Random graphs from minor-closed classes

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Question

Let $\mathcal{G}$ be a class of (simple) graphs closed under isomorphism, eg the class $\mathcal{P}$ of planar graphs.

$\mathcal{G}_n$ is the set of graphs in $\mathcal{G}$ on vertices $1, \ldots, n$.

$R_n \in_u \mathcal{G}$ means that R_n is picked uniformly at random from $\mathcal{G}_n$.

What are typical properties of R_n?

usually a giant component? probability of being connected?

Generating functions

For a class $\mathcal{G}$ of graphs, the exponential generating function (egf) is

$$G(x) = \sum_n |\mathcal{G}_n| x^n / n!.$$

ρ_G or $\rho_{\mathcal{G}}$ is the radius of convergence (where $0 \leq \rho_{\mathcal{G}} \leq \infty$).

For suitable classes, we can relate the egfs (or two variable versions) of all graphs, connected graphs, 2-connected graphs and 3-connected graphs.

If we know enough about the 3-connected graphs (as we do for planar graphs, thanks to Tutte and others) then we may be able to extend to all graphs.

We aim to proceed in greater generality.

Minors

H is a minor of G if H can be obtained from a subgraph of G by edge-contractions. $\mathcal{G}$ is minor-closed if

$$G \in \mathcal{G},\ H \text{ a minor of } G \quad \Rightarrow\ H \in \mathcal{G}$$

Examples:

forests, series-parallel graphs, and more generally graphs of treewidth $\leq k$;

outerplanar graphs, planar graphs, and more generally graphs embeddable on a given surface;

graphs with at most k (vertex) disjoint cycles.

Minors

$Ex(\mathcal{H})$ is the class of graphs with no minor a graph in $\mathcal{H}$. For example: series-parallel graphs $= \mathrm{Ex}(K_4)$, planar graphs $= \mathrm{Ex}(\{K_5, K_{3,3}\})$, graphs with no two disjoint cycles $= \mathrm{Ex}(2C_3)$.

Easy to see: $\mathcal{G}$ is minor-closed iff $\mathcal{G} = \mathrm{Ex}(\mathcal{H})$ for some class $\mathcal{H}$.

By Robertson and Seymour's graph minors theorem (once Wagner's conjecture), if $\mathcal{G}$ is minor-closed then it is $\mathrm{Ex}(\mathcal{H})$ for some **finite** class $\mathcal{H}$.

The unique minimal such $\mathcal{H}$ consists of the excluded minors for $\mathcal{G}$.

Minors

Mostly we shall assume that $\mathcal{G}$ is minor-closed and proper (that is, not empty and not all graphs).

For such $\mathcal{G}$, there is a $c = c(\mathcal{G})$ such that the average degree of each graph in $\mathcal{G}$ is at most c (Mader). Thus our graphs are sparse. For $\mathrm{Ex}(K_t)$ the maximum average degree is of order $t\sqrt{\log t}$ (Kostochka, Thomason).

By definition, the radius of convergence $\rho_{\mathcal{G}}$ is > 0 if and only if $\exists c$ such that $|\mathcal{G}_n| \leq c^n n!$.

Norine, Seymour, Thomas and Wollan (2006), and Dvorák and Norine (2010), showed that each proper minor-closed graph class $\mathcal{G}$ has $\rho_{\mathcal{G}} > 0$.

Decomposable

If a graph is in $\mathcal{G}$ if and only if each component is, then we call $\mathcal{G}$ decomposable.

For example the class of planar graphs is decomposable but the class of graphs embeddable on the torus is not.

A minor-closed class is decomposable iff each excluded minor is connected (Exercise 1(a)).

Bridge-addable and addable

$\mathcal{G}$ is bridge-addable if whenever $G \in \mathcal{G}$ and u and v are in different components of G then $G + uv \in \mathcal{G}$.

$\mathcal{G}$ is addable if it is decomposable and bridge-addable.

