# The triangle-free process 

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Nice summer school
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## A note on the notes

Quite a lot of these notes come from previous talks. You will notice the notes get more technical later on, and there may be a little repetition.

The notes are also incomplete at some points, including the parts we discussed on the board.

The exercises are included at the end.

## The Plan:

Lecture 1: Introduction to the triangle-free process, and a discussion of Bohman's result from 2008

Lecture 2: How to go further and achieve a result which is best possible (asymptotically).

## Ramsey Theory

Ramsey Theory is the theory of inevitable structure.
Let $R(k, \ell)$ be the least $n$ such that every graph on $n$ vertices contains either a clique on $k$ vertices or an independent set of $\ell$ vertices.

The diagonal Ramsey numbers satisfy:

$$
(\sqrt{2})^{k} \leqslant R(k, k) \leqslant 4^{k}
$$

Our focus will be on the off-diagonal Ramsey numbers $R(3, k)$

## Upper bounds on $R(3, k)$

Easy: $R(3, k) \leqslant k^{2}$

Proof: If there is a vertex of degree at least $k$, we either obtain a triangle or an independent set of size $k$, as required. If not then can construct an independent set of size $k$ inductively.

Ajtai, Komlos and Szemerédi (1980): $R(3, k) \leqslant 100 k^{2} / \log k$
Shearer (1983): $R(3, k) \leqslant(1+o(1)) k^{2} / \log k$

## Lower bounds on Ramsey numbers

Erdős proved the lower bound

$$
R(k, k) \geqslant 2^{k / 2}
$$

on the diagonal Ramsey number.
The idea: consider a random graph
How about $R(3, k)$ ?

## Lower bounds on $R(3, k)$

Either:

Generate a random graph and then remove the triangles........
Erdős (1961): $R(3, k) \geqslant c k^{2} /(\log k)^{2}$
Or:

Consider a random graph process that never contains a triangle in the first place

Kim (1995) and Bohman (2008): $R(3, k) \geqslant c k^{2} / \log k$

## The triangle-free process

The process starts with $G_{0}$ being the empty graph on $n$ vertices.
At each step thereafter $G_{m+1}$ is obtained from $G_{m}$ by adding an edge selected uniformly at random from those which would not create a triangle.

Let $G_{n, \Delta}$ denote the (random) final graph obtained by this process.

An instance of the triangle-free process

An instance of the triangle-free process


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An instance of the triangle-free process


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## Natural Questions:

1) How many edges does $G_{n, \triangle}$ have?
2) What does $G_{m}$ look like?
3) How large are the independent sets of $G_{n, \Delta}$ ?

## Bohman's Results

Theorem 1: The triangle-free process runs for at least $c n^{3 / 2} \sqrt{\log n}$ steps with high probability.

Theorem 2: $\alpha\left(G_{n, \Delta}\right) \leqslant C n^{1 / 2} \sqrt{\log n}$ with high probability.
Let us focus first on Theorem 1.
Challenges:
Going beyond the usual time scale of the differential equations method Tracking a large family of random variables (processes)

## What must we track?

Let $G_{m}$ be the graph after $m$ steps. We write $Q(m)$ for the number (and the set of) open edges in $G_{m}$. We use a time scaling $t=m n^{-3 / 2}$.
Why should we expect $Q(m)$ to be approximately $\binom{n}{2} e^{-4 t^{2}}$ ? (We will set $q=q(m)=e^{-4 t^{2}}$.)
Let us define for each open edge $u v \in Q(m)$ the values $X_{u, v}(m)$ and $Y_{u, v}(m)$ as the number of open edges incident to $u v$ in which the third side is open/an edge respectively.
What should be the behaviour of $X_{u, v}(m)$ and $Y_{u, v}(m)$ ?
Can we formalise our guesses in terms of differential equations?

## Relations between $X_{u, v}, Y_{u, v}$ and $Q$

$$
\begin{gathered}
\mathbb{E}\left[Q(m+1)-Q(m) \mid G_{m}\right]=-\frac{1}{Q(m)} \sum_{f \in Q(m)} Y_{f}(m) . \\
\mathbb{E}\left[Y_{u, v}(m+1)-Y_{u, v}(m) \mid G_{m}\right]=\frac{1}{Q(m)}\left(X_{u, v}(m)-\sum_{f \in Y_{u, v}(m)} Y_{f}(m)\right) . \\
\mathbb{E}\left[X_{u, v}(m+1)-X_{u, v}(m) \mid G_{m}\right]=1 \frac{1}{Q(m)} \sum_{f \in X_{u, v}(m)} Y_{f}(m) .
\end{gathered}
$$

