The Chromatic Thresholds of graphs

Simon Griffiths

Based on joint work with

Peter Allen, Julia Böttcher, Yoshiharu Kohayakawa, Rob Morris

What is your favourite triangle-free graph?

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Complete bipartite! It has the most edges!

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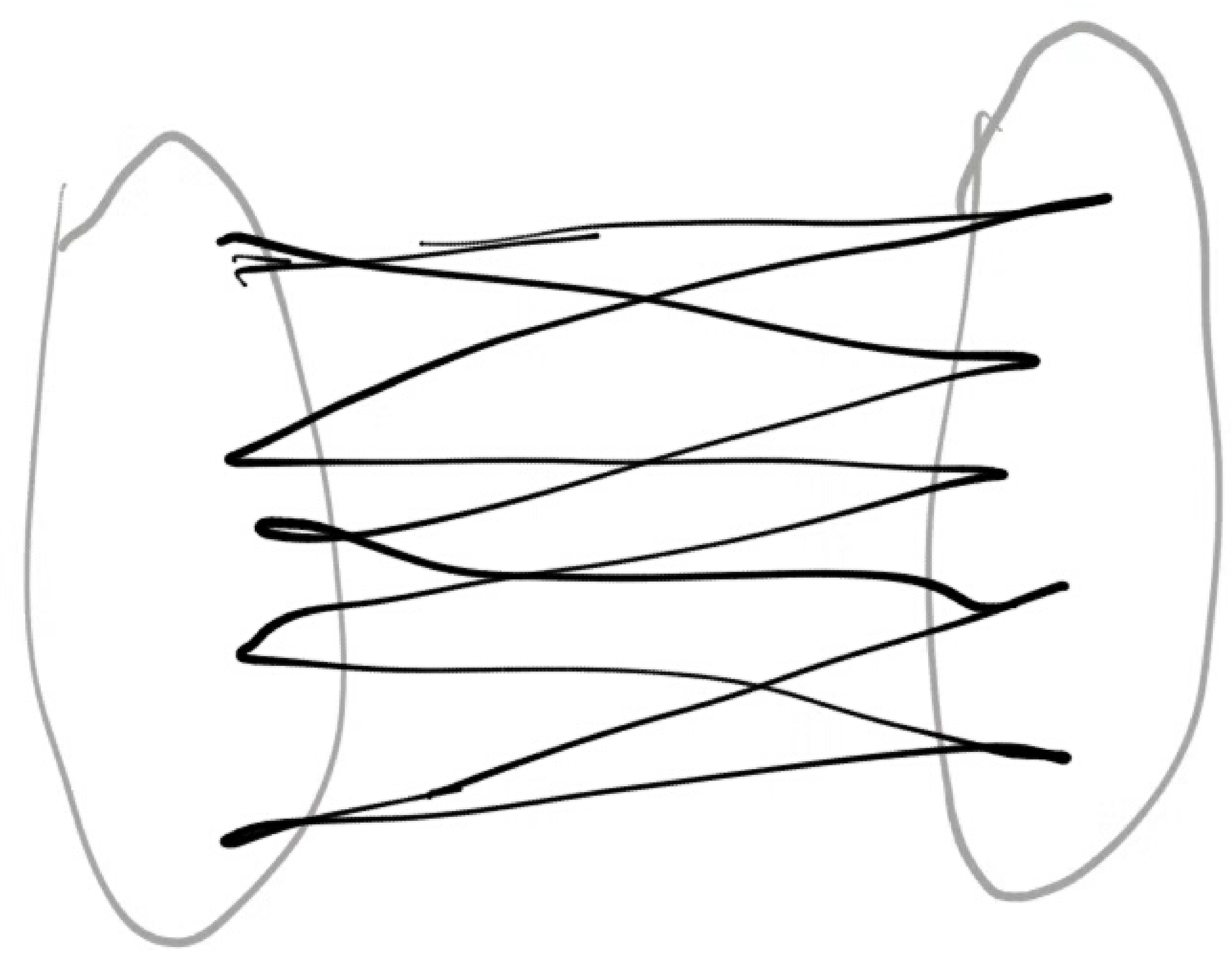

$K_{\frac{n}{2},\frac{n}{2}}$

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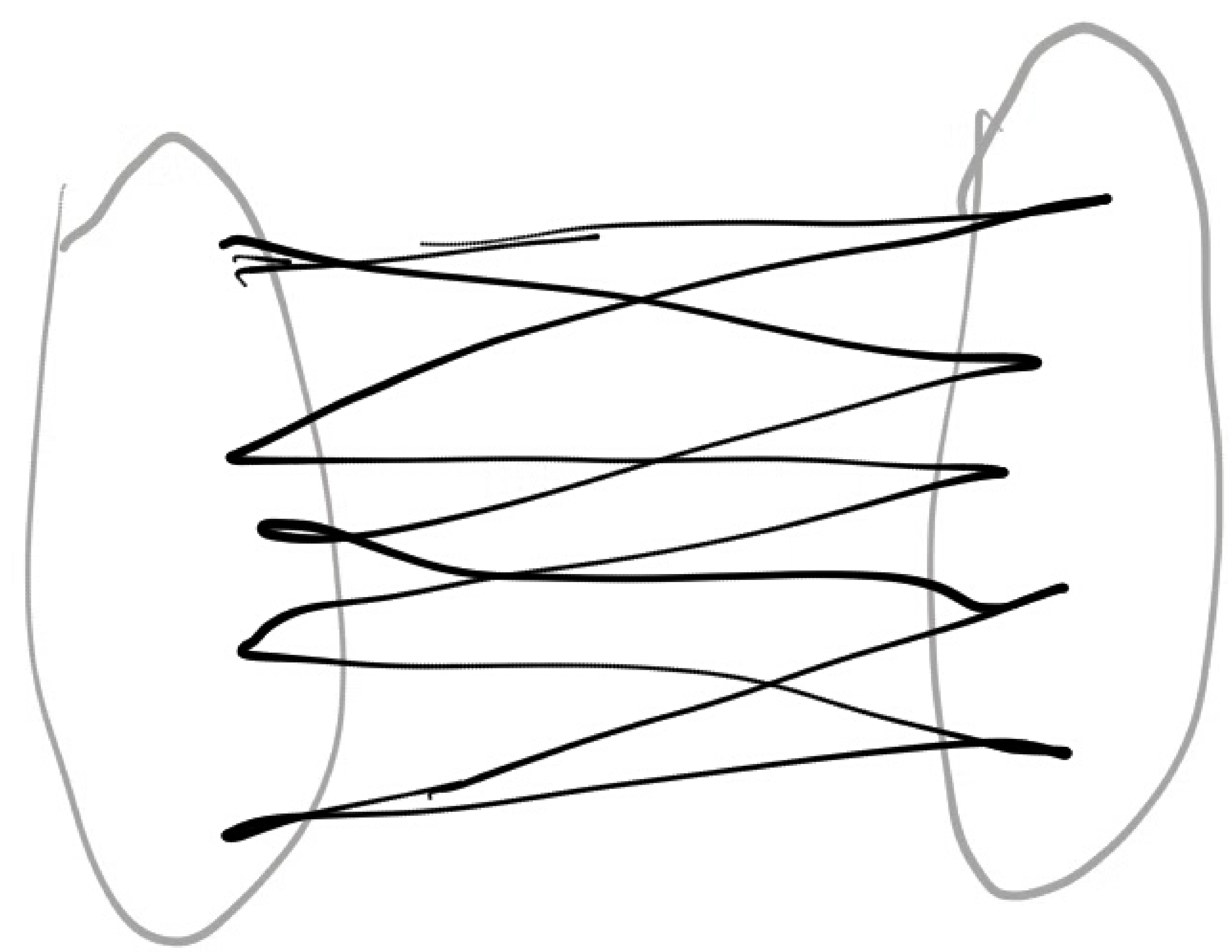

$K_{\frac{n}{2}, \frac{n}{2}}$

But it's boring! Small chromatic number!

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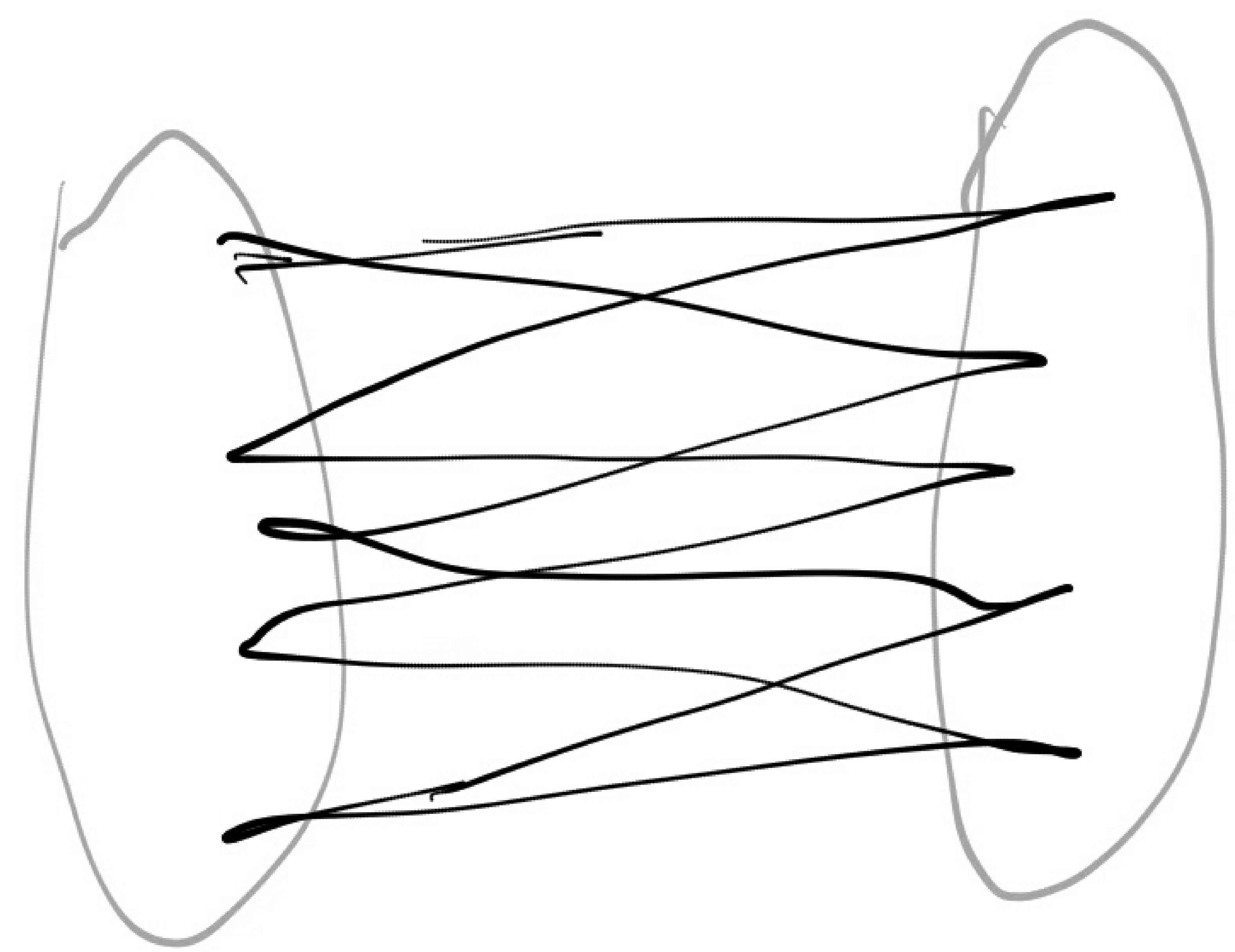

Δ-free
large χ

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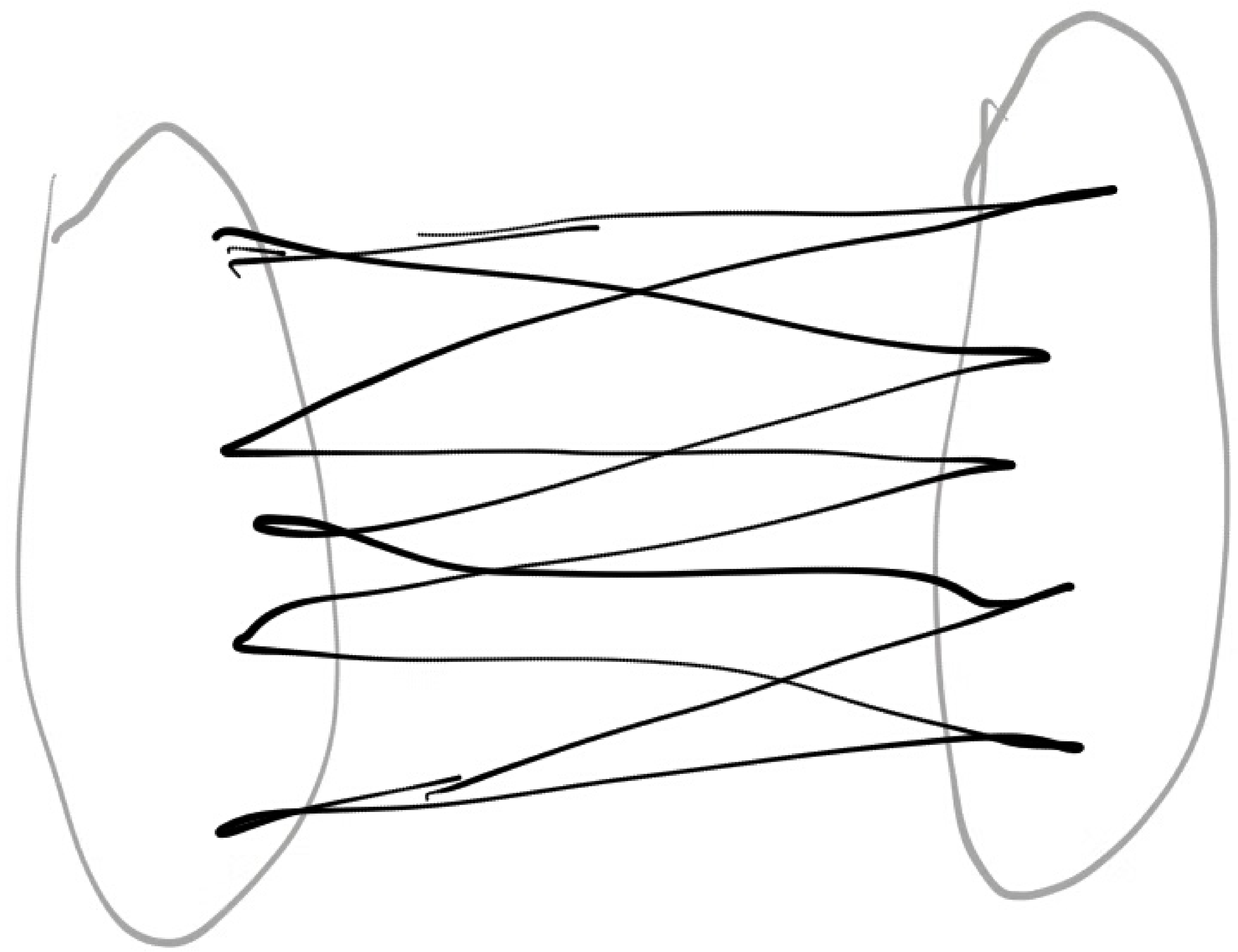

Δ-free
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Not so interesting....