A minor-closed class $\mathcal{G}$ is addable iff each excluded minor is 2-connected (Exercise 1(b)).

Given a surface S, let $\mathcal{G}^S$ denote the class of graphs embeddable in S. $\mathcal{G}^S$ is bridge-addable but not decomposable except in the planar case.

Bridge-addability and being connected

McD, Steger and Welsh (2005) used double counting to show:

Lemma

If $\mathcal{G}$ is bridge-addable and $R_n \in_u \mathcal{G}$ then

$$\mathbb{P}(R_n \textit{ is connected}) > 1/e \approx 0.3679.$$

Proof: Exercise 2.

For trees $\mathcal{T}$ and forests $\mathcal{F}$, $|\mathcal{T}_n| = n^{n-2}$ and $|\mathcal{F}_n| \sim e^{\frac{1}{2}} n^{n-2}$. Thus for $F_n \in_u \mathcal{F}$,

$$\mathbb{P}(F_n \text{ is connected}) \sim e^{-\frac{1}{2}} \approx 0.6065.$$

Bridge-addability and being connected

McD, Steger and Welsh (2006) conjectured that, if $\mathcal{G}$ is bridge-addable, then

$$\mathbb{P}(R_n \text{ is connected}) \geq e^{-\frac{1}{2}+o(1)}. \tag{1}$$

Balister, Bollobás and Gerke (2008, 2010) improved the lower bound. Under the extra condition that $\mathcal{G}$ is also closed under deleting bridges, Addario-Berry, McD and Reed (2012), and Kang and Panagiotou (2013) proved the conjecture (1).

The full conjecture was recently proved by Chapuy and Perarnau (JCTB to appear).

Big component

The big component $\mathrm{Big}(G)$ of a graph G is the (lex first) component with most vertices.

The fragment 'left over', $\mathrm{Frag}(G)$, is the subgraph induced on the vertices not in the big component.

Theorem

If $\mathcal{G}$ is bridge-addable then $\mathbb{E}[v(\mathrm{Frag}(R_n))] < 2$.

Proof: Exercise 3.

Growth constant

To go further, we need to know that the numbers $|\mathcal{G}_n|$ do not jump around too much. $\mathcal{G}$ has a growth constant if $0 < \rho_{\mathcal{G}} < \infty$, and

$$(|\mathcal{G}_n|/n!)^{1/n} \to \rho_{\mathcal{G}}^{-1} \quad \text{as } n \to \infty,$$

that is, if

$$|\mathcal{G}_n| = (1 + o(1))^n \rho_{\mathcal{G}}^{-n} \; n!.$$

If $\mathcal{G}$ contains arbitrarily long paths then clearly $\rho_{\mathcal{G}} \leq 1$.

Bernardi, Noy and Welsh 2010: if $\mathcal{G}$ is monotone and does not contain all paths then $\rho_{\mathcal{G}} = \infty$.

When is there a growth constant?

minor-closed and addable – and $\mathcal{G}^S$

McD, Steger and Welsh (2005), McD (2009):

Theorem

Each addable proper minor-closed class $\mathcal{G}$ has a growth constant.

This follows from supermultiplicativity, together with $\rho_{\mathcal{G}} > 0$.

In particular the class $\mathcal{P}$ of planar graphs has a growth constant. Indeed, each surface class $\mathcal{G}^S$ has a growth constant, the same as for $\mathcal{P}$ (McD 2008).

Bernardi, Noy and Welsh (2010) asked:
does every proper minor-closed class of graphs have a growth constant?

Having a growth constant yields ..

Pendant appearances theorem - introduction

Let H be a connected graph.

G has a pendant appearance of H if G has a bridge e with H at one end.

H is attachable to $\mathcal{G}$ if whenever we have a graph G in $\mathcal{A}$ and a disjoint copy of H, and we add an edge between a vertex in G and the root of H, then the resulting graph must be in $\mathcal{G}$.

For an addable minor-closed class $\mathcal{G}$, the attachable graphs are the connected graphs in $\mathcal{G}$.