We see that these random variables form a closed system and one may check that the differential equations method would suggest that they track

$$
\tilde{Q}(m)=\binom{n}{2} e^{-4 t^{2}} \quad, \quad \tilde{\gamma}=4 t n^{1 / 2} e^{-4 t^{2}} \quad \text { and } \quad \tilde{X}=2 n e^{-8 t^{2}} .
$$

## Error terms and martingales

We may use Freedman's inequality, or a special case which Bohman uses, if a supermartingale $M_{i}$ may decrease by at most $c_{-}$and increase by at most $c_{+}$then

$$
\mathbb{P}\left(M_{m}-M_{0}>\alpha\right) \leqslant \exp \left(\frac{-\alpha^{2}}{2 c_{-} c_{+} m}\right) .
$$

One may prove that the values $c_{-}=(\log n)^{2}$ and $c^{+}=4 / n$ may be used in this case.

## Error terms and martingales

We use the error terms:

$$
f_{q}(m)=\tilde{Q}(m) n^{-1 / 6} t^{-1} e^{41 t^{2}+40 t} \quad, \quad f_{y}(m)=\tilde{Y}(m) n^{-1 / 6} e^{41 t^{2}+40 t} \quad \text { and }
$$

$$
q_{x}(m)=\tilde{X}(m) n^{-1 / 6} e^{37 t^{2}+40 t}
$$

For example, one may verify that

$$
Y_{u, v}(m)-\tilde{Y}(m)-f_{y}(m)
$$

is a supermartingale. (Ok, really we should consider a stopped version, and all starting times $b$.)
One may prove that the values $c_{-}=(\log n)^{2}$ and $c^{+}=4 / n$ may be used in this case.

Therefore a deviation of the size of $f_{y}(m)$ has probability at most

$$
\exp \left(\frac{-f_{y}(m)^{2}}{n^{1 / 2}(\log n)^{3}}\right) \ll \exp \left(-n^{1 / 10}\right)
$$

provided $m$ is at most a small multiple of $n^{3 / 2}(\log n)^{1 / 2}$.

## Independent sets in $G_{m}$

The idea is to track the number of open edges in sets of size $k=C n^{1 / 2}(\log n)^{1 / 2}$.

Problem: Sometimes big shifts can occur.
Ideas?

## Lecture 2: The triangle-free process all the way.....

We have already seen Bohman's analysis of the triangle-free process
How long should the process last?
We guess until $m \approx(2 \sqrt{2})^{-1} n^{3 / 2} \sqrt{\log n}$.

## Results

Theorem 1: With high probability the triangle-free process lasts for

$$
\frac{(1+o(1))}{2 \sqrt{2}} n^{3 / 2} \sqrt{\log n}
$$

steps.
Theorem 2: With high probability the final graph has

$$
\alpha\left(G_{n, \Delta}\right) \leqslant(\sqrt{2}+\varepsilon) \sqrt{n \log n} .
$$

Corollary: The Ramsey number $R(3, k)$ satisfies:

$$
\frac{1-o(1)}{4} \frac{k^{2}}{\log k} \leqslant R(3, k) \leqslant(1+o(1)) \frac{k^{2}}{\log k} .
$$

Joint with Gonzalo Fiz Pontiveros and Rob Morris. Independently by Tom Bohman and Peter Keevash.

## Ideas involved in proof of Theorem

Theorem 1: With high probability the triangle-free process lasts for

$$
\frac{(1+o(1))}{2 \sqrt{2}} n^{3 / 2} \sqrt{\log n}
$$

steps.
Naturally, we must prove that $Q(m)$ stays close to the function $\tilde{Q}(m)=\binom{n}{2} e^{-4 t^{2}}$.
For that we need stability!
Not just stability, but STABILITY!

## Stability in Dynamical Systems

Consider the trajectory $x(t): t \geqslant 0$ (in $\mathbb{R}^{d}$ ) obtained when $x(t)$ starts at an initial position $x(0) \in \mathbb{R}^{d}$ and evolves according to some rule, e.g., according to a system of differential equations.