Are there triangle-free graphs G with:

- $\delta(G) \geq dn$
- $\chi(G)$ large

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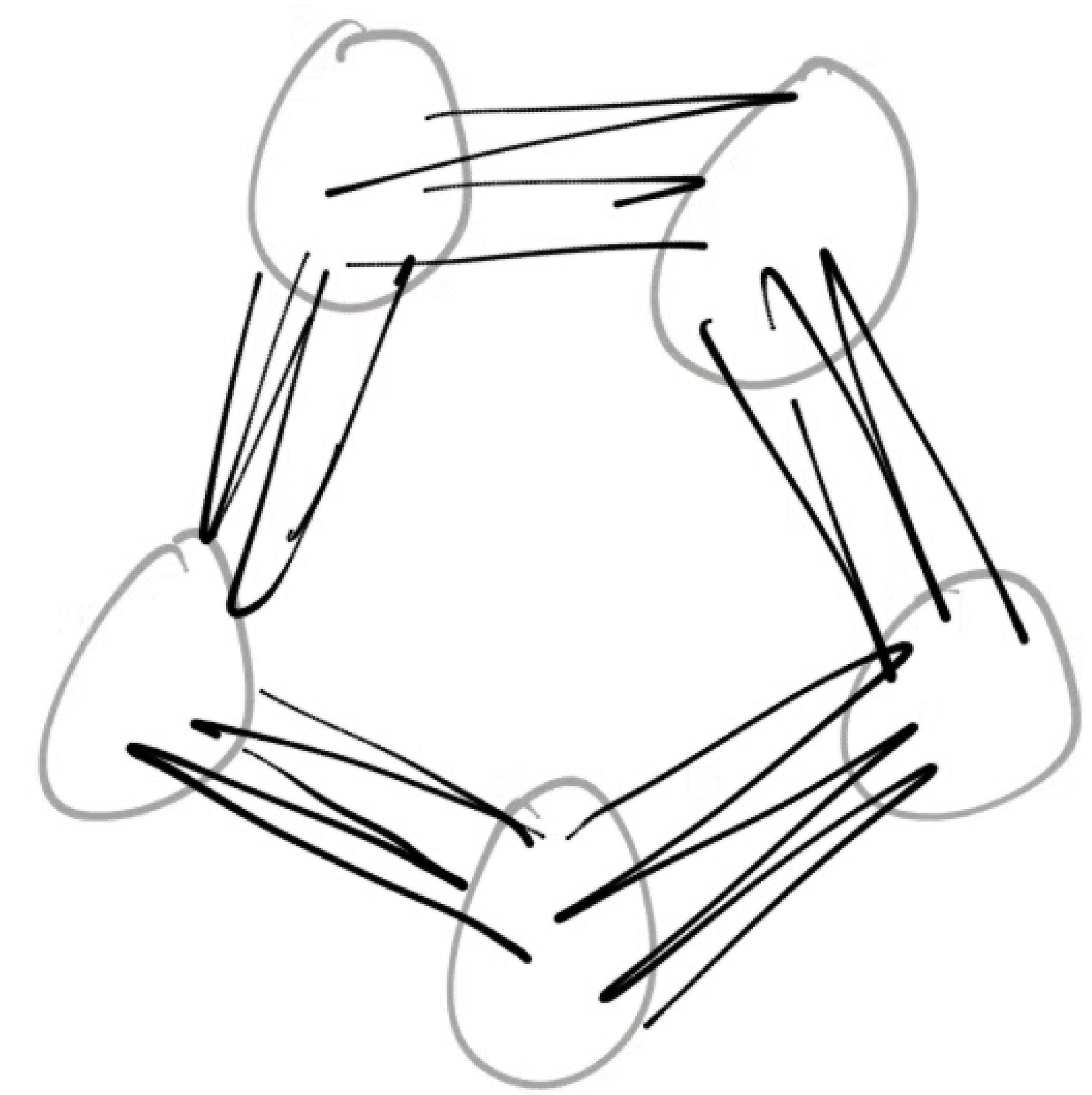

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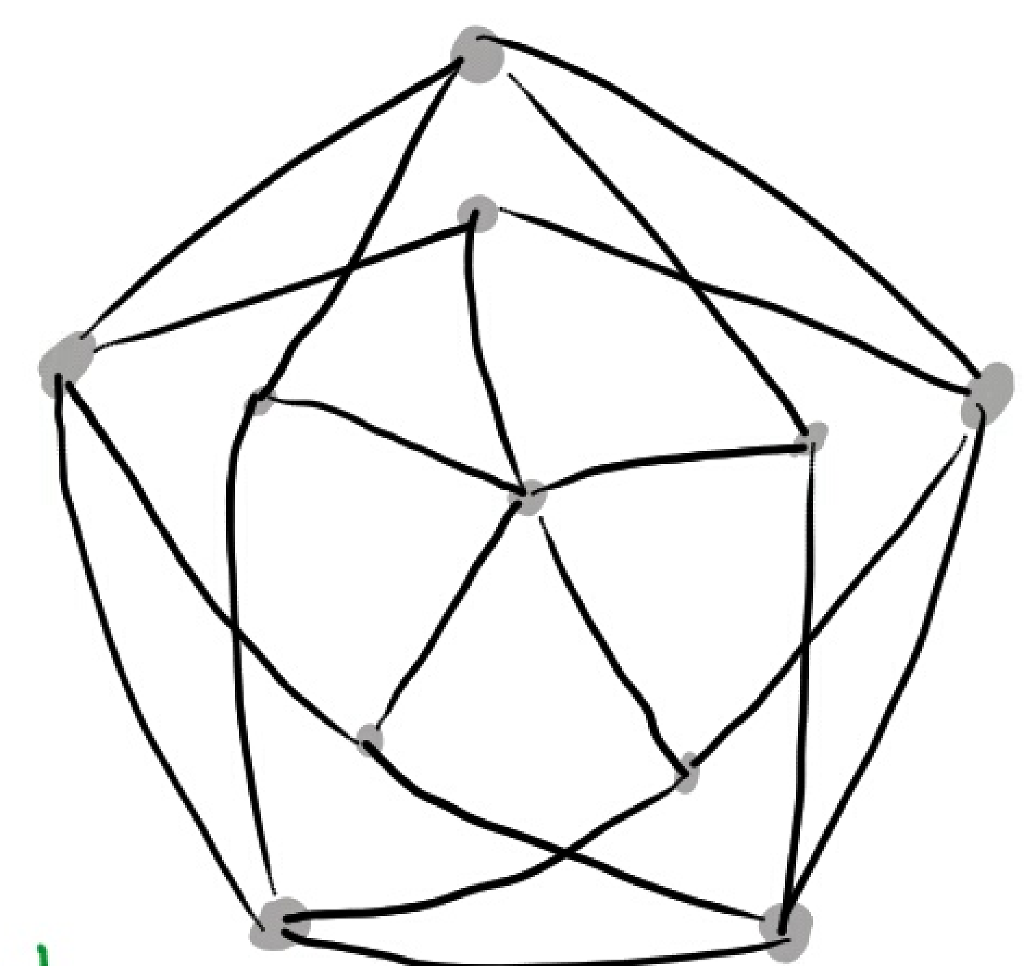
Extremal example :
Unbalanced blow up
of Grötzsch graph :



(Häggkvist 1982)

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$d < \frac{1}{3}$: $\chi(G)$ unbounded (Hajnal 1973)

General question (Erdős, Simonovits 1973): For each graph H, determine:

$$\delta_\chi(H) := \inf\{d : \exists C, \text{ all } H\text{-free graphs with } \delta(G) > dn \text{ have } \chi(G) \le C\}$$

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We have seen that $\delta_\chi(K_3) = \frac{1}{3}$ (Thomassen 2002)

Also: $\delta_\chi(C_k) = 0$ odd $k \geq 5$ (Thomassen 2007)

$\delta_\chi(K_r) = \frac{2r-5}{2r-3}$ $r \geq 3$ (Goddard, Lyle 2011)

General result for H with $\chi(H) = 3$:

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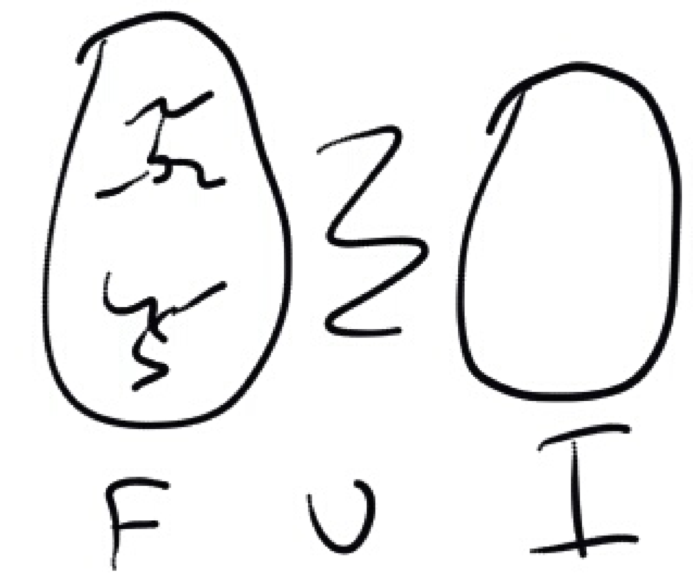
Call H a **forest graph** if $H =$ 

Call H **near-acyclic** if

also no odd cycle C has $|C \cap I| = 1$

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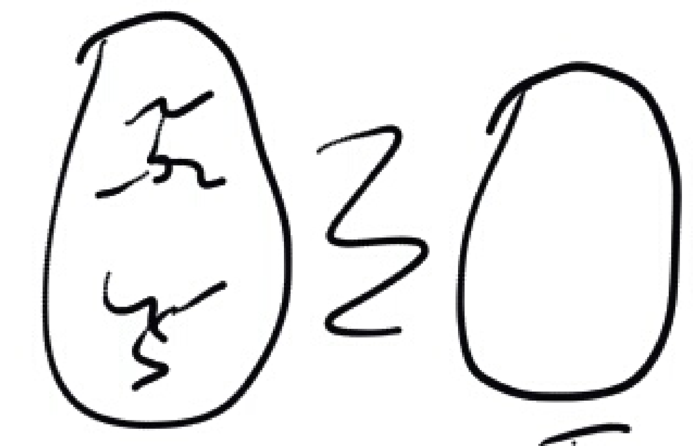
Call H a forest graph if $H =$ 
$F \cup I$

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Theorem (Allen, Böttcher, G., Kohayakawa, Morris 2013)

For H with $\chi(H)=3$:

$$\delta_\chi(H) \in \left\{0, \frac{1}{3}, \frac{1}{2}\right\}$$

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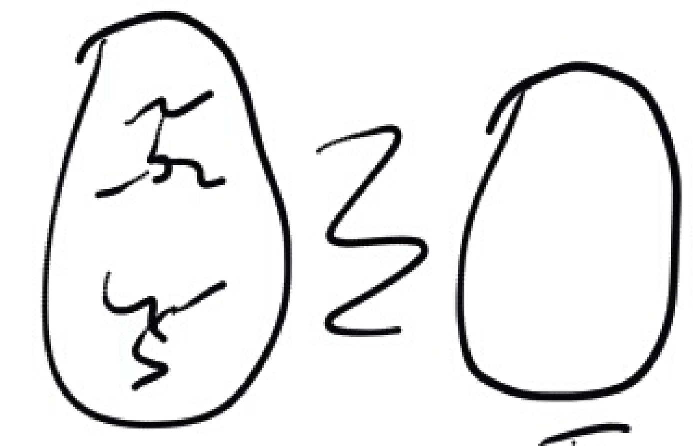
Call H a forest graph if $H =$  $F \cup I$

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Conjectured by Łuczak & Thomassé

STANDING ON THE SHOULDERS OF GIANTS

Erdős and Simonovits asked a wonderful question

Our results rely on many other advances.

The upper bound at $\frac{1}{3}$ is influenced by Goddard and Lyle.

Most directly, Łuczak and Thomassé conjectured the answer, proved the LB at $\frac{1}{3}$ and introduced Boosters!