For $\mathcal{G}^S$, the attachable graphs are the connected planar graphs.

Pendant Appearances Theorem

McD, Steger and Welsh (2005), (amended 2014+):

Theorem

Let the connected graph H be attachable to $\mathcal{G}$, where $0 < \rho_{\mathcal{G}} < \infty$. Then there exists $\alpha > 0$ such that, if $\mathcal{H}$ is the set of $G \in \mathcal{G}$ with less than $\alpha\, v(G)$ pendant appearances of H, then $\rho_{\mathcal{H}} > \rho_{\mathcal{G}}$.

Proof idea. From $G \in \mathcal{H}_n$, by adding δn pendant appearances of H we may construct many graphs G' in $\mathcal{G}_{n'}$, where $n' = n + v(H)\delta n$; and since each G had few pendant appearances of H, each graph G' is not constructed very often, so if $\mathcal{H}_n$ is big we get too many graphs in $\mathcal{G}_{n'}$.

Pendant Appearances Theorem

Theorem

Let the connected graph H be attachable to $\mathcal{G}$, where $0 < \rho_{\mathcal{G}} < \infty$. Then there exists $\alpha > 0$ such that, letting $\mathcal{H}$ be the set of $G \in \mathcal{G}$ with less than $\alpha\, v(G)$ pendant appearances of H, we have $\rho_{\mathcal{H}} > \rho_{\mathcal{G}}$.

Corollary

If $\mathcal{G}$ has a growth constant, then for $R_n \in_u \mathcal{G}$

$$\Pr\left(R_n \text{ has } < \alpha n \text{ pendant appearances of } H\right) = e^{-\Omega(n)}.$$

For

$$\frac{|\mathcal{H}_n|}{|\mathcal{G}_n|} \leq \frac{(\rho_{\mathcal{H}}^{-1} + o(1))^n}{(\rho_{\mathcal{G}}^{-1} + o(1))^n} = e^{-\Omega(n)}.$$

Pendant Appearances Theorem - an application

Suppose that $\mathcal{G}$ has a growth constant.

If the 3-vertex star (cherry) is attachable to $\mathcal{G}$, then whp there are at least αn pendant cherries, and thus at least $2^{\alpha n}$ automorphisms.

This can be used to show that the unlabelled planar graphs have strictly bigger (unlabelled) growth constant than the labelled planar graphs $\mathcal{P}$.

Unlabelled graphs

Think of an unlabelled graph as an equivalence class of labelled graphs. Write $\tilde{\mathcal{G}}$ for the set of unlabelled graphs in $\mathcal{G}$,

We say that $\mathcal{G}$ (or $\tilde{\mathcal{G}}$) has *unlabelled growth constant* $\tilde{\gamma} = \tilde{\gamma}_{\mathcal{G}}$ if $|\tilde{\mathcal{G}}_n|^{1/n} \to \tilde{\gamma}$ as $n \to \infty$.

The earlier result showing that minor-closed classes are small was stated in terms of labelled graph classes, but in fact the 'smallness' result of Dvorák and Norine is for unlabelled graph classes.

For each proper minor-closed class $\mathcal{G}$ of graphs there is a constant c such that $|\tilde{\mathcal{G}}_n| \leq c^n$ for each n.

This result immediately implies the earlier upper bound, since $|\mathcal{G}_n| \leq n! \cdot |\tilde{\mathcal{G}}_n|$.

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Unlabelled graphs: growth constant

Theorem

Let $\mathcal{G}$ be a proper addable minor-closed class of graphs, and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{G}$. Then $\mathcal{G}$ and $\mathcal{C}$ have an unlabelled growth constant, and $\tilde{\gamma}_{\mathcal{G}} = \tilde{\gamma}_{\mathcal{C}} > \gamma_{\mathcal{G}}$ $(= \gamma_{\mathcal{C}})$.