A point $z$ is called as equilibrium point if

$$
x(0)=z \quad \Rightarrow \quad x(t)=z \quad \text { for all } t \geqslant 0 .
$$

Stable equilibrium: An equilibrium $z$ is said to be stable if the implication

$$
\|x(0)-z\| \leqslant \varepsilon \quad \Rightarrow \quad x(t) \rightarrow z \quad \text { as } t \rightarrow \infty
$$

holds for some $\varepsilon>0$.
A stronger notion of stability would be that $x(t) \rightarrow z$ whatever the start point $x(0)$.

Examples


$$
\binom{\dot{x}}{\dot{y}}=\left(\begin{array}{cc}
0 & 1 \\
-1 & -2
\end{array}\right)\binom{x}{y}
$$

Examples


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$$

Examples


$$
\binom{\dot{x}}{\dot{y}}=\left(\begin{array}{ll}
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1 & 0
\end{array}\right)\binom{x}{y}
$$

Examples


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1 & 0 \\
0 & 1
\end{array}\right)\binom{x}{y}
$$

Examples


$$
\binom{\dot{x}}{\dot{y}}=\left(\begin{array}{cc}
-1 & 0 \\
0 & -1
\end{array}\right)\binom{x}{y}
$$

There is a theorem about stability in dynamical systems like these.

Theorem
Let $A$ be a $d \times d$ matrix with real entries. Then the origin is a stable point of the vector field on $R^{d}$ defined by $\dot{x}=A x$ if and only if all eigenvalues of $A$ have negative real part.

Remark: Rotation in $\mathbb{R}^{2}$, where both eigenvalues are pure imaginary, is a boundary case. The origin is not stable, yet the trajectory does not diverge. This system contains periodic orbits.

We do note give a proof of the theorem. You can get an intuitive idea by looking at the examples. The proof of stability in the case that all eigenvalues have negative real part depends on defining a Lyapunov function.

## Applications

Many stochastic processes are best understood as dynamical systems.
Random processes with re-inforcement including random walks with re-inforcement and generalised Pólya urns.
Benaïm proved that if the "randomness" of a stochastic process $X(0), X(1), \ldots$ is diminishing sufficiently quickly then the process converges to a trajectory in a dynamical system.
In fact the assumption is that

$$
\mathbb{E}[\Delta X(m)]=\frac{F(X(m))}{n} \pm R(m)
$$

for a summable sequence $R(m)$, and there exists a constant $K$ such that

$$
\operatorname{Var} \Delta X(m) \leqslant \frac{K}{n^{2}} \quad \text { for all } m \geqslant 1
$$

## Summary

Stability


1 -dimensional stability is simply self-correction.
A simple case of stability is that of conditional self-correction in every co-ordinate.

## Back to the triangle-free process

We want to discuss the evolution of $Q(m)$. We have

$$
\mathbb{E}\left[Q(m+1)-Q(m) \mid G_{m}\right]=-\bar{Y}(m)
$$

where $\bar{Y}(m)$ is the average of $Y_{f}(m)$ over $f \in Q(m)$.

$$
\mathbb{E}\left[\bar{Y}(m+1)-\bar{Y}(m) \mid G_{m}\right] \approx \frac{-\bar{Y}(m)^{2}+\bar{X}(m)-2 \operatorname{Var} Y_{e}(m)}{Q(m)}
$$

Let us ignore (for a moment) the $\bar{X}$ and variance terms. We have

$$
\mathbb{E}\left[\bar{Y}(m+1)-\bar{Y}(m) \mid G_{m}\right] \approx \frac{-\bar{Y}(m)^{2}}{Q(m)}
$$

The relationship between $Q$ and $\bar{Y}$ may be described by the following vector field.


Define

$$
g(t):=n^{-1 / 4} e^{2 t^{2}}(\log n)^{2}
$$

We prove that

$$
Q(m) \in(1 \pm g(t)) \tilde{Q}(m)
$$

and

$$
\bar{Y}(m) \in(1 \pm g(t)) \tilde{Y}(m)
$$

throughout the process.
Define $Q^{*}(m)$ and $Y^{*}(m)$ by

$$
Q(m)=\left(1+g(t) Q^{*}(m)\right) \tilde{Q}(m)
$$

and

$$
\bar{Y}(m)=\left(1+g(t) Y^{*}(m)\right) \tilde{Y}(m)
$$

The evolution of $Q^{*}(m)$ and $Y^{*}(m)$ is described by:

$$
\mathbb{E}\left[\Delta\binom{\dot{Q}^{*}}{\dot{Y^{*}}}\right]=\frac{4 t}{n^{3 / 2}}\left(\begin{array}{cc}
1 & -2 \\
2 & -3
\end{array}\right)\binom{x^{*}}{y^{*}}
$$

(up to terms that we are assuming are small)


Stable!
And using martingales we may prove that the probability of deviations in such a system is very small.