Thm: $\delta_\chi(H) \in \{0, \frac{1}{3}, \frac{1}{2}\}$, $\leq \frac{1}{3} \iff$ forest graph

$(\chi(H)=3)$ $= 0 \iff$ near-acyclic

Thm: $(\chi(H)=3)$ $\delta_\chi(H) \in \{0, \frac{1}{3}, \frac{1}{2}\}$, $\le \frac{1}{3} \iff$ forest graph

$= 0 \iff$ near-acyclic

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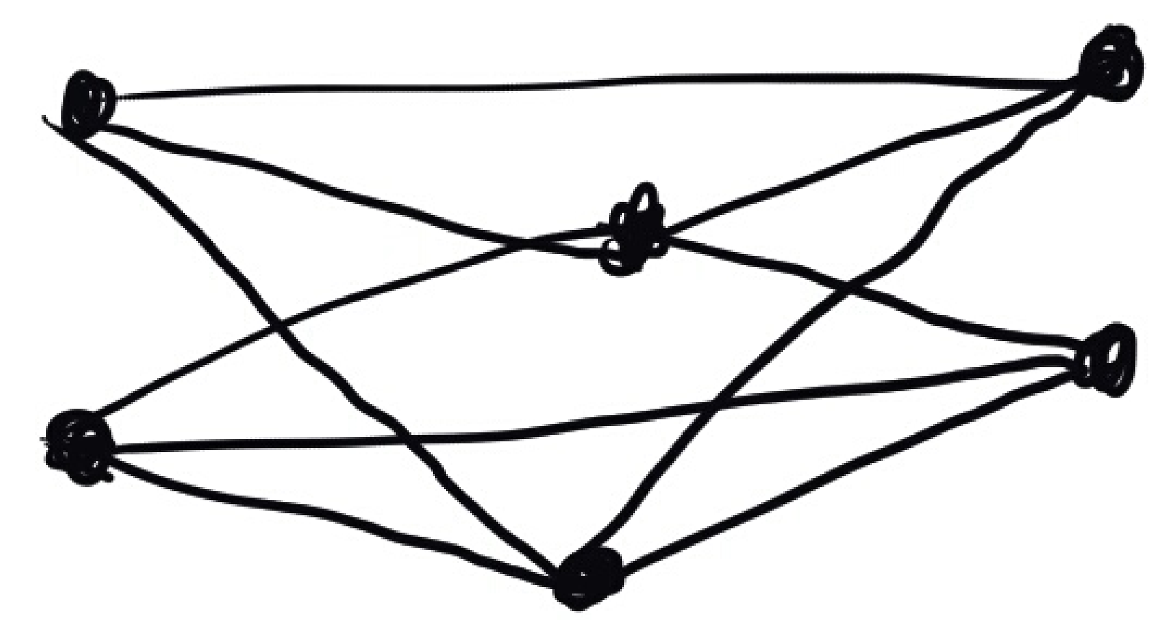

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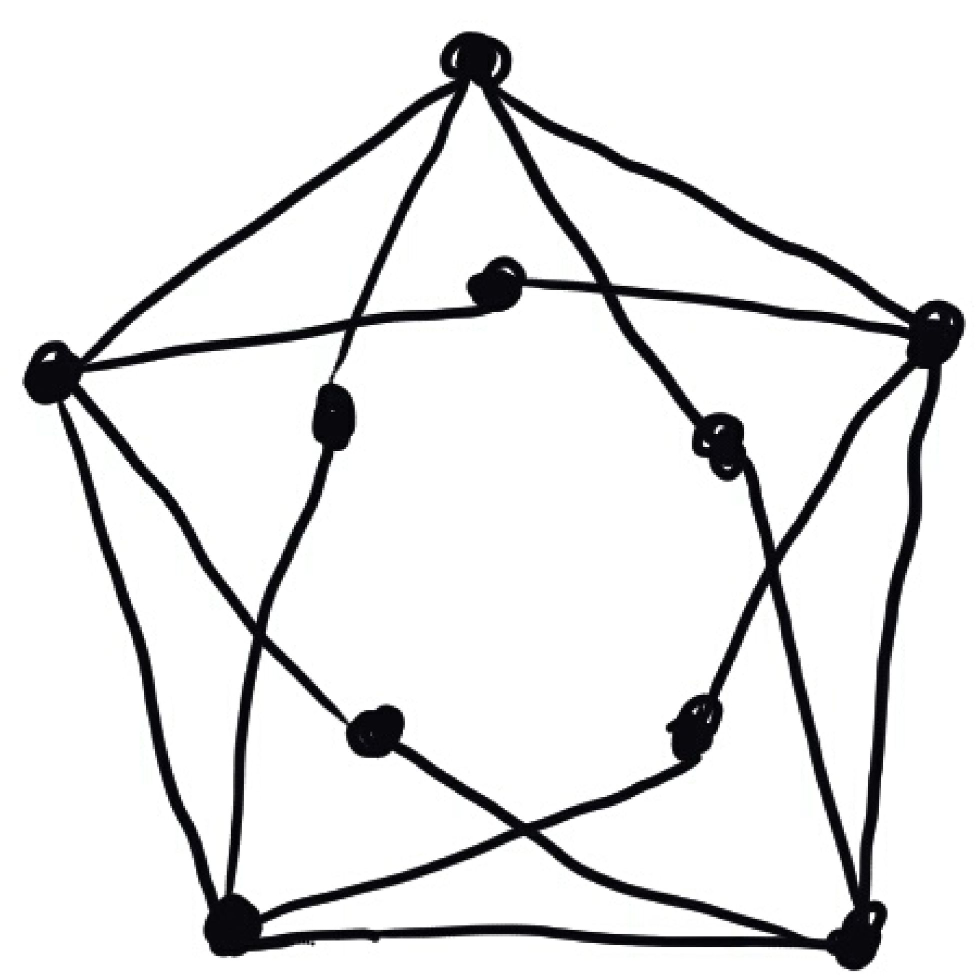

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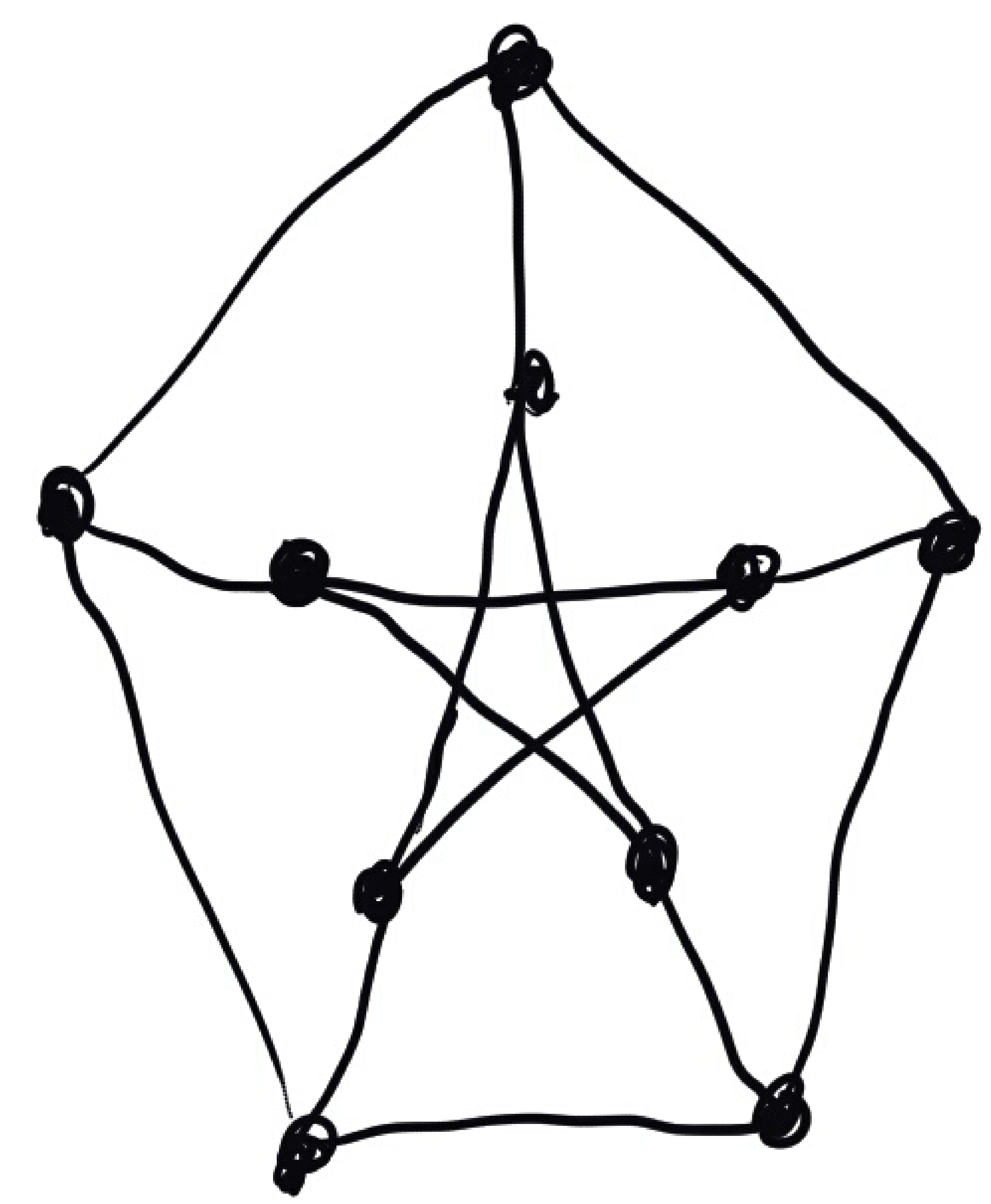

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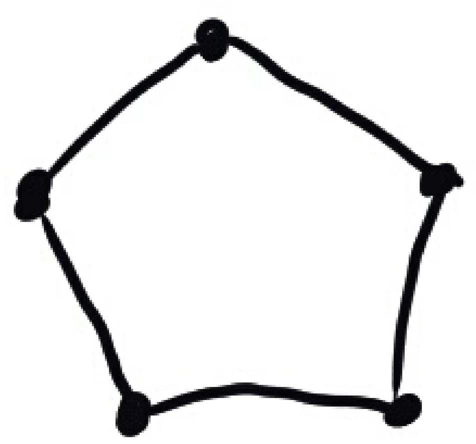
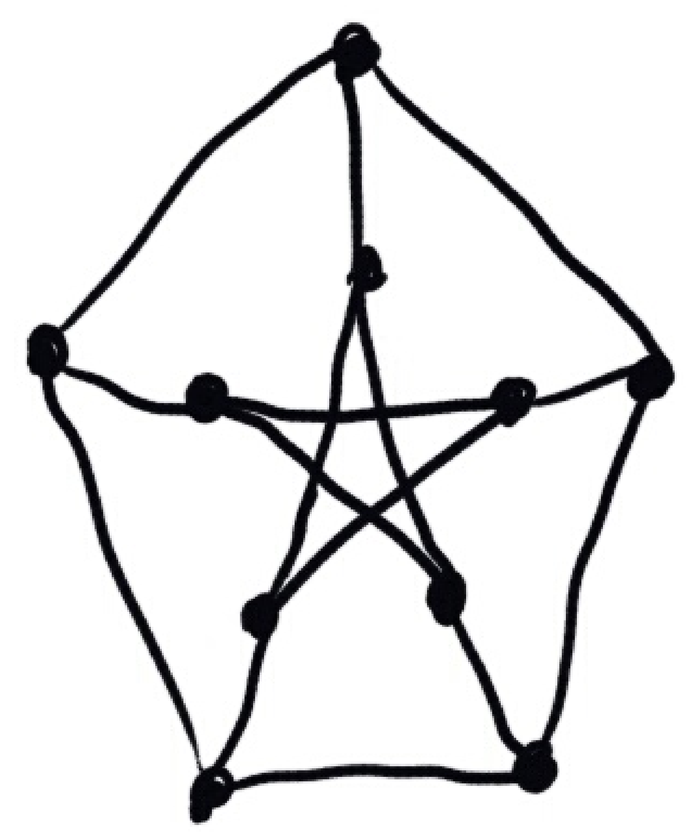
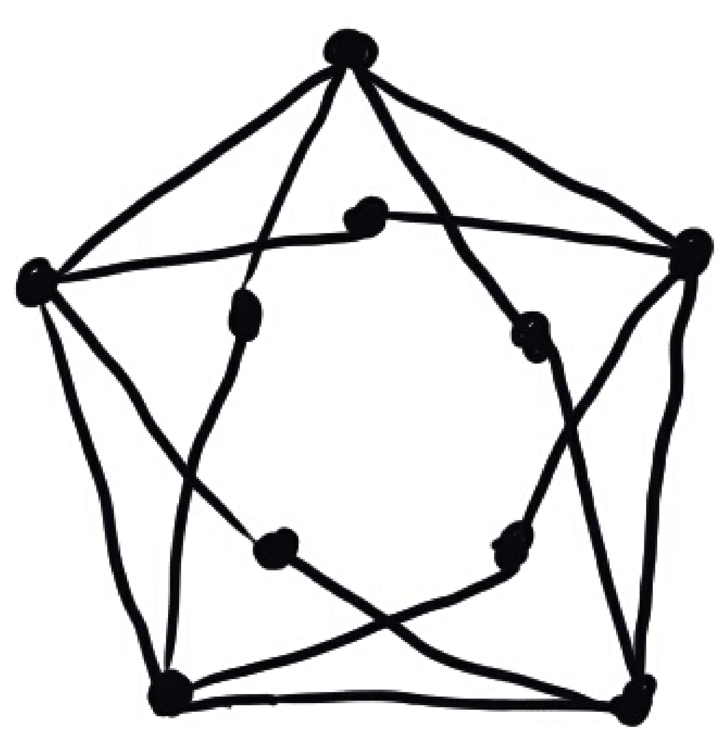
Examples

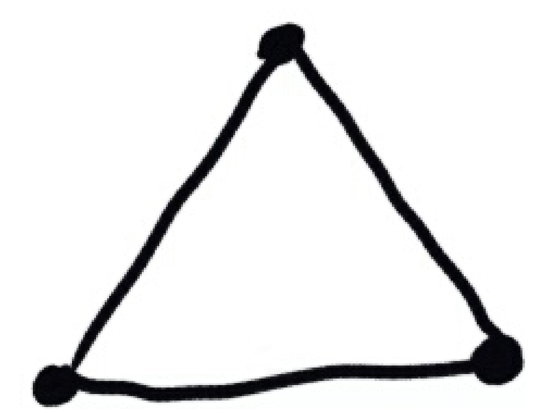
$\delta_\chi(H) = 0$







$\delta_\chi(H) = \frac{1}{3}$



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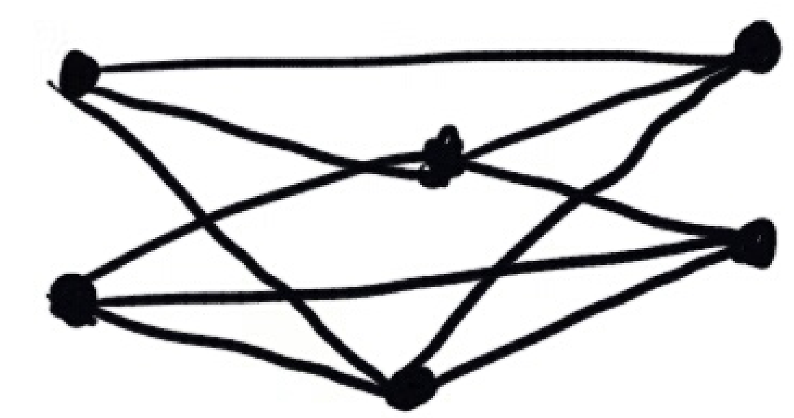

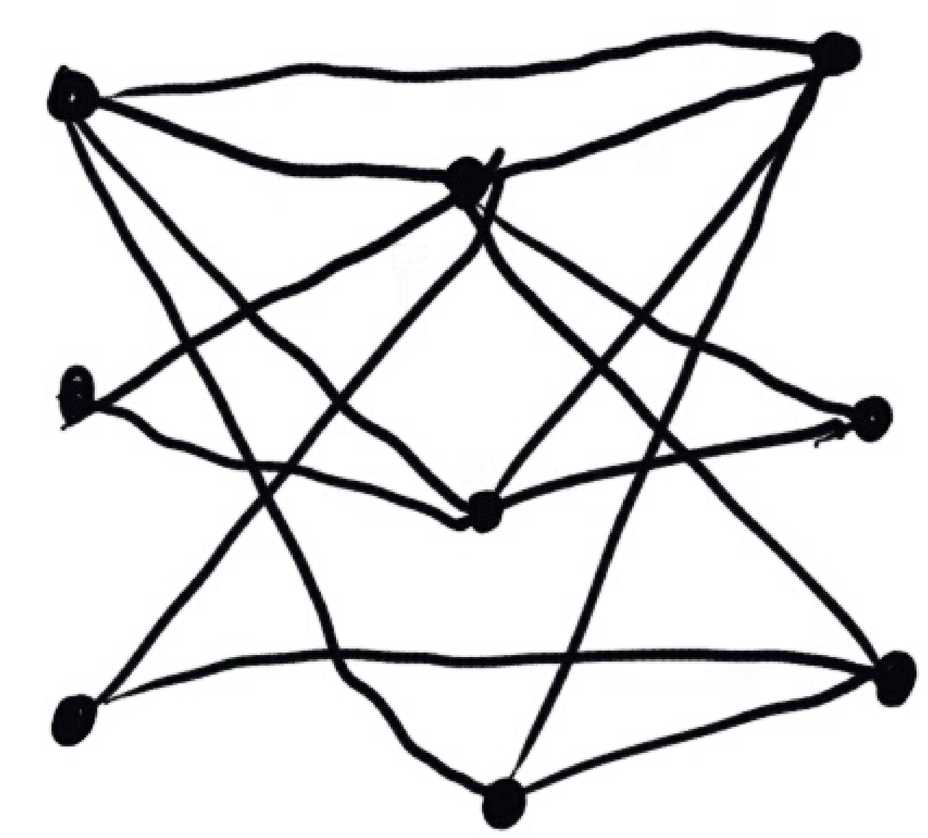

Thm: $\delta_\chi(H) \in \{0, \frac{1}{3}, \frac{1}{2}\}$, $\leq \frac{1}{3} \iff$ forest graph
$(\chi(H)=3)$ $\qquad\qquad\qquad\qquad = 0 \iff$ near-acyclic

How do you prove a lower bound?