Idea of proof. Let $\tilde{\mathcal{C}}^{\bullet}$ denote the set of (vertex-) rooted graphs in $\tilde{\mathcal{C}}$. Then $f(n) = |\tilde{\mathcal{C}}^{\bullet}_n|$ is supermultiplicative; that is, for positive integers a and b

$$|\tilde{\mathcal{C}}^{\bullet}_{a+b}| \geq |\tilde{\mathcal{C}}^{\bullet}_a| \cdot |\tilde{\mathcal{C}}^{\bullet}_b|. \tag{2}$$

Now use Fekete's Lemma and 'smallness' to show that $\tilde{\mathcal{C}}^{\bullet}$ has unlabelled growth constant. Hence $\tilde{\mathcal{C}}$ and $\tilde{\mathcal{G}}$ have the same unlabelled growth constant.

Unlabelled graphs: showing $\tilde{\gamma}_{\mathcal{G}} > \gamma_{\mathcal{G}}$

The appearances theorem gives $\alpha > 0$ such that: $\geq \frac{1}{2}$ of $G \in \mathcal{G}_n$ have $\geq 2^{\alpha n}$ automorphisms, and thus are in isomorphism classes $\tilde{G}$ of size $\leq 2^{-\alpha n} n!$.

Hence

$$|\tilde{\mathcal{G}}_n| \geq \frac{1}{2}|\mathcal{G}_n|/2^{-\alpha n} n!$$

and so

$$\gamma_{\mathcal{G}} \leq 2^{-\alpha} \tilde{\gamma}_{\mathcal{G}}.$$

Unlabelled graphs: showing $\tilde{\gamma}_{\mathcal{G}} > \gamma_{\mathcal{G}}$

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Smoothness

Given $\mathcal{G}$ let $r_n = \frac{n|\mathcal{G}_{n-1}|}{|\mathcal{G}_n|}$. Call $\mathcal{G}$ smooth if $r_n \to \rho_{\mathcal{G}}$ as $n \to \infty$.

All the classes for which we know an asymptotic counting formula are smooth, for example forests, series-parallel graphs, $\mathcal{P}$, $\mathcal{G}^S$, .. (apart from cubic planar graphs,..).

Showing smoothness is often a crucial step in proving results about $R_n \in_u \mathcal{G}$.

When is $\mathcal{G}$ smooth?

Bender, Canfield and Richmond (2008):

Theorem

$\mathcal{G}^S$ is smooth for any surface S.

The proof did not involve an asymptotic counting formula, and indeed none was then known.

We knew at that time that $\mathcal{G}^S$ has growth constant $\rho_{\mathcal{P}}^{-1} \approx 27.226878$. Counting $\mathcal{G}^S$ was greatly improved in 2011 by Chapuy, Fusy, Giménez, Mohar and Noy, and by Bender and Gao, to give an asymptotic formula for $|\mathcal{G}_n^S|$.

Idea of proof of smoothness

The key is to consider the core (2-core).

The core of G, $\mathrm{core}(G)$, is the unique maximal subgraph H such that the minimum degree $\delta(H) \geq 2$. We obtain the core by repeatedly trimming off leaves.

Let $\mathcal{G}^{\delta \geq 2}$ denote the class of graphs in $\mathcal{G}$ with minimum degree $\delta \geq 2$.

The idea is that if $\mathcal{G}^{\delta \geq 2}$ grows reasonably smoothly then rooting trees in the core leads to a smooth class $\mathcal{G}$.

Addable minor-closed classes

The proof method yields further results. In particular (McD 2009):

Theorem

Let $\mathcal{G}$ be a proper addable minor-closed class of graphs and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{G}$. Then $\mathcal{G}$ and $\mathcal{C}$ are smooth.

Sketch proof: $\mathcal{A}$ is smooth (1/3)

Let $\mathcal{A} \neq \mathcal{F}$ be an addable minor-closed class. Let us show that $\mathcal{A}$ is smooth.

