H be obtained from a $K_{2s,2s}$ by adding a matching of size s on one side.

M(H) consists of those graphs with no matching of size s.

A graph which is M(H)-free has a cover of size at most 2s-2 and only s-1 vertices of degree $\geq$2s.

There are at most $\binom{k}{2s-2}$ $2^{2s-2}\binom{k}{2s}^{s}2^{(s-1)(k)}$ $=2^{(s-1)k+o(k)}$ such graphs on k vertices.

Consider graphs G with 2k vertices consisting of a stable set S of λ vertices, s.t. G-S has a bipartition (U,W) and at most s-1 edges from each vertex of V-S to S. Such graphs are H-free. An appropriate choice of s and l, show we cannot make B zero in the previous theorem. Balogh, Bollobas, Simonovits 2013

An Open Question

What is the slowest growing function f_H such that almost every G in $IForb_H^n$ contains a set Z of at most f(H) vertices s.t. G-Z has an H-freeness witnessing partition.

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What is the slowest growing function f_H such that almost every G in $Forb_H^n$ contains a set Z of at most f(H) vertices s.t. G-Z can be partitioned into $X_1,\ldots,X_{\chi(G)-1}$ s.t.

(a) For any partition of V(H) into $\{S_1,\ldots,S_{\chi(G)-1}\}$ there is some i s.t. $H[S_i]$ is not a subgraph of $G[X_i]$, and

(b) for all i, $|X_i|=(1+o(1))\frac{n}{wpn(H)}$

The Number of H-free Graphs.

Definiton $Forb_H^n$: H-free graphs on V_n ={1,…,n}

Observation: If $\chi(\mathrm{H}) = c$ then:

$$|\ Forb_H^n| > 2^{\frac{c-2}{c-1}\binom{n}{2}}$$

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Proof: For all $\varepsilon > 0$ there is an n_0 s. t. for $n>n_0$, every G in $Forb_H^n$ can be made c-1 chromatic by deleting at most εn^2 edges.

The Number of $\mathcal{H}$-*free* Graphs.

Definiton $IForb_H^n$: $\mathcal{H}$-*free* graphs on V_n ={1,…,n}

Observation: If $\mathrm{wpn(H)} = t$ then:

$$|\ IForb_H^n|>2^{\frac{t-1}{t}\binom{n}{2}}$$

Theorems: If $\mathrm{wpn(H)} = t$ then:

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$\exists \alpha > 0\ s.t.\ |IForb_H^n| < 2^{\frac{t-1}{t}\binom{n}{2}+n^{2-\alpha}}$ ***ABBM 2009***

VC-Dimension and Sauer’s Lemma

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A multiset $\mathcal{F}$ of subsets of a ground set S *shatters* some subset D of S if:

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Sauers Lemma: If $\mathcal{F}$ has VC-dimension d and S has size m then the number of distinct elements in $\mathcal{F}$ is

$$\sum_{i=0}^{d} \binom{m}{d} < m^{d+1}$$

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Corollary: If G is U(K)-free then $\forall S \subseteq V$ of size m, the multiset $\{N(v) \cap S \mid v \text{ in } V-S\}$ has fewer than m^{d+1} distinct elements.

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Corollary: If G is U(K)-free then $\forall S \subseteq$ V of size m, the multiset {N(v) ∩ S|v in V-S} has fewer than m^{d+1} distinct elements.

=> $\forall \mathrm{k} \exists \alpha > 0\ s.t$ the number of U(k) free graphs on {1,..,n}=$O(2^{n^{2-\alpha}})$.

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Theorem: ∀H, large k & $\delta > 0$ ∃α>0 & b s.t. a.e G∈ $IForb_n^H$ contains a subset Z with |Z|≤ $n^{1-\alpha}$, a set B with |B|≤ b such that G-Z has an H-freenesswitnessing partition $\{S_1,\ldots,S_{wpn(H)}\}$ and Z has a partition $\{Z_1,\ldots,Z_{wpn(H)}\}$ for which: ∀v∈ $S_i \cup Z_i$, ∃w∈B s.t.

$|(N(v)\Delta N(w))\cap (S_i \cup Z_i)|<\delta n$

R. & Scott 2013

Theorem: ∀H, large k & $\delta > 0$ ∃$\alpha > 0$ & b s.t. a.e G in $IForb_b^H$ has a partition into $X_1,\ldots,X_{wpn(H)},Z_1,\ldots,Z_{wpn(H)}$ and a set B of ≤b vertices s.t.

$|\bigcup_{i=1}^{wpn(H)} Z_i| \leq n^{1-\alpha}$

each X_i is U[k]-free & has $(1+o(1))\frac{n}{wpn(H)}$ vertices and for which:

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ABBM 2009

Thank You For Your Attention!