Sequence of H-free graphs G_n with:

- $\delta(G_n) \geq \delta n$
- $\chi(G_n) \longrightarrow \infty$

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How do you prove a lower bound?

Sequence of H-free graphs G_n with:

- $\delta(G_n) \geq dn$
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How do you prove an upper bound?

Show $\delta(G) \geq dn$ and $\chi(G)$ large

$\Rightarrow G \supseteq H$

LOWER BOUNDS

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G with:

$\delta(G) \geq dn$

$\chi(G)$ large

$G \not\supseteq H$

LOWER BOUNDS:

$\delta_x(H) = \frac{1}{2}$ if H not a forest graph:

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$\delta_\chi(H) = \frac{1}{2}$ if H <u>not</u> a forest graph:

Can you construct G with:

$$\delta(G) \geq \frac{n}{2}$$

$$\chi(G) \text{ large}$$

and $G \not\supseteq H$?

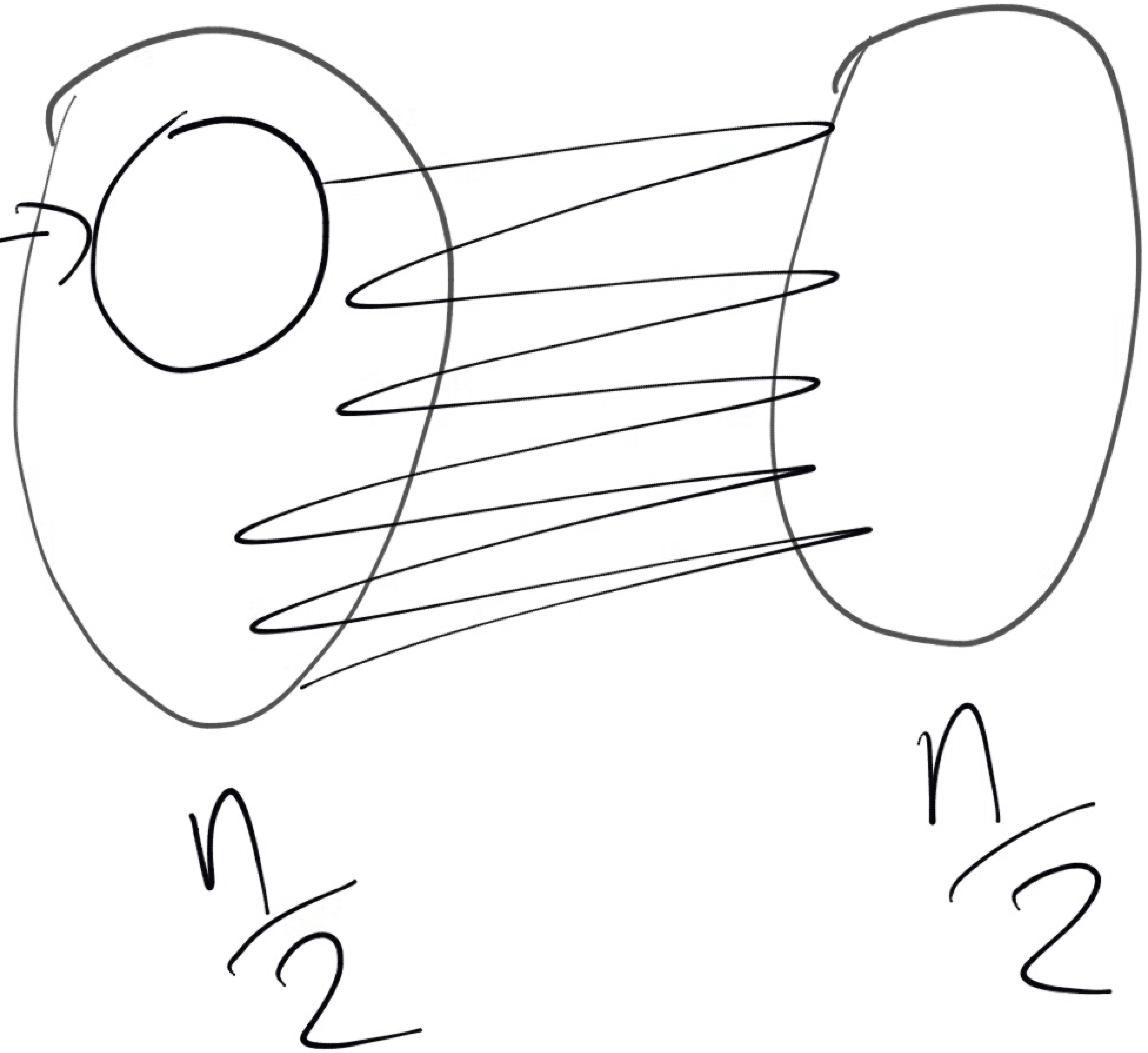
LOWER BOUNDS:

$\delta_\chi(H) = \frac{1}{2}$ if H not a forest graph:

An Erdős graph,

$\chi \to \infty$

girth $\to \infty$



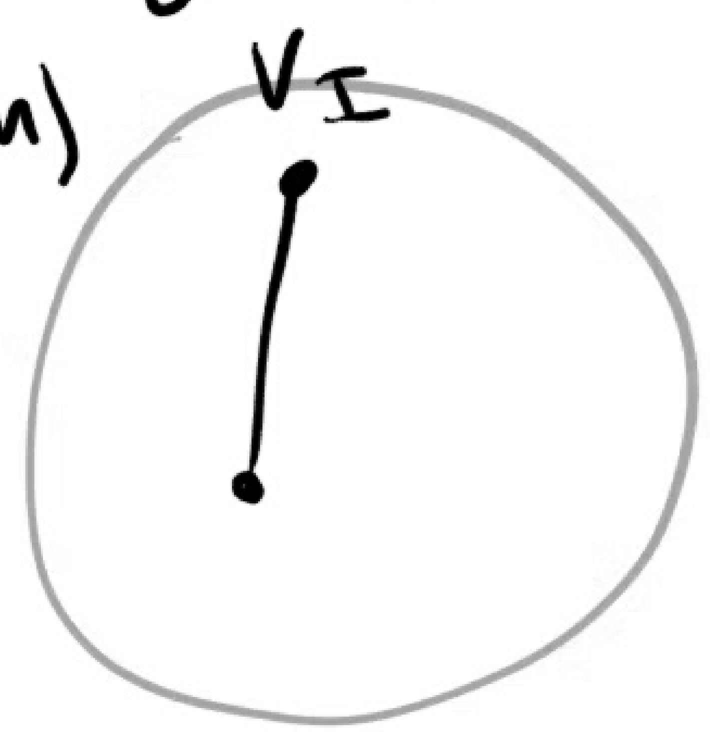
MORE DIFFICULT: $\delta_x(H) \geq \frac{1}{3}$ if H not near-acyclic

MORE DIFFICULT: $\delta_\chi(H) \geq \frac{1}{3}$ if H <u>not</u> near-acyclic

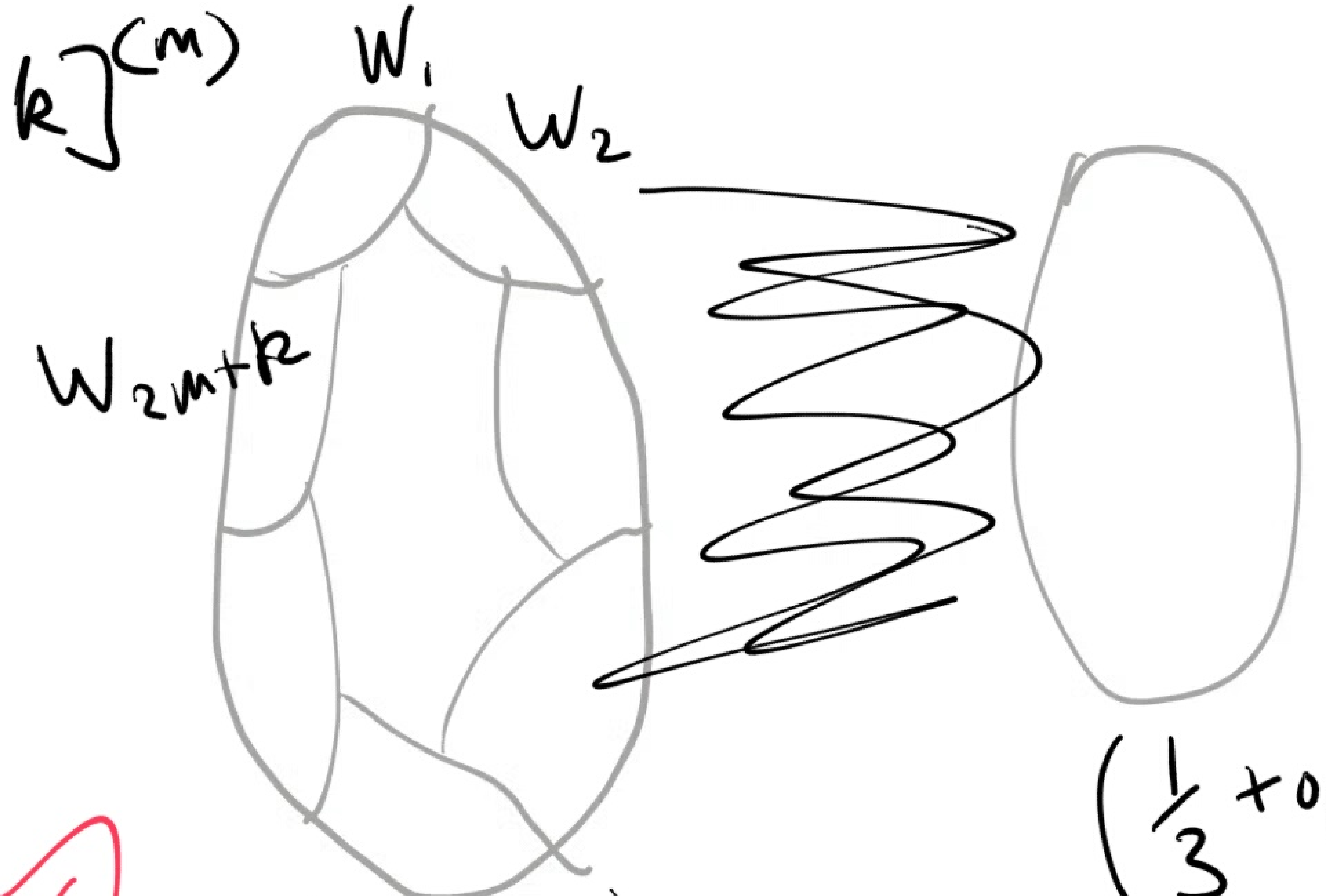
HAJNAL'S PROOF THAT $\delta_\chi(K_3) \geq \frac{1}{3}$