So, does that complete the proof??

## No!

We must also prove that $\operatorname{Var}\left(Y_{e}(m)\right)=o\left(\bar{Y}(m)^{2}\right)$ until $m=(1 / 2 \sqrt{2}-\varepsilon) n^{3 / 2} \sqrt{\log n}$.
Define $g_{y}(t):=n^{-1 / 4}(\log n)^{4} e^{2 t^{2}}$.
It suffices to prove that $Y_{e}(m) \in\left(1 \pm g_{y}(t)\right) \tilde{Y}$ for every open edge $e$.

Proof that $Y_{e}(m) \in\left(1 \pm g_{y}(t)\right) \tilde{Y}$ for every open edge $e$ Idea (as always): Self-correction!
In general, for $\sigma \in\{L, R\}^{|\sigma|}$, we may define $V_{e}^{\sigma}$ to be the average of $Y_{f}$, over $f$ that may be reached by a $\sigma$-walk from $e$ in the $Y$-graph.
Counting with multiplicity of course.
We can control $V_{e}^{\sigma}$ provided we can control (even more tightly) $V_{e}^{\sigma L}$ and $V_{e}^{\sigma R}$. But we cannot carry on like that forever.
Let $k=\left\lceil 3 \varepsilon^{-1}\right\rceil$. Say $\sigma$ is $k$-short if $\sigma$ contains no LLLL $\ldots L$, no $R R R R \ldots R$, and no subsequence $L \ldots R \ldots L \ldots \ldots R$.
We only claim we can track $V_{e}^{\sigma}$ if $\sigma$ is $k$-short.

We must prove that if $\sigma$ is (for example) $L R L R L R \ldots L R$, then taking a $\sigma$-walk from $e$ we arrive at an (approximately) uniform random open edge $f$.
Indeed this would imply that $\left|V_{e}^{\sigma}-\bar{Y}\right|=o\left(g_{y} \bar{Y}\right)$.


## Tracking everything

Definition: A graph structure $F$ has edges $E(F)$ and open edges $O(F)$. For each graph structure $F$ we will claim we can track the number of copies of $F$ in $G(m)$.
Well, actually we require that $E(F)$ contains no triangle and there is no triangle in $F$ using two edges and one open. If $F$ contains no such triangles we say $F$ is permissible.
Definition: A graph structure pair $(F, A)$ consists of a graph structure $F$ and a subset $A \subset V(F)$ which contains no edges or open edges.
Definition: A graph structure triple $(F, A, \phi)$ consists of a graph structure pair $(F, A)$ and an injective function $\phi: A \rightarrow V(G)$.

The function $\phi: A \rightarrow V(G)$ is called faithful if $F \cup E\left(G_{m}[\phi(A)]\right)$ is permissible.

## Examples:

## Examples:



## Examples:



## Examples:



## Examples:



## Tracking everything

Definition The tracking function of a graph structure pair $(F, A)$ is

$$
\tilde{N}(F, A)=n^{V(F)-|A|} p^{e(F)} q^{o(F)},
$$

where $p=2 m / n^{2}$ and $q=e^{-4 t^{2}}=e^{-4 m^{2} / n^{3}}$.
Definition The tracking time of a graph structure pair $(F, A)$ is the minimum $t$ such that

$$
\tilde{N}\left(F^{\prime}, A\right)=1
$$

for some substructure $F^{\prime} \subseteq F$.

Recall: A graph structure triple $(F, A, \phi)$ consists of a graph structure pair $(F, A)$ and an injective function $\phi: A \rightarrow V(G)$. We denote by $t_{A}(F)$ its tracking time.
Theorem: With high probability the following holds simultaneously for all ${ }^{\star}$ graph structure triples $(F, A, \phi)$.
(i) If $n^{3 / 2}<m \leqslant t_{A}(F) n^{3 / 2}$, then

$$
N_{\phi}(F)(m) \in(1 \pm o(1)) \tilde{N}_{A}(F)(m)
$$

(ii) If $m>t_{A}(F) n^{3 / 2}$, then

$$
N_{\phi}(F)(m) \leqslant(\log n)^{\Delta(F, H, A)} \tilde{N}_{H}(F)(m),
$$

where $A \subsetneq H \subseteq F$ is minimal such that $t<t_{H}(F)$.
Example: $X_{e}$ counts (twice)