Note that $\mathcal{F} \subseteq \mathcal{A}$, and C_3 is attachable to $\mathcal{A}$. Hence by the Pendant Appearances Theorem $\rho(\mathcal{A}) < \rho(\mathcal{F})$ $(= 1/e)$.

Let $\mathcal{A}^-$ be the class of graphs $G \in \mathcal{A}$ with no tree components. We may think of $\mathcal{A}$ as $\mathcal{A}^- \times \mathcal{F}$. But $\rho(\mathcal{A}^-) < \rho(\mathcal{F})$, so it suffices to show that $\mathcal{A}^-$ is smooth.

$\mathcal{A}^{\delta \geq 2}$ is addable and small, and so it has a growth constant.

Sketch proof: $\mathcal{A}$ is smooth (2/3)

Denote $\mathcal{A}^{\delta \geq 2}$ by $\mathcal{B}$, and let $\mathcal{T}^{\bullet}$ denote the class of rooted trees. Graphs G in $\mathcal{A}^{-}$ are obtained by starting with a graph in $\mathcal{B}$ and rooting a tree at each vertex. Then

$$A^{-}(x) = B(T^{\bullet}(x)).$$

Now

$$\begin{aligned} |\mathcal{A}_n^{-}|/n! = [x^n]A^{-}(x) &= [x^n] \sum_{k=0}^{n} |\mathcal{B}_k| \, (T^{\bullet}(x))^k / k! \\ &\sim [x^n] \sum_{k:|k-\alpha n|<\epsilon n} |\mathcal{B}_k|/k! \, (T^{\bullet}(x))^k, \end{aligned}$$

where $\alpha = 1 - \rho_2$, $0 < \alpha < 1$. (Recall that $\rho_2 = \rho(\mathcal{G}^{\delta \geq 2})$.)

Sketch proof: $\mathcal{A}$ is smooth (3/3)

We have

$$\frac{|\mathcal{A}_n^-|}{n!} \sim \sum_{k:|k-\alpha n|<\epsilon n} |\mathcal{B}_k|/k! \; [x^n] T^\bullet(x)^k$$

and

$$\frac{|\mathcal{A}_{n+1}^-|}{(n+1)!} \sim \sum_{k:|k-\alpha n|<\epsilon n} |\mathcal{B}_k|/k! \; [x^{n+1}] T^\bullet(x)^k.$$

Each term

$$[x^{n+1}] T^\bullet(x)^k / [x^n] T^\bullet(x)^k$$

is close to $\rho_{\mathcal{A}}^{-1}$. Hence

$$\frac{|\mathcal{A}_{n+1}^-|}{(n+1)|\mathcal{A}_n^-|} \sim \rho_{\mathcal{A}}^{-1}.$$

Addable minor-closed classes

We have now shown:

Theorem

Let $\mathcal{G}$ be a proper addable minor-closed class of graphs and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{G}$. Then $\mathcal{G}$ and $\mathcal{C}$ are smooth.

Conjecture: is every proper minor-closed class smooth?

What do we get from smoothness?

Boltzmann Poisson random graph

Let $\mathcal{G}$ be decomposable.

Fix $\rho > 0$ such that $G(\rho)$ is finite; and let

$$\mu(H) = \frac{\rho^{v(H)}}{\text{aut}(H)} \text{ for each } H \in \tilde{\mathcal{G}}.$$

Here $\tilde{\mathcal{G}}$ denotes the unlabelled graphs in $\mathcal{G}$. Easy to check:

$$G(\rho) = \sum_{H \in \tilde{\mathcal{G}}} \mu(H).$$

Boltzmann Poisson random graph

The Boltzmann Poisson random graph $R = BP(\mathcal{G}, \rho)$ takes values in $\tilde{\mathcal{G}}$, with

$$\mathbb{P}[R = H] = \frac{\mu(H)}{G(\rho)} \quad \text{for each } H \in \tilde{\mathcal{G}}.$$

Let $\mathcal{C}$ denote the class of connected graphs in $\mathcal{G}$. For each $H \in \tilde{\mathcal{C}}$ let $\kappa(G, H)$ denote the number of components of G isomorphic to H.