Kneser graph $[2m+k]^{(m)}$

$I \in [2m+k]^{(m)}$



$\chi = k+2$



$\left(\frac{2}{3}+o(1)\right)n$

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2m+k parts

- Large χ
- $\delta \geq \left(\frac{1}{3}-o(1)\right)n$
- $G \not\supseteq \Delta$

v_I —— $\bigcirc$ W_i $\quad \forall i \in I$

Generalisation: Borsuk–Hajnal graphs by Łuczak & Thomassé

UPPER BOUNDS

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Show that

$$\delta(G) \geq dn$$

and $\chi(G)$ large

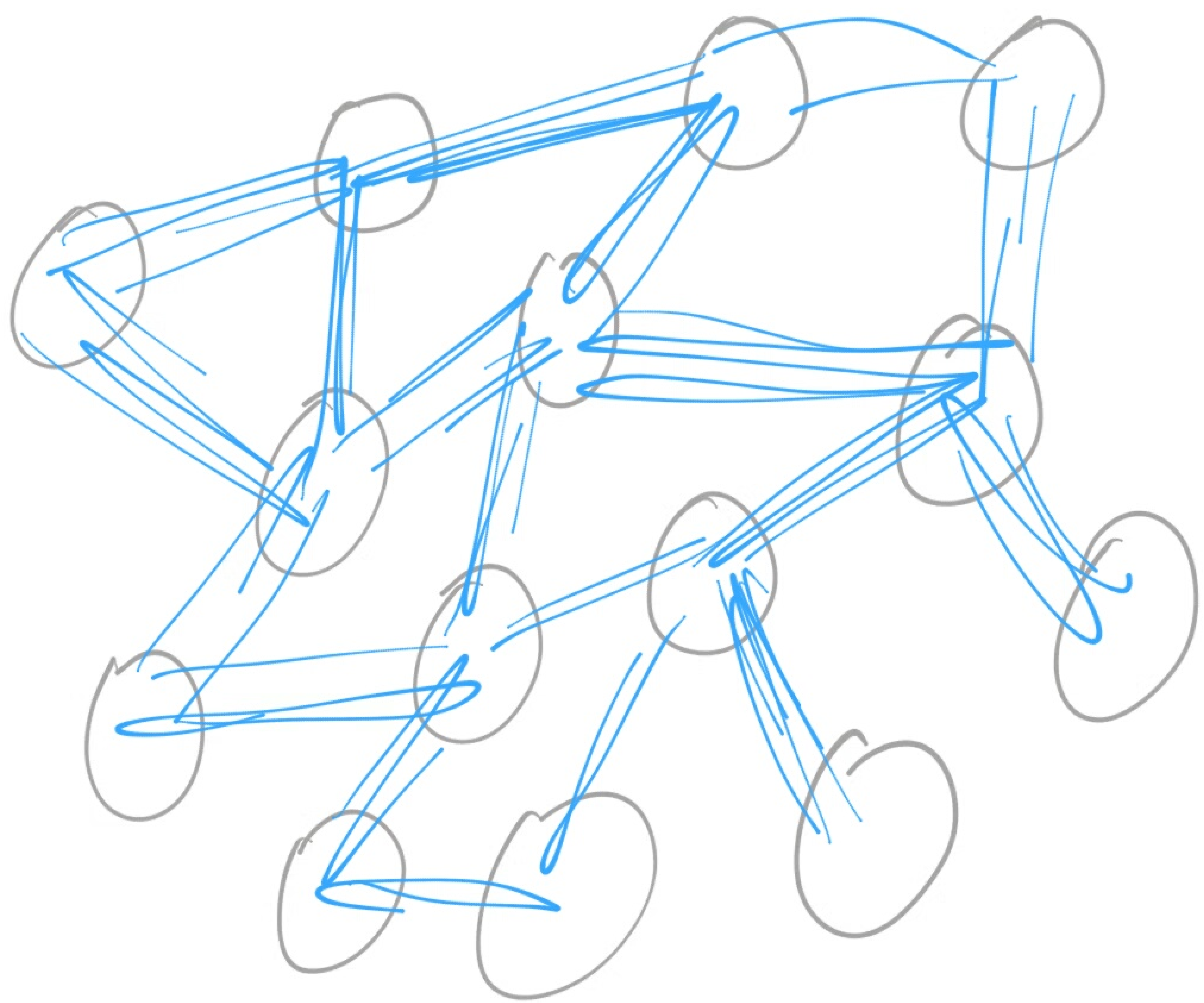
implies

$$G \supseteq H$$

UPPER BOUNDS

A key tool: Minimum degree version of Szemerédi's Regularity Lemma

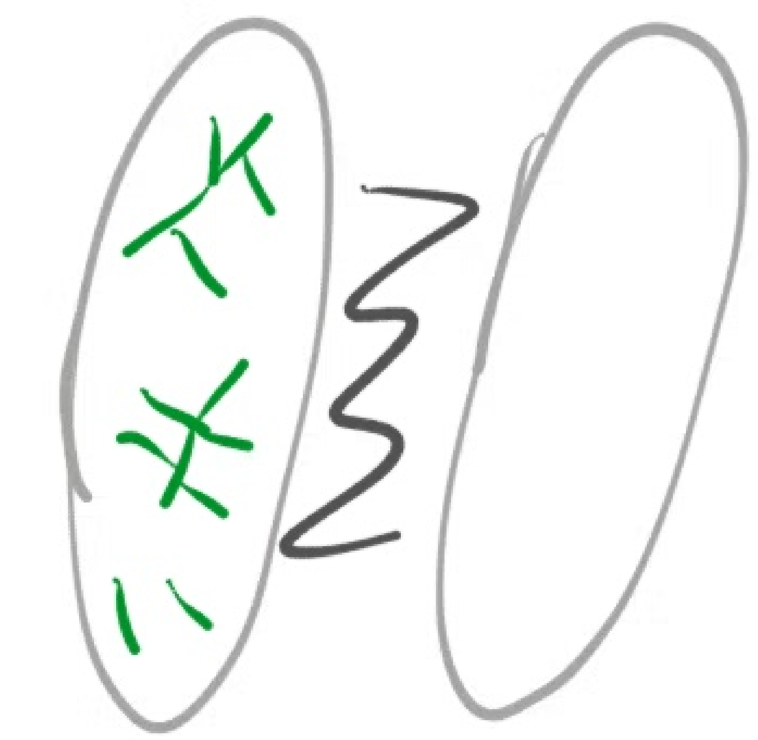
(See Komlós, Simonovits 1996)



For G with $\delta(G) \geq dn$ we obtain a reduced graph R with $\delta(R) \geq (d-\varepsilon)|V(R)|$

UPPER BOUNDS

$\delta_\chi(H) \le \frac{1}{3}$ for H a forest graph



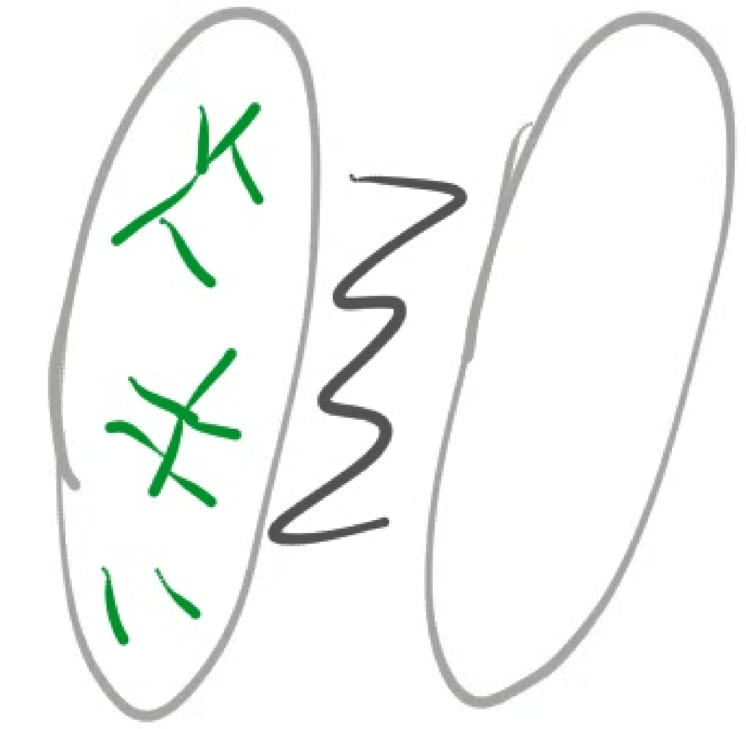
$F \cup I$

UPPER BOUNDS

$\delta_\chi(H) \le \frac{1}{3}$ for H a forest graph

Let G have $\delta(G) \ge \left(\frac{1}{3} + 10\delta\right) n$
and large $\chi(G)$:

We must prove that $G \supseteq H$



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We show that $G \supseteq H$

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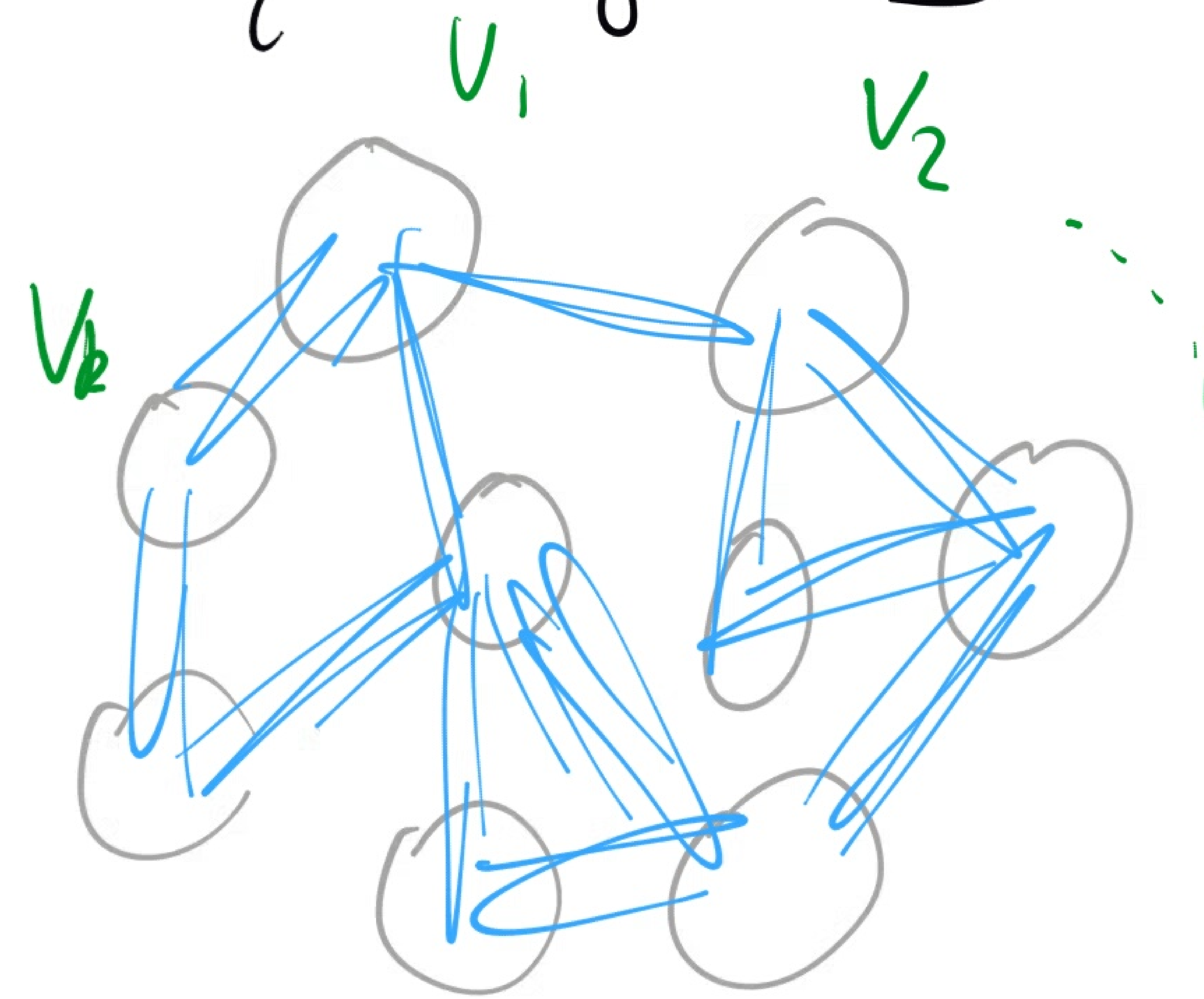

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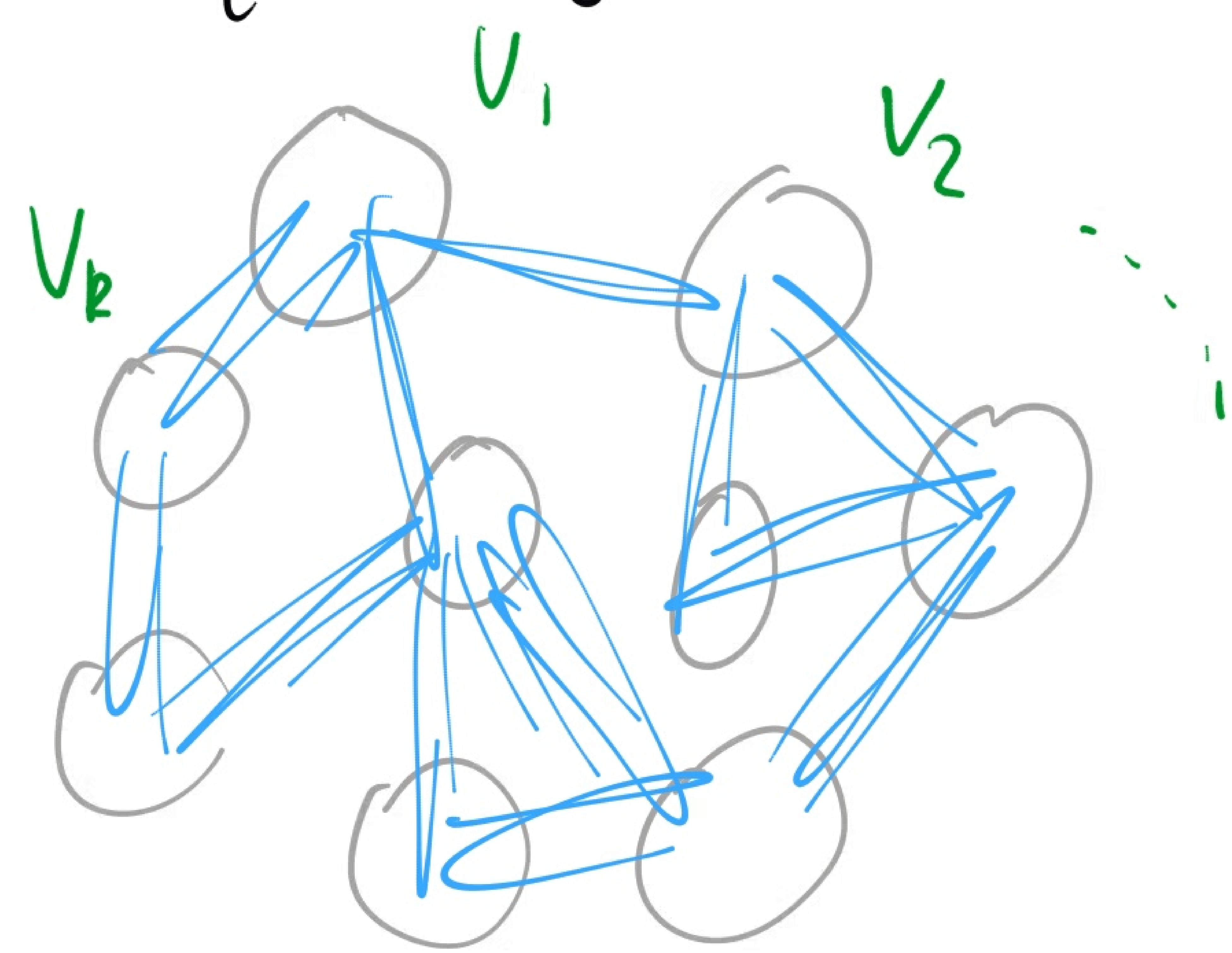
1) Apply min. deg. version of Regularity Lemma

