The Mixing Time of The Giant Component $H_{n,p}$ of $G_{n,p}$

The Main Result

For every $\varepsilon > 0$, if $p>(1+\varepsilon)/n$ then for d=pn, the mixing time of the uniform random walk on the giant component $H_{n,p}$ of $G_{n,p}$ is almost surely

$$\Theta \left(\max \left\{ \frac{\ln n}{\ln d}, \left(\frac{\ln n}{d} \right)^2 \right\} \right)$$

Fountoulakis and Reed 2008

The Lower Bounds

The Lower Bounds

Long Induced Paths

 $\mathbf{P}(d(v)=2) = \binom{n}{2}p^2(1-p)^{n-2} \approx d^2e^{-d}$

So the probability that an edge goes to such a vertex is $2de^{-d}$ and the expected number of induced paths of length 1 is $dn(2de^{-d})^{1}$.

For d>1 (2de^{-d}) lies between e^{-d} and e^{-d/4} and we can show a.s. the giant component of $G_{n,p}$ has induced path of length $\Omega(\ln n/d)$.

It follows $G_{n,p}$ has mixing time $\Omega((\ln n/d)^2)$.

The Lower Bounds

Long Induced Paths

 $\mathbf{P}(d(v)=2) = \binom{n}{2}p^2(1-p)^{n-2} \approx d^2e^{-d}$

So the probability that an edge goes to such a vertex is $2de^{-d}$ and the expected number of induced paths of length 1 is $dn(2de^{-d})^{l}$.

For d>1 (2de^{-d}) lies between e^{-d} and e^{-d/4} and we can show a.s. the giant component of $G_{n,p}$ has induced path of length $\Omega(\ln n/d)$.

It follows $G_{n,p}$ has mixing time $\Omega((\ln n/d)^2)$.

Diameter

Letting $N_i = N_i(v)$ be the size of the vertices at distance i from v we see that $E(|N_i|)$ given $N_1,...,N_{i-1}$

 $=d|N_i|(|V-N_1-N_2-..-N_{i-1}|/n) < d|N_i|$

Furthermore while, as we see in the other panel, $|N_i|$ can be far from its expectation when $|N_{i-1}|$ is small simple concentration results show

that a.s. for every v and i s.t.

 $|N_{i-1}| > \log^2 n$, $|N_i| < d^2 \log n$.

So the diameter of the giant component of $G_{n,p}$ and hence its mixing time is $\Omega(\ln n/\ln d)$

 $\Phi(S) = \frac{\pi(i)p_{i,j}}{\pi(S)}, \ \Phi(M) = \min_{\{S \mid \pi(S) \le \frac{1}{2}\}} \Phi(S)$

Jerrum and Sinclair won the Godel prize for proving: For time-reversible lazy \mathcal{M} , $T_{mix}(\mathcal{M}) = O(\Phi(G)^{-2}(-\log (\pi_{min})))$

$$\begin{split} \Phi(\mathsf{S}) &= \frac{\pi(i)p_{i,j}}{\pi(\mathsf{S})}, \ \Phi(M) = \min_{\left\{ S \middle| \pi(S) \leq \frac{1}{2} \right\}} \Phi(S) \\ \text{Jerrum and Sinclair won the Godel prize for proving:} \\ \text{For time-reversible lazy } \mathcal{M}, \ \mathsf{T}_{\mathsf{mix}}(\mathcal{M}) &= O(\Phi(G)^{-2}(-\log(\pi_{min}))) \\ \text{For } \mathcal{M} \text{ the (lazy) uniform random walk on (the giant component of) } \mathsf{G}_{\mathsf{n},\mathsf{p}} \\ \pi_{min} \text{ is } \Theta(\frac{1}{d|V(H_{n,p})|}) \text{ and } \Phi(\mathsf{G}) \text{ is } \Theta(\frac{d}{\ln n}). \text{ So } \mathsf{T}_{\mathsf{mix}} = O((\frac{\ln n}{d})^2 \ln(\mathsf{n})). \end{split}$$

 $\Phi(S) = \frac{\pi(i)p_{i,j}}{\pi(S)}, \ \Phi(M) = \min_{\{S \mid \pi(S) \le \frac{1}{2}\}} \Phi(S)$

Jerrum and Sinclair won the Godel prize for proving:

For time-reversible lazy \mathcal{M} , $T_{mix}(\mathcal{M}) = O(\Phi(G)^{-2}(-\log (\pi_{min})))$

For \mathcal{M} the (lazy) uniform random walk on (the giant component of) $G_{n,p}$

$$\pi_{min}$$
 is $\Theta(\frac{1}{d|V(H_{n,p})|})$ and $\Phi(G)$ is $\Theta(\frac{d}{\ln n})$. So $T_{mix} = O((\frac{\ln n}{d})^2 \ln(n))$