Proposition

The random variables $\kappa(R, H)$ for $H \in \tilde{\mathcal{C}}$ are independent, with $\kappa(R, H) \sim \mathrm{Po}(\mu(H))$. In particular

$$\mathbb{P}(R = \emptyset) = e^{-C(\rho)}.$$

Fragments theorem for an addable minor-closed class

Recall that the fragment $\mathrm{Frag}(G)$ of a graph G is the subgraph remaining when you discard the largest component.

Theorem

Let $\mathcal{G}$ be a proper addable minor-closed class.

Then $0 < \rho_{\mathcal{G}} < \infty$ and $G(\rho_{\mathcal{G}})$ is finite;

and for $R_n \in_u \mathcal{G}$, the random unlabelled fragment converges in distribution to $BP(\mathcal{G}, \rho_{\mathcal{G}})$.

We may deduce that

$$\mathbb{P}(R_n \text{ is connected }) \to e^{-C(\rho_{\mathcal{G}})}.$$

Fragments theorem - proof idea (1/3)

Recall $\mu(H) = \rho^{v(H)}/\mathrm{aut}(H)$ for $H \in \tilde{\mathcal{G}}$, and $r_n = n|\mathcal{G}_{n-1}|/|\mathcal{G}_n|$.

Lemma

Let $\mathcal{G}$ be a decomposable class of graphs and let $\rho > 0$. Let $H_1, \ldots, H_m$ be pairwise non-isomorphic connected graphs in $\mathcal{G}$. Let $k_1, \ldots, k_m$ be non-negative integers, and let $K = \sum_{i=1}^{m} k_i v(H_i)$. Then for $R_n \in_u \mathcal{G}_n$,

$$\mathbb{E}\left[\prod_{i=1}^{m} (\kappa(R_n, H_i))_{k_i}\right] = \prod_{i=1}^{m} \mu(H_i)^{k_i} \prod_{j=1}^{K} (r_{n-j+1}/\rho).$$

Fragments theorem - proof idea (2/3)

Lemma

Let $\mathcal{G}$ be decomposable and smooth, and let $\rho = \rho_{\mathcal{G}}$. Let $H_1, \ldots, H_m$ be pairwise non-isomorphic connected graphs in $\mathcal{G}$. Then the m-tuple $\kappa(R_n, H_1), \ldots, \kappa(R_n, H_m)$ converges in distribution to $\mathrm{Po}(\mu(H_1)) \otimes \cdots \otimes \mathrm{Po}(\mu(H_m))$.

Proof.

Since $r_n \to \rho$ as $n \to \infty$, by the last lemma

$$\mathbb{E}\left[\prod_{i=1}^{m} (\kappa(R_n, H_i))_{k_i}\right] \to \prod_{i=1}^{m} \mu(H_i)^{k_i},$$

for all non-negative integers $k_1, \ldots, k_m$. Now we may use a standard result in the method of moments. $\square$

Fragments theorem - proof idea (3/3)

Lemma

Let $\mathcal{G}$ be decomposable and smooth, and let $\rho = \rho_{\mathcal{G}}$. Let $H_1, \ldots, H_m$ be pairwise non-isomorphic connected graphs in $\mathcal{G}$. Then the m-tuple $\kappa(R_n, H_1), \ldots, \kappa(R_n, H_m)$ converges in distribution to $\mathrm{Po}(\mu(H_1)) \otimes \cdots \otimes \mathrm{Po}(\mu(H_m))$.

Putting this lemma together with

$$\mathbb{E}[v(\mathrm{Frag}(R_n))] < 2$$

yields the fragments theorem.

More on $\mathcal{G}^{\delta \geq 2}$

We saw earlier that, if $\mathcal{G}$ is a proper addable minor-closed class of graphs, and $\mathcal{C}$ is the class of connected graphs in $\mathcal{G}$, then $\mathcal{G}$ and $\mathcal{C}$ are smooth.