2) For $x \in V$ let

$$I(x) = \{ i : |\Gamma(x) \cap V_i| > \delta |V_i| \}$$



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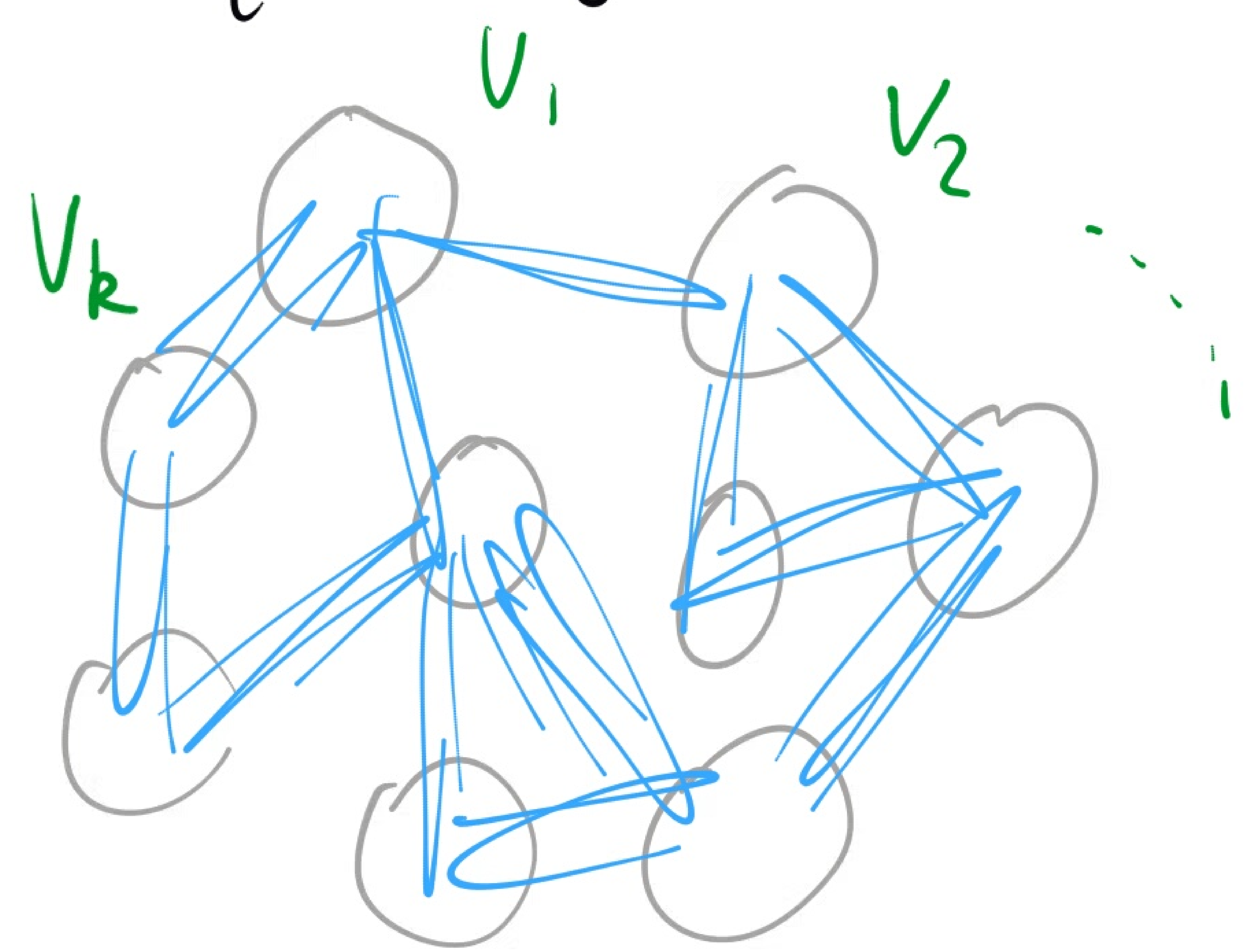

3) Partition

$$V(G) = \bigcup_{I \subseteq [k]} X_I \quad \text{where } X_I = \{x : I_x = I\}$$

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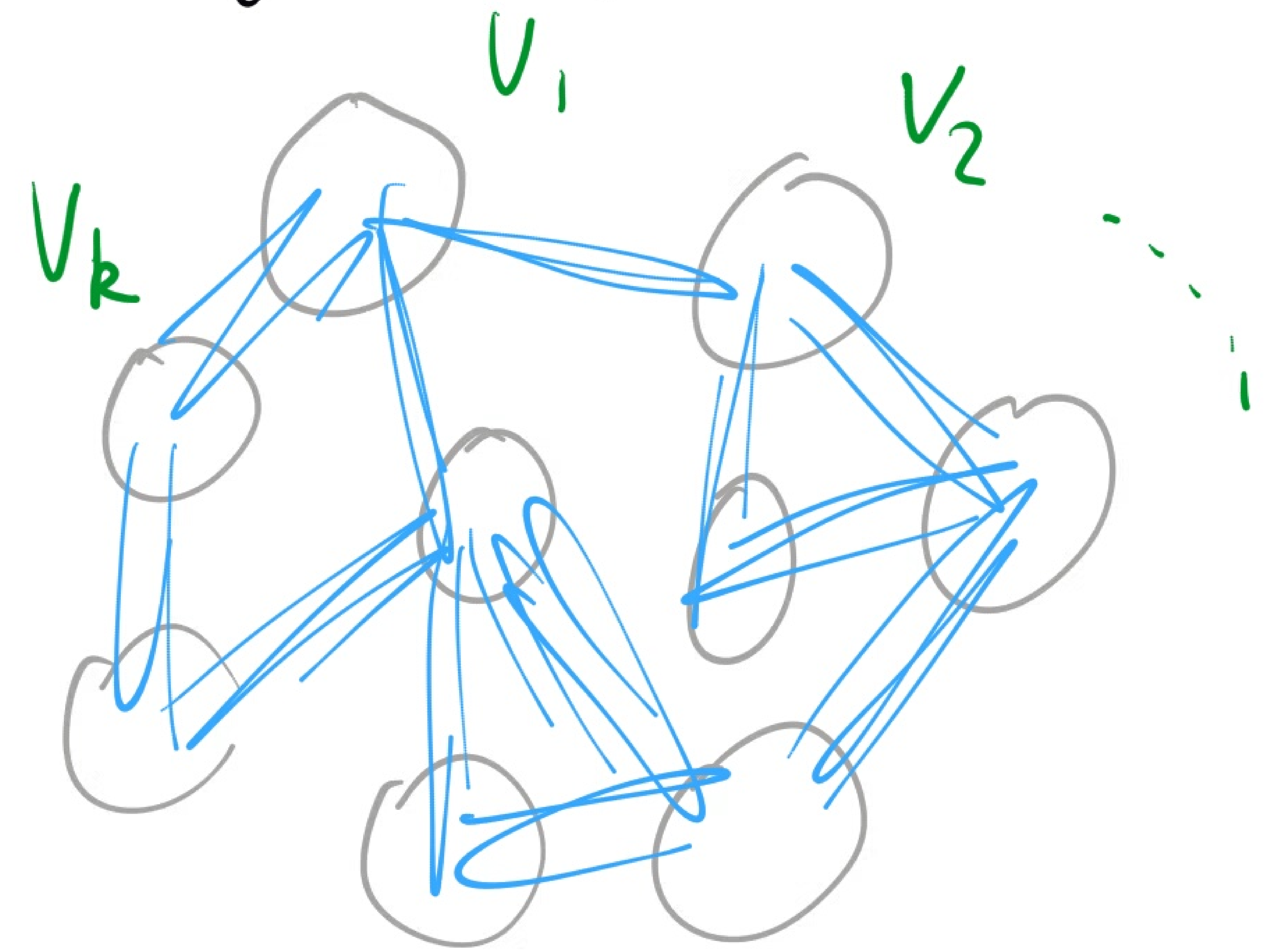
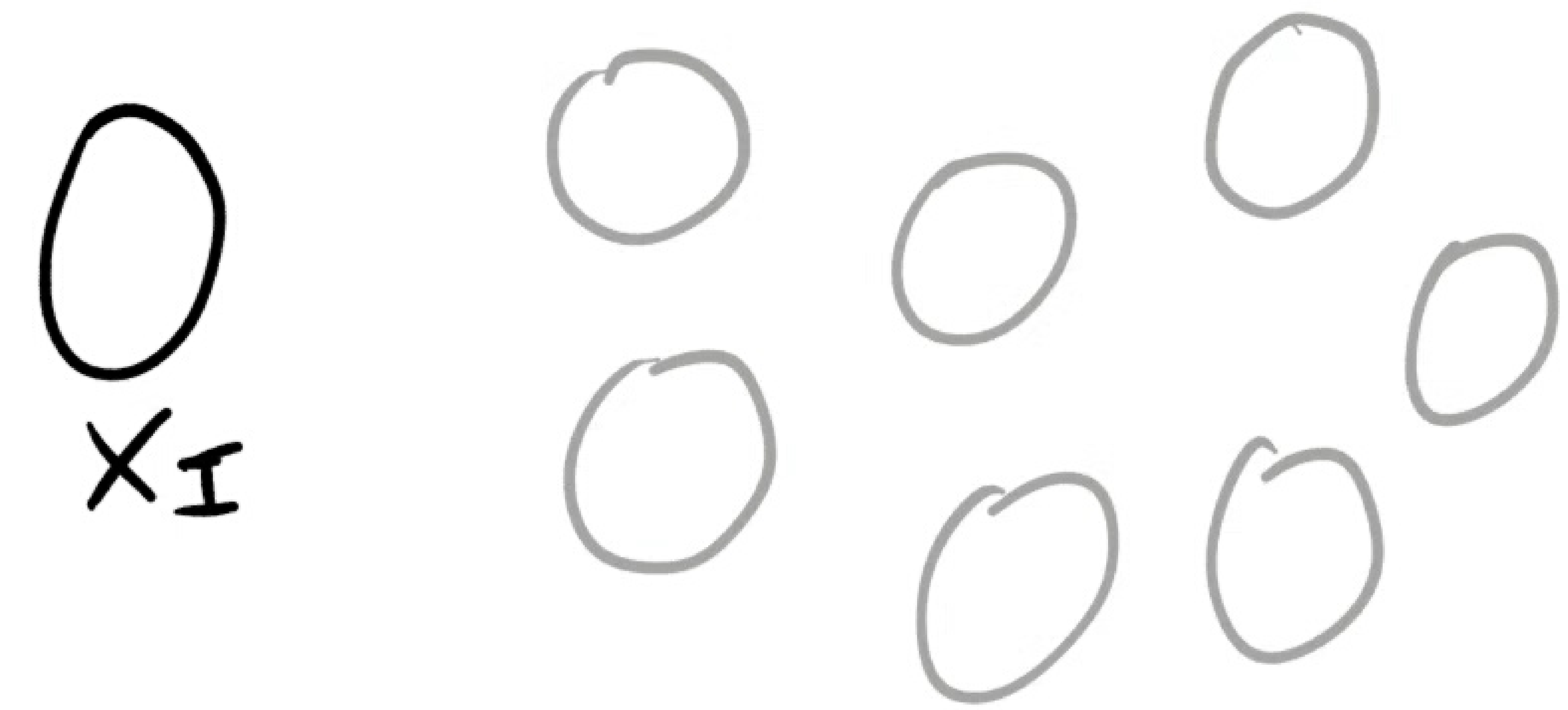


3) Partition

$$V(G) = \bigcup_{I \subseteq [k]} X_I \quad \text{where } X_I = \{x : I_x = I\}$$

4) Some $G[X_I]$ has large χ.

CASE I: $|I| \geq \frac{2k}{3}$



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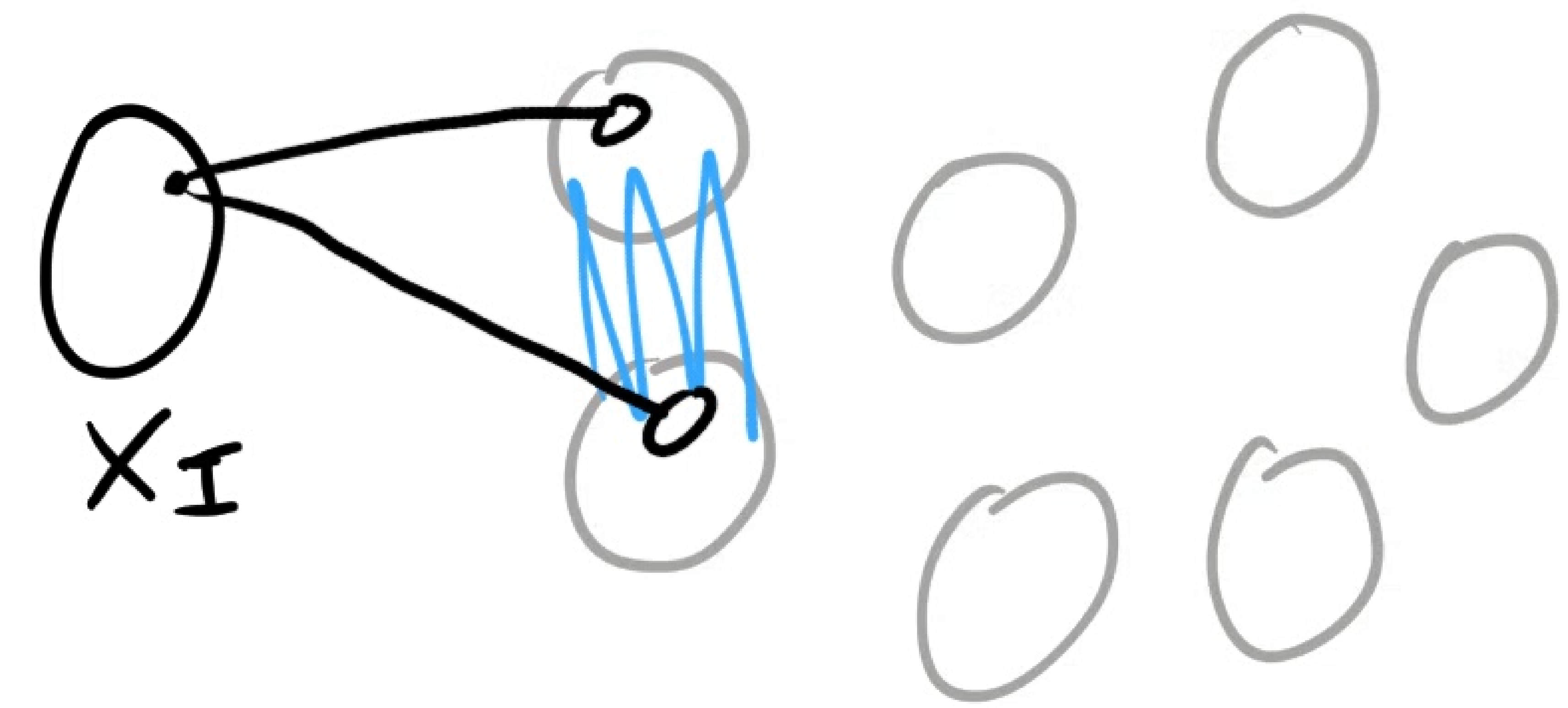