Standing on the shoulders of Kannan & Lovasz, Fountalakis & Reed strengthened Jerrum and Sinclair's result to show:

Letting
$$\Phi(p) = \min_{\{S \mid \frac{p}{2} \le \pi(S) \le p, S \text{ connected}\}} \Phi(S), \mathcal{T}_{mix} = O(\sum_{i=1}^{|-\log \pi_{min}| - 1} \Phi(2^{-j})^{-2}).$$

 $\Phi(S) = \frac{\pi(i)p_{i,j}}{\pi(S)}, \ \Phi(M) = \min_{\{S \mid \pi(S) \le \frac{1}{2}\}} \Phi(S)$

Jerrum and Sinclair won the Godel prize for proving:

For time-reversible lazy \mathcal{M} , $T_{mix}(\mathcal{M}) = O(\Phi(G)^{-2}(-\log (\pi_{min})))$

For \mathcal{M} the (lazy) uniform random walk on (the giant component of) $G_{n,p}$

$$\pi_{min}$$
 is $\Theta(\frac{1}{d|V(H_{n,p})|})$ and $\Phi(G)$ is $\Theta(\frac{d}{\ln n})$. So $T_{mix} = O((\frac{\ln n}{d})^2 \ln(n))$

Standing on the shoulders of Kannan & Lovasz, Fountalakis & Reed strengthened Jerrume and Sinclair's result to show:

Letting
$$\Phi(p) = \min_{\{S \mid \frac{p}{2} \le \pi(S) \le p, S \text{ connected}\}} \Phi(S), \mathcal{T}_{mix} = O(\sum_{i=1}^{|-\log \pi_{min}| - 1} \Phi(2^{-j})^{-2}).$$

They also proved that a.s. for $p < \frac{\sqrt{(\log n)}}{n} H_{n,p} \sum_{i=1}^{\left[-\log \pi_{min}\right]-1} \Phi(2^{-j})^{-2} = O(\ln n + (\frac{\ln n}{d})^2)$. The main result follows. For larger p the proof is easier and uses different techniques.

The Conductance of H_{n,p}



Conductance in $H_{n,p}$ $\left(\frac{1+\varepsilon}{n}$ $For <math>e_{out}(S) = \{xy, x \in S, y \notin S\} e(S) = \{xy \mid x, y \in S\}$, and $d(S) = e_{out}(S) + 2e(S)$, $\Phi(S) = \frac{\varepsilon_{out}(S)}{2e(S) + \varepsilon_{out}(S)}$. Let Y_s^a be the number of connected sets of size s with e(S) > a, and Z_s^b be $|\{S \text{ s.t. S connected}, |S| = s, d(S) \leq |E(H_{n,p})|, e_{out}(S) < b\}$. Conductance in $H_{n,p}$ $\left(\frac{1+\varepsilon}{n} < \mathbf{p} < \frac{\sqrt{\{\ln n\}}}{n}\right)$ For $e_{out}(S) = \{xy, x \in S, y \notin S\} e(S) = \{xy \mid x, y \in S\}$, and $d(S) = e_{out}(S) + 2e(S)$, $\Phi(S) = \frac{\varepsilon_{out}(S)}{2e(S) + e_{out}(S)}$. Let Y_s^a be the number of connected sets of size s with e(S) > a, and Z_s^b be $|\{S \text{ s.t. } S \text{ connected}, |S| = s, d(S) \leq |E(H_{n,p})|, e_{out}(S) < b\}$. Theorem 1: A.s. $\forall s = o((\ln n)^2) Y_s^{2s} = 0$, and $\exists c > 1 \text{ s.t. } a.s. \forall s, Y_s^{cds} = 0$ Theorem 2: $\exists 1 > \delta > 0 \ A > 0 \text{ s.t. } a.s. \forall s > \frac{A(\ln n)}{d} Z_s^{\delta ds} = 0$