Theorem: With high probability, for all $n^{3 / 2}<m \leqslant(1 / 2 \sqrt{2}-\varepsilon) n^{3 / 2} \sqrt{\log n}$ we have

$$
X_{e}(m) \in\left(1 \pm g_{x}(m)\right) \tilde{X}(m)
$$

for all pairs $e$ open in $G_{m}$, where

$$
\tilde{X}(m):=2 n e^{-8 t^{2}}
$$

and

$$
g_{x}(m):=10 n^{-1 / 4} e^{2 t^{2}}(\log n)^{4}=10 g_{y} .
$$

Observe that

$$
\mathbb{E}\left[\Delta X_{e}(m)\right] \in-\frac{2}{Q(m)} \sum_{f \in X_{e}(m)}\left(Y_{f}(m) \pm(\log n)^{3}\right)
$$

Set

$$
X_{e}^{*}(m):=\frac{X(m)-\tilde{X}(m)}{g_{x}(m) \tilde{X}(m)}
$$

(and assume $Y_{e}^{*}(m)$ and $Q^{*}(m)$ similarly defined) With this re-normalisation

$$
\mathbb{E}\left[\Delta X_{e}^{*}(m)\right] \in \frac{4 t}{n^{3 / 2}}\left(-X_{e}^{*}(m) \pm \frac{1}{10}\right),
$$

assuming everything else is still tracking.

Proof that $X_{e}(m) \in\left(1 \pm g_{x}(m)\right) \tilde{X}(m)$
Let us bound for all $b, s$ the probability of an escape $\left(\left|X_{e}^{*}(m)\right|>1\right)$ after $b+s$ steps, where $b$ was the last time we crossed the $\left|X_{e}^{*}(m)\right|=1 / 2$ line. We shall use the martingale bound

$$
\mathbb{P}\left(A_{s}-A_{0} \geqslant \frac{1}{2}\right) \leqslant \exp \left(\frac{-1}{8 \alpha \beta s+\alpha}\right)
$$

where

$$
\left|A_{i+1}-A_{i}\right| \leqslant \alpha \quad \text { a.s. }
$$

and

$$
\mathbb{E}\left[\left|A_{i+1}-A_{i}\right|\right] \leqslant \beta \quad \text { a.s. }
$$

The worst case for us is $s=n^{3 / 2} / t$. So it suffices to prove that

$$
8 \alpha \beta n^{3 / 2} \leqslant(\log n)^{-2}
$$

It suffices to prove that

$$
\max \left|\Delta X_{e}^{*}(m)\right| \cdot \max \left\lvert\, \mathbb{E}\left[\left|\Delta X_{e}^{*}(m)\right|\right] \leqslant \frac{1}{n^{3 / 2}(\log n)^{2}}\right.
$$

A bound on $\left|\Delta X_{e}^{*}(m)\right|$
It follows that

$$
\left|\Delta X_{e}(m)\right| \leqslant 4 t n^{1 / 2} e^{-8 t^{2}}
$$

for $n^{3 / 2} \leqslant m \leqslant \frac{1}{4} n^{3 / 2} \sqrt{\log n}$, and

$$
\left|\Delta X_{e}(m)\right| \leqslant(\log n)^{C}
$$

for $\frac{1}{4} n^{3 / 2} \sqrt{\log n} \leqslant m \leqslant\left(\frac{1}{2 \sqrt{2}}-\varepsilon\right) n^{3 / 2} \sqrt{\log n}$.
Thus

$$
\left|\Delta X_{e}^{*}(m)\right| \leqslant \frac{t n^{-1 / 4} e^{-2 t^{2}}}{\log n^{4}}
$$

for $n^{3 / 2} \leqslant m \leqslant \frac{1}{4} n^{3 / 2} \sqrt{\log n}$, and

$$
\left|\Delta X_{e}(m)\right| \leqslant \frac{(\log n)^{C}}{n^{3 / 4}\left(\log n^{4}\right) e^{-6 t^{2}}}
$$

for $\frac{1}{4} n^{3 / 2} \sqrt{\log n} \leqslant m \leqslant\left(\frac{1}{2 \sqrt{2}}-\varepsilon\right) n^{3 / 2} \sqrt{\log n}$.