Each $x \in X_I$ sees +ve prop. of all $K_{t,t}$

By double counting $\exists K_{t,t}$ seen by t vertices

$$\Rightarrow \quad G \supseteq K_{t,t,t} \supseteq H$$

CASE II: $|I| < \frac{2k}{3}$ Let $V_I = \bigcup_{i \in I} V_i$

then $|\Gamma(x) \cap V_I| > \left(\frac{1}{2} + \delta\right)|V_I|$

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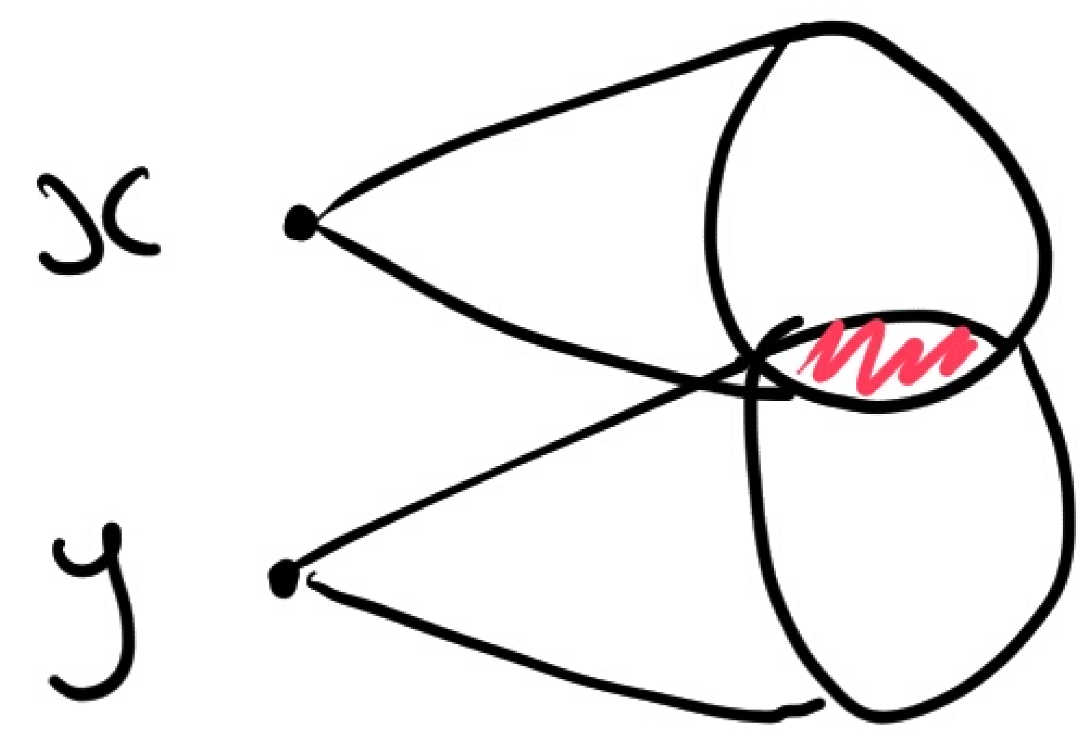

$\Rightarrow \forall x, y \in X_I \quad |\Gamma(x) \cap \Gamma(y)| > \delta n$

$\Rightarrow$ all edges xy in $G[X_i]$ see +ve prop. of t-sets

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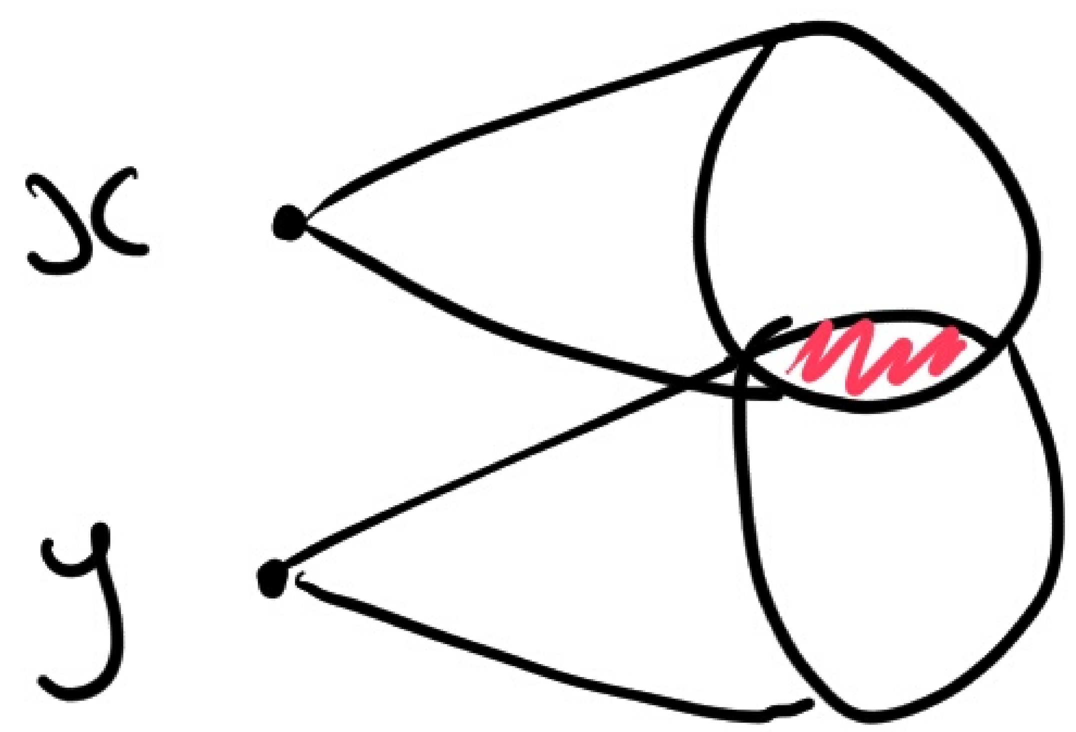

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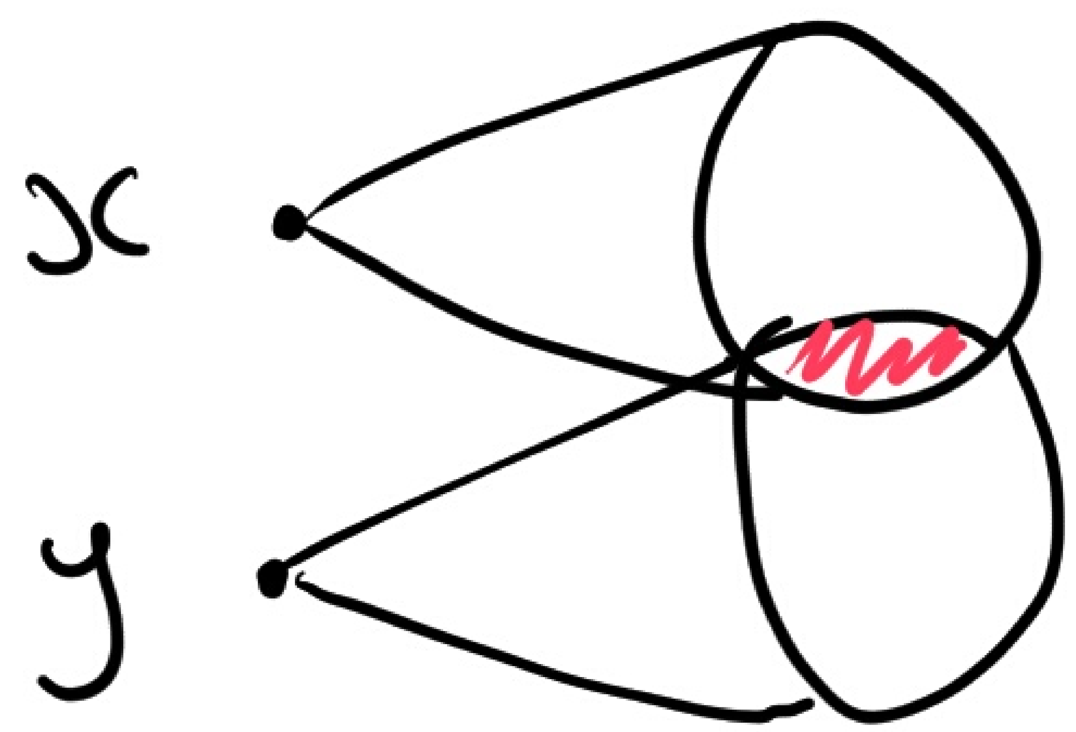

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$\Rightarrow \exists$ t-set seen by all edges of $G' \subseteq G[X_i]$ with $\delta(G')$ large

As $F \subseteq G'$ we have $H \subseteq G$

UPPER BOUNDS:

$$\delta_x(H) = 0 \quad \text{if } H \text{ is near-acyclic}$$

UPPER BOUNDS:

$\delta_\chi(H) = 0$ if H is near-acyclic

MORE CHALLENGING!

Need $\delta_\chi(G) \geq \delta n$ and $\chi(G)$ large
implies $G \supseteq H$.

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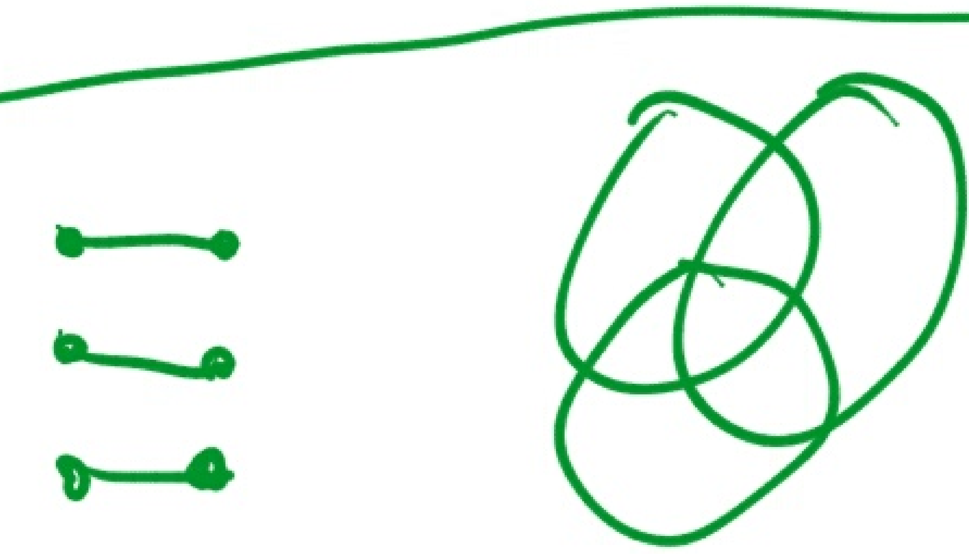
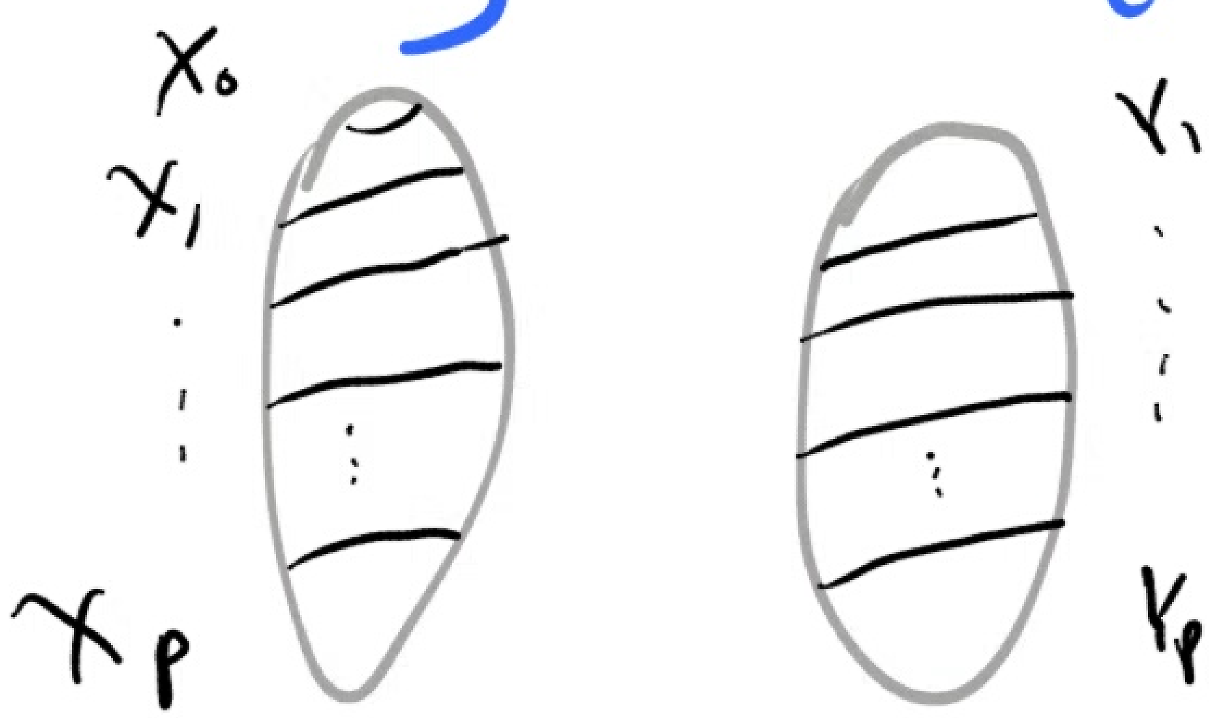