Conductance in H_{n.p} $\left(\frac{1+\varepsilon}{n}$ For $e_{out}(S) = \{xy, x \in S, y \notin S\}$ $e(S) = \{xy \mid x, y \in S\}$, and $d(S) = e_{out}(S) + 2e(S)$, $\Phi(S) = \frac{e_{out}(S)}{2e(S) + e_{out}(S)}$. Let Y_s^a be the number of connected sets of size s with e(S) > a, and Z_s^b be $|\{S \text{ s.t. } S \text{ connected}, |S|=s, d(S) \leq |E(H_{n,p})|, e_{out}(S) < b\}.$ **Theorem 1:** A.s. $\forall s = o((\ln n)^2)$ $Y_s^{2s} = 0$, and $\exists c > 1$ s.t. a.s. $\forall s$, $Y_s^{cds} = 0$ **Theorem 2:** $\exists 1 > \delta > 0$ A > 0 s.t.a.s. $\forall s > \frac{A(\ln n)}{d} Z_s^{\delta ds} = 0$ $\begin{array}{l} \textbf{Corollary:} \forall \ connected \ S \ \Phi(S) > \frac{1}{2|S|} \ \& \frac{8A \ln \frac{a}{n}}{2d|E(H_{n,p})|} < \pi(S) < \frac{1}{2} = > \Phi(S) > \frac{\delta}{3c} \\ \sum_{i=1}^{\lceil \log \pi_{min} \rceil - 1} \Phi(2^{-j})^{-2} \le \frac{(2c)^2 \log(4dn)}{\delta^2} + \sum_{j=1}^{\lceil \log(8A \ (\ln n)/d) \rceil} \frac{1}{2(2^j) + 1} \end{array}$ $= 0((\ln n) + \frac{(\ln n)^2}{d^2})$

The Core of The Proof($p < \frac{d_o}{n}$ for constant d_0)

The core $C_{n,p}$ of $H_{n,p}$ is its minimal subgraph of minimum degree 2.

This is the subdivision of a multigraph $K_{n,p}$ of minimum degree 3.

To obtain $H_{n,p}$ from $C_{n,p}$, we decorate it by adding a tree at each of its vertices.

Prove results on conductance of $C_{n,p}$ via the configuration model.

Prove that number of vertices and total degree of a decorated piece is not that different from that for the piece itself.

A General Bound on Mixing Time From Conductance

The Bound

Letting
$$\Phi(p) = \min_{\{S \mid \frac{p}{2} \le \pi(S) \le p, S \text{ connected}\}} \Phi(S),$$

For time reversible chains:

$$T_{mix} = O(\sum_{i=1}^{\left[-\log \pi_{min}\right] - 1} \Phi(2^{-j})^{-2}).$$

A Different Mixing Time

A stopping rule Γ is a rule for stopping our walk where the probability we stop at any time depends only on the sequence of states we have seen so far. We focus on stopping rules where we stop in π .

H(i, π) is the minimum expected time to stop for such a stopping rule starting in i. \mathcal{H} is the maximum of these values.

Theorem: For time-reversible Chains, \mathcal{H} is $\Theta(T_{mix})$ Aldous(1982)

Bounding ${\cal H}$

Choose i_0 & a stopping ruling Γ for i_0 , s.t. the time it takes to stop is $H(i_0,\pi)=\mathcal{H}$.

Choose i_0 & a stopping ruling Γ for i_0 , s.t. the time it takes to stop is $H(i_0,\pi)=\mathcal{H}$.

 x_i = expected exits of i before we halt.

$$y_{i} = \frac{x_{i}}{\pi(i)}$$

$$\mathcal{H} = \sum_{i \text{ in } \Omega} x_{i} = \sum_{i \text{ in } \Omega} \pi(i) y_{i}.$$

Choose i_0 & a stopping ruling Γ for i_0 , s.t. the time it takes to stop is $H(i_0,\pi)=\mathcal{H}$.

 x_i = expected exits of i before we halt.

 $y_{i} = \frac{x_{i}}{\pi(i)}$ $\mathcal{H} = \sum_{i \ in \ \Omega} x_{i} = \sum_{i \ in \ \Omega} \pi(i) y_{i}.$ $\exists h \ s. \ t. \ x_{h} = y_{h} = 0 \text{ Lovasz & Winkler}$ **Claim 1:** y_{i} is maximized at $i = i_{0}$

Choose i_0 & a stopping ruling Γ for i_0 , s.t. the time it takes to stop is $H(i_0,\pi)=\mathcal{H}$.

 x_i = expected exits of i before we halt.

 $y_{i} = \frac{x_{i}}{\pi(i)}$ $\mathcal{H} = \sum_{i \ in \ \Omega} x_{i} = \sum_{i \ in \ \Omega} \pi(i) y_{i}.$ $\exists h \ s. \ t. \ x_{h} = y_{h} = 0 \text{ Lovasz & Winkler}$ **Claim 1:** y_{i} is maximized at $i = i_{0}$ WLOG: $0=y_1 \le y_2 \le \dots \le y_n$ C_i -component of $G[\bigcup_{j\le i} y_j]$ containing y_1 $i^*=\min \{i \mid \pi(C_i) > \frac{1}{2}\}.$ We bound: $\sum_{i=1}^n \pi(i) y_{i^*} + \sum_{i=1}^n \pi(i) (y_i - y_{i^*})$