We can go further if $\mathcal{G}$ is not just the forests $\mathcal{F}$.

Theorem

Let $\mathcal{G} \neq \mathcal{F}$ be a proper addable minor-closed class of graphs, and let $\mathcal{C}$ be the class of connected graphs in $\mathcal{G}$. Then $\mathcal{G}^{\delta \geq 2}$ and $\mathcal{C}^{\delta \geq 2}$ are smooth.

Further, let ρ_2 be the radius of convergence of $\mathcal{G}^{\delta \geq 2}$. Then $G^{\delta \geq 2}(\rho_2) < \infty$; and both the core of the unlabelled fragment of $R_n \in_u \mathcal{G}$, and the unlabelled fragment of $R_n \in_u \mathcal{G}^{\delta \geq 2}$, converge in distribution to $BP(\mathcal{G}^{\delta \geq 2}, \rho_2)$.

Idea of proof that $\mathcal{G}^{\delta\geq 2}$ is smooth

Observe that C_3 is *freely attachable* to $\mathcal{G}^{\delta\geq 2}$ (that is, $G \in \mathcal{G}^{\delta\geq 2}$ iff $G' \in \mathcal{G}^{\delta\geq 2}$, where G' is G plus a pendant copy of C_3).

Let $\mathcal{D}$ be the class of graphs in $\mathcal{G}^{\delta\geq 2}$ with no pendant C_3 and no component C_3. Given a graph G, the inner core $\mathrm{icore}(G)$ is the unique maximal subgraph of G in $\mathcal{D}$.

Now argue roughly as before (to show that $\mathcal{G}$ is smooth) but with core replaced by inner core.

Like $\mathcal{G}^S$ and $\mathcal{P}$

We can also generalise (and thus better understand?) the relationship between $\mathcal{G}^S$ and $\mathcal{P}$.

Call $\mathcal{G}$ *trimmable* when for any graphs G and H

$$G \in \mathcal{G} \text{ and } \mathrm{core}(H) = \mathrm{core}(G) \implies H \in \mathcal{G}.$$

If a trimmable class contains a forest then it must contain all forests; and each surface class $\mathcal{G}^S$ is trimmable. Also a minor-closed class of graphs is trimmable if and only if each excluded minor has minimum degree at least two.

Let D be the diamond (C_4 plus a chord). Then $\mathrm{Ex}(D)$ is addable and minor-closed. Let $\mathcal{G}^{(k)}$ be the graphs with at most k edge-disjoint subgraphs contractible to D. Then $\mathcal{G}^{(k)}$ is trimmable and $\rho(\mathcal{G}^{(k)}) = \rho(\mathrm{Ex}(D))$.

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Like $\mathcal{G}^S$ and $\mathcal{P}$

Theorem

Let $\mathcal{A} \neq \mathcal{F}$ be a proper addable minor-closed class of graphs; let $\mathcal{G}$ be a trimmable class of graphs with $\mathcal{G} \supseteq \mathcal{A}$ and $\rho_{\mathcal{G}} = \rho_{\mathcal{A}}$; and let $\mathcal{C}$ be the connected graphs in $\mathcal{G}$. Then:

(a) $\mathcal{G}$ and $\mathcal{C}$ are smooth.

(b) $\mathcal{G}^{\delta \geq 2}$ and $\mathcal{C}^{\delta \geq 2}$ have radius of convergence $\rho_2 := \rho(\mathcal{A}^{\delta \geq 2})$; and for $R_n \in_u \mathcal{G}$ or for $R_n \in_u \mathcal{C}$

$$\mathbb{P}(|v(\mathrm{core}(R_n)) - (1 - \rho_2)n| > \epsilon n) = e^{-\Omega(n)}.$$

(c) If some cycle is freely attachable to $\mathcal{G}$, then $\mathcal{G}^{\delta \geq 2}$ and $\mathcal{C}^{\delta \geq 2}$ are smooth.

That's it

Thanks for your attention!