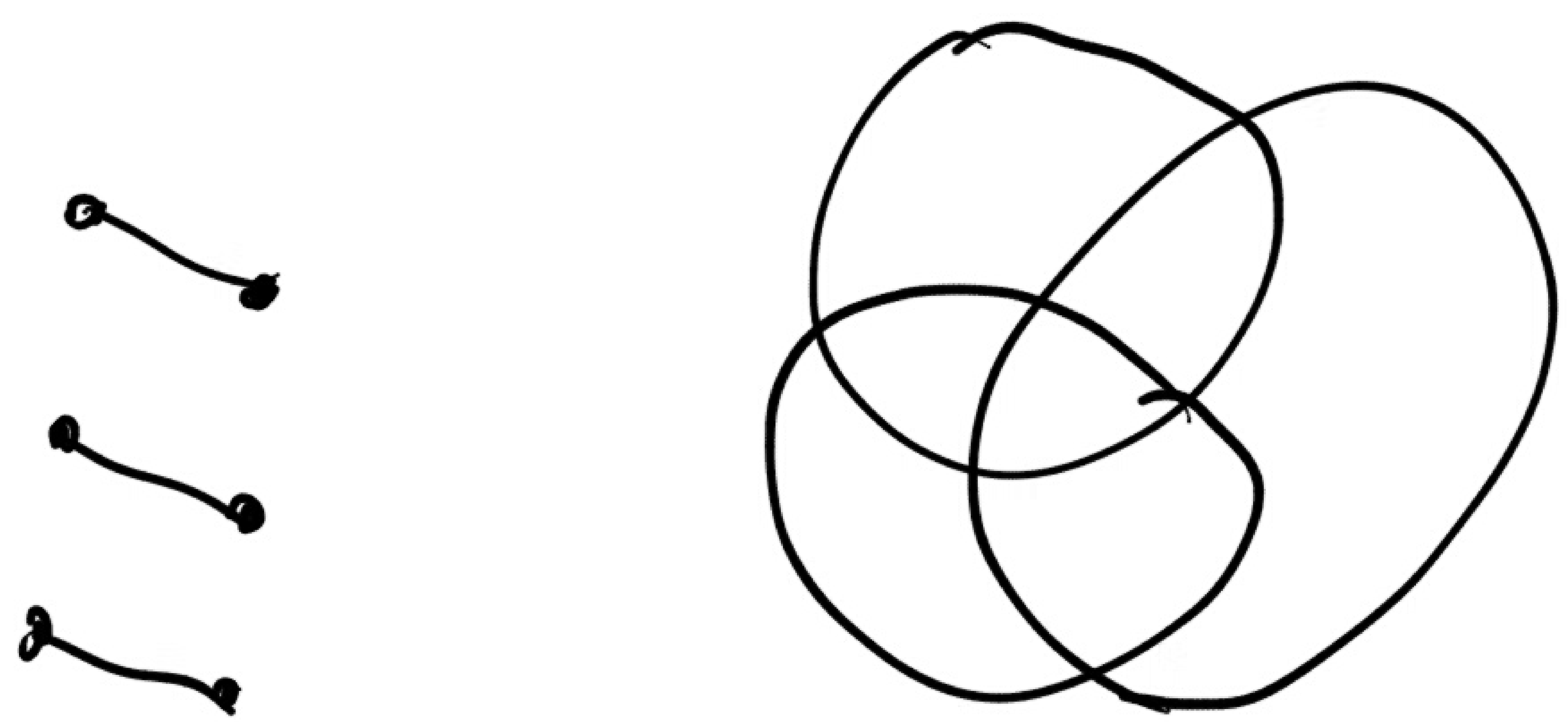
$\triangle$-free

Key idea of LT: Boosters



$$|\Gamma(x_i) \cap Y_i| \geq (1+\eta)\,\delta\,|Y_i|$$

If ∄ booster then $G \geq$



which gives all near-bipartite,

REAL CHALLENGE: DO WITH FOREST IN PLACE OF MATCHING

Thm: $\delta_\chi(H) \in \{0, \frac{1}{3}, \frac{1}{2}\}$, $\quad \leq \frac{1}{3} \iff$ forest graph

$(\chi(H) = 3)$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad = 0 \iff$ near-acyclic

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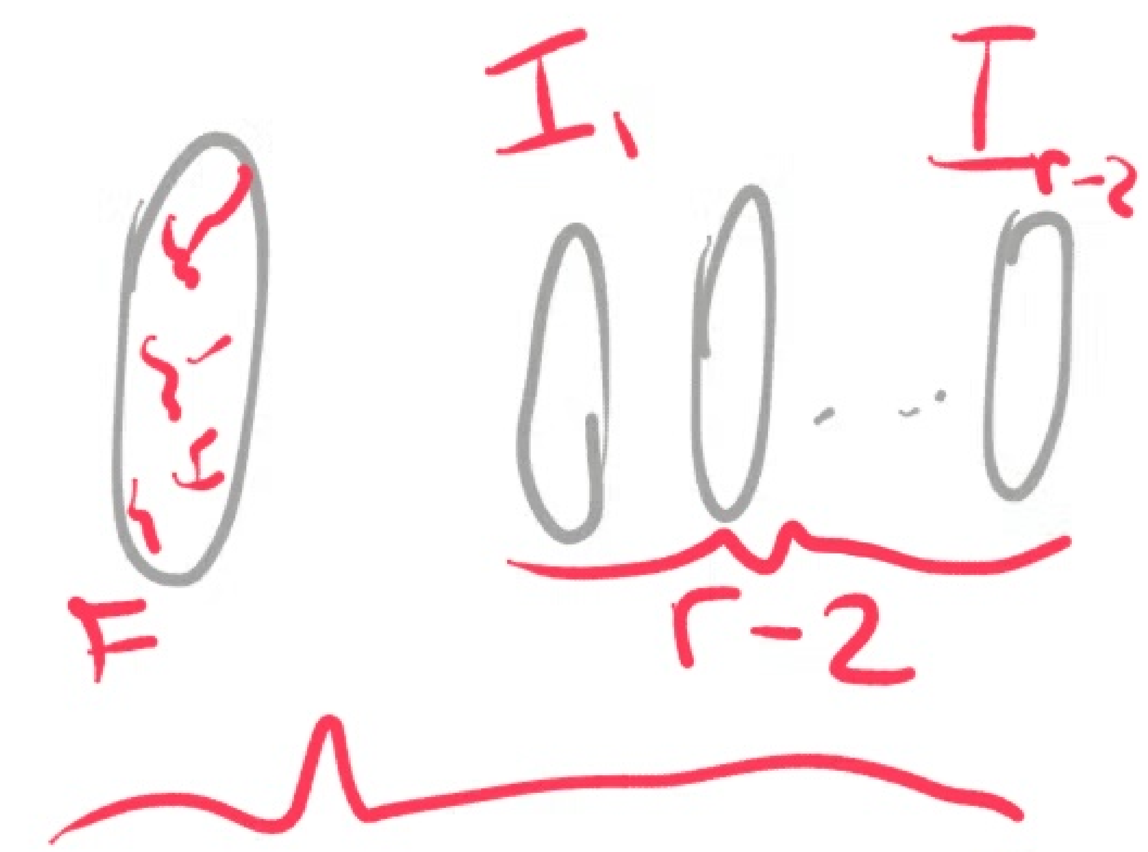
This result generalises to



Thm: $\delta_\chi(H) \in \left\{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1}\right\}$,
$(\chi(H)=r\geq 3)$

$\le \frac{2r-5}{2r-3}$ if H r-forest graph

$= \frac{r-3}{r-2}$ if H r-near-acyclic

no odd cycle with 1 vertex in I_1

Related open problems:

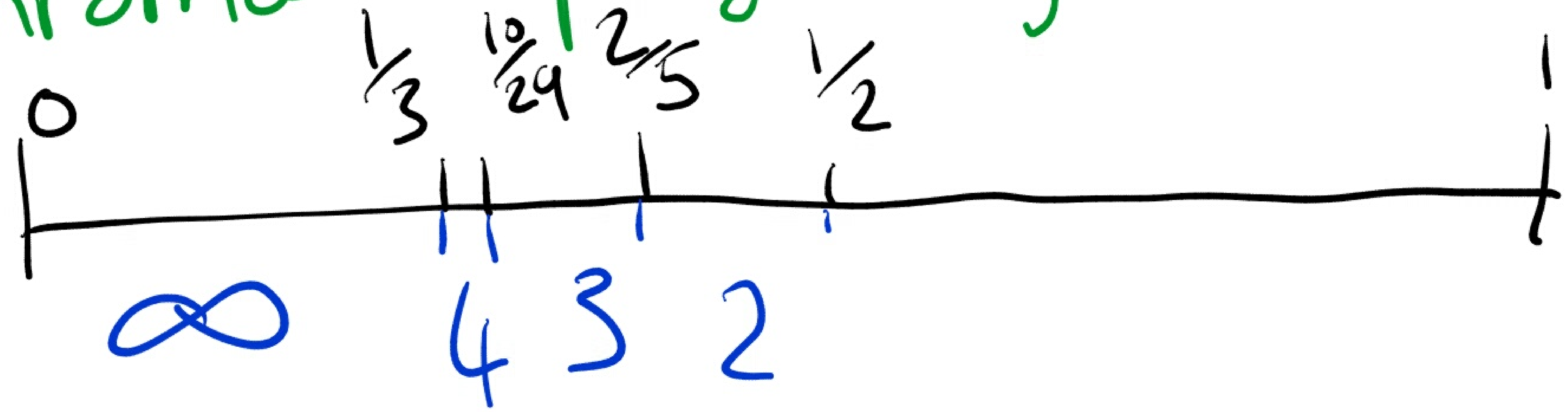
1) For H a forest graph, is $\max\left\{\chi(G) : \begin{array}{l} G \not\supseteq H \\ \delta(G) \geq \frac{n}{3} \end{array}\right\}$ always bounded? What bound?

Related open problems:

1) For H a forest graph, is $\max\left\{\chi(G) : \begin{array}{c} G \not\supseteq H \\ \delta(G) \geq \frac{n}{3} \end{array}\right\}$ always bounded? What bound?

2) Determine chromatic profile for H.

For Δ:



Related open problems:

1) For H a forest graph, is $\max\{\chi(G) : G \not\supseteq H,\ \delta(G) \geq \frac{n}{3}\}$ always bounded? What bound?

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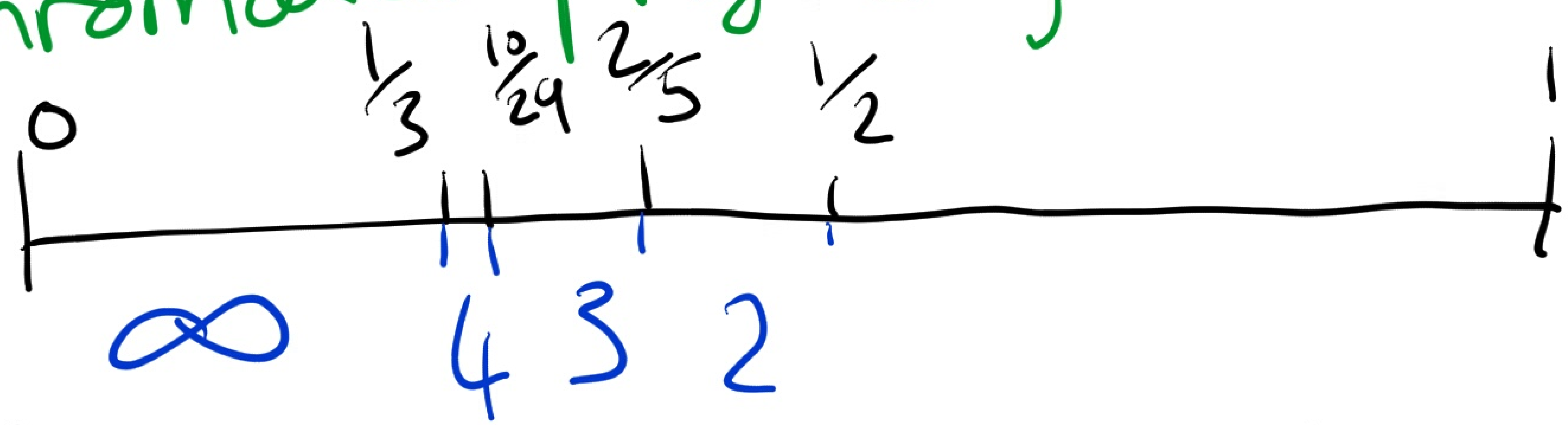

3) $\delta_\chi(\mathcal{F})$ for monotone families of graphs

$\delta_\chi(\mathcal{H})$ for hypergraphs (see Ballogh, Butterfield, Hu, Lenz, Mubayi 2016)

Chromatic thresholds in random graphs

Let $\delta_{\chi,p}(H) = \inf\{ d : \exists C,$ w.h.p. all H-free $G \subseteq G(n,p)$ with $\delta(G) \geq dpn$ have $\chi(G) \leq C \}$

Chromatic thresholds in random graphs

Let $\delta_{\chi,p}(H) = \inf\{ d : \exists C,\ \text{w.h.p. } \chi(G) \le C \text{ for all } H\text{-free } G \subseteq G_{n,p} \text{ with } \delta(G) \ge dpn \}$

General Results:

- $\delta_{\chi,p}(H) = \delta_\chi(H)$ for p constant

- $$\delta_{\chi,p}(H) = \begin{cases} \delta_\chi(H) & p = \Theta(1) \\ \frac{r-2}{r-1} & n^{-\frac{1}{m_2(H)}} \ll p \ll 1 \\ 1 & \frac{\log n}{n} \ll p \ll n^{-\frac{1}{m_2(H)}} \end{cases}$$

where $\chi(H) = r \ge 5$

SPECIFIC RESULTS:

$$\delta_{x,p}(K_3) = \begin{cases} 1/3 & p = \Theta(1) \\ 1/2 & n^{-1/2} \ll p \ll 1 \\ 1 & \frac{\log n}{n} \ll p \ll 1 \end{cases}$$

$$\delta_{x,p}(C_5) = \begin{cases} 0 & p = \Theta(1) \\ 1/3 & n^{-1/2} \ll p \ll 1 \\ 1/2 & n^{-3/4} \ll p \ll n^{-1/2} \\ 1 & \frac{\log n}{n} \ll p \ll n^{-3/4} \end{cases}$$

OPEN PROBLEMS:

- Determine $\delta_{\chi,p}(C_{2k+1})$, $k \geq 3$ and $n^{-\frac{2k-3}{2k-2}} \ll p \ll n^{-\frac{1}{2}}$
- For which graphs H do we have
  $$\delta_{\chi,p}(H) = 0 \quad \text{for } n^{-o(1)} < p = o(1) \quad ?$$
- What happens around thresholds?
  e.g. $\delta_{\chi,p}(H)$ for $p = c\, n^{-\frac{1}{m_2(H)}}$

¡شكرا

GRACIAS!

THANK YOU!

OBRIGADO!

MERCI!

ありがとう

Ευχαριστώ!

Dziękuję

DANKE!

Tack!

Ďakujem

謝謝

Grazie!

תודה

Köszönöm

고마워

СПАСИБО