We bound:

$$\sum_{i=1}^{n} \pi(i) y_{i^*} + \sum_{i=1}^{n} \pi(i) (y_i - y_{i^*})$$

$$\begin{split} & m_{0}=y_{0}=0, \ Z_{0}=\{1\}, \ m_{i}=m_{i-1}+\frac{4}{\Phi(Z_{i-1})}, \ j_{i}=\max\{j\mid y_{j}\leq m_{i}\}, Z_{i}=C_{j_{i}} \\ & L=\max\{i\mid m_{i}$$

We bound: $\sum_{i=1}^{n} \pi(i) y_{i^*} + \sum_{i=1}^{n} \pi(i) (y_i - y_{i^*})$

$$\begin{split} n_{0} = n, S_{0} = n \\ For \ i \geq 0, n_{i} = \min \{j \mid \pi(\{j, j+1, ..., n\}) \geq \pi(S_{i-1})(1 - \frac{\Phi(S_{i-1})}{4}) \ S_{i} = \{n_{i}, ..., n\}. \\ L = \min \{i \mid n_{i+1} < i^{*} + 1\} \\ \sum_{i=1}^{n} \pi(i)(y_{i} - y_{i^{*}}) \leq \sum_{k=0}^{L} (y_{n_{k}} - y_{n_{k+1}}) \pi(S_{k+1} - n_{k+1}) \\ Claim 3: (y_{n_{k}} - y_{n_{k+1}}) \pi(S_{k+1} - n_{k+1}) \leq \frac{3}{\Phi(S_{k})} \\ By \ definition, \ there \ are \leq \frac{4}{\Phi(2^{-j})} S_{k} \ s. \ t \ 2^{-j-1} \leq \pi(S_{k}) \leq 2^{-j} \\ => \sum_{i=1}^{n} \pi(i)(y_{i} - y_{i^{*}}) \leq \sum_{k=1}^{L+1} \frac{3}{\Phi(S_{k})} \leq \sum_{i=1}^{\lceil \log \pi_{min} \rceil - 1} 12 \ \Phi(2^{-j})^{-2} \end{split}$$

Proof of Claims

A Simple Observation

Observation 1: For all S not containing i_{0,}

 $\pi(S) = \sum_{i \text{ in } S, j \text{ not in } S} \pi(i) p_{i,j}(y_j - y_i)$

Proof: The sum counts the number of times we enter S minus the number of times we leave S. Since we stop in stationary the result follows.

Corollorary (Claim 1): For every a, the graph induced by $\{i | y_i \ge a\}$ is either empty or contains i_0 and is connected.

Proof: for any component S of this graph not containing i_0 the summands in observation 1 are negative but the sum is positive.

Proof of Claim 2

By Observation 1 $\pi(Z_k) = \sum_{i \text{ in } Z_k, j \text{ not in } Z_k} \pi(i) p_{i,j}(y_j - y_i)$ Since Z_k is a component of {i | $y_i \leq a$ } for some a $\pi(Z_k) \geq \sum_{i \in Z_k, j \notin Z_{k+1}} \pi(i) p_{i,j}(y_j - y_i) \leq \sum_{i \in Z_k, j \notin Z_{k+1}} \pi(i) p_{i,j} \frac{4}{\Phi(Z_k)}$ Letting Q(A,B) $= \sum_{i \text{ in } A, j \text{ in } B} \pi(i) p_{i,j}$ so $\Phi(S) = \frac{Q(S, \Omega - S)}{\pi(S)}$, we have: $\pi(Z_k) \geq Q(Z_k, \Omega - Z_{k+1}) \frac{4}{\Phi(Z_k)} = Q(Z_k, \Omega - Z_{k+1}) \frac{4\pi(Z_k)}{Q(Z_k, \Omega - Z_k)}$ Hence, $Q(Z_k, \Omega - Z_{k+1}) \leq \frac{1}{4} Q(Z_k, \Omega - Z_k)$. Since $Q(Z_k, \Omega - Z_k) = Q(Z_k, \Omega - Z_{k+1}) + Q(Z_k, Z_{k+1} - Z_k)$, $Q(Z_k, Z_{k+1} - Z_k) \geq \frac{3}{4} Q(Z_k, \Omega - Z_k) = \frac{3\pi(Z_k)\Phi(Z_k)}{4}$ Now, $Q(Z_k, Z_{k+1} - Z_k)$ is the probability that in stationary we move into $Z_{k+1} - Z_k$ from Z_k so: $\pi(Z_{k+1} - Z_k) \geq \frac{3\pi(Z_k)\Phi(Z_k)}{4}$ and $\pi(Z_{k+1}) \geq (1 + \frac{3\Phi(Z_k)}{4}) \pi(Z_k)$

We omit the proof of Claim 3 it can be found in the references.

References

Fountoulakis and Reed, The evolution of the mixing rate..., Random Structures and algorithms 2008.

Fountoulakis and Reed, Faster Mixing and Smaller Bottlenecks, Probability Theory and Related Fields, 2007.

Thank You For Your Attention!