Bounding $\mathbb{E}\left[\left|\Delta X_{e}^{*}(m)\right|\right]$ is easier
We have

$$
\mathbb{E}\left[\left|\Delta X_{e}^{*}(m)\right|\right] \leqslant \frac{C}{g_{x}(m)} \cdot \frac{\log n}{n^{3 / 2}}
$$

And so the product $\left|\Delta X_{e}^{*}(m)\right| \cdot \mathbb{E}\left[\left|\Delta X_{e}^{*}(m)\right|\right]$ is at most

$$
\frac{t n^{-1 / 4} e^{-2 t^{2}}}{\log n^{4}} \cdot \frac{C}{g_{x}(m)} \cdot \frac{\log n}{n^{3 / 2}} \leqslant \frac{1}{n^{3 / 2}(\log n)^{2}}
$$

for $n^{3 / 2} \leqslant m \leqslant \frac{1}{4} n^{3 / 2} \sqrt{\log n}$, and

$$
\frac{(\log n)^{C}}{n^{3 / 4}\left(\log n^{4}\right) e^{-6 t^{2}}} \cdot \frac{C}{g_{x}(m)} \cdot \frac{\log n}{n^{3 / 2}} \leqslant \frac{(\log n)^{C}}{n^{2} e^{-4 t^{2}}}
$$

for $\frac{1}{4} n^{3 / 2} \sqrt{\log n} \leqslant m \leqslant\left(\frac{1}{2 \sqrt{2}}-\varepsilon\right) n^{3 / 2} \sqrt{\log n}$.

## Upper bounds

We prove that degrees may only increase by $o(\sqrt{n} \sqrt{\log n})$ after we stop tracking the process.
We may also prove that

$$
\alpha\left(G_{n, \Delta}\right) \leqslant \alpha\left(G_{m *}\right) \leqslant(\sqrt{2}+o(1)) \sqrt{n} \sqrt{\log n} .
$$

## Ramsey Theory

Our bound on the independence number of $G_{n, \Delta}$ implies that

$$
R(3, k) \geqslant\left(\frac{1}{4}-o(1)\right) \frac{k^{2}}{\log k} .
$$

Together with the upper bound of Shearer, this gives that

$$
\left(\frac{1}{4}-o(1)\right) \frac{k^{2}}{\log k} \leqslant R(3, k) \leqslant(1+o(1)) \frac{k^{2}}{\log k} .
$$

## Open Problems

Improve the upper bound on $R(3, k)$.
Obtain similar results for the $H$-free process.

## Exercises

The exercises start on the next page. This initial discussion may be helpful.
One thing we will see on Monday is that it is sometimes necessary to go beyond the usual scaling of time that occurs in the differential equations method. Indeed, this was the key to the advance achieved by Bohman.
In the first exercise we shall do a similar analysis for a slightly contrived problem. There then follow some variants of this problem. Finally, there is an exercise which is very similar in nature to the triangle-free process and will allow us to practise applying the approach shown in the course to a similar problem.
We will generally use Freedman's inequality rather than Hoeffding-Azuma. The sum which occurs in the denominator in the
Hoeffding-Azuma inequality is $\sum_{i=1}^{m} c_{i}^{2}$ where the martingale increments $X_{i}$ are such that $\left|X_{i}\right| \leqslant c_{i}$ almost surely. This sum is an upper bound on the variance of the martingale. In Freedman's inequality we effectively get to replace this sum by a quantity related to variance.
Freedman's inequality Let $\left(S_{m}\right)_{m=0}^{M}$ be a supermartingale with increments $\left(X_{i}\right)_{i=1}^{M}$ with respect to a filtration $\left(\mathcal{F}_{m}\right)_{m=0}^{M}$, let $R \in \mathbb{R}$ be such that $\max _{i}\left|X_{i}\right| \leqslant R$ almost surely, and let

$$
V(m):=\sum_{i=1}^{m} \mathbb{E}\left[\left|X_{i}\right|^{2} \mid \mathcal{F}_{i-1}\right]
$$

Then, for every $\alpha, \beta>0$, we have

$$
\mathbb{P}\left(S_{m}-S_{0} \geqslant \alpha \quad \text { and } \quad V(m) \leqslant \beta \quad \text { for some } m\right) \leqslant \exp \left(\frac{-\alpha^{2}}{2(\beta+R \alpha)}\right)
$$

1. Consider a random Battle Royale which starts with $N_{0}=N$ participants numbered 1, .., N. At each time step a number $k \in[N]=\{1, \ldots, N\}$ is selected uniformly at random, and player $k$ (if they are still alive) shoots one other player. If player $k$ has already died then nothing happens. Let us write $N_{i}$ for the number of participants alive after $i$ steps.
(a) We shall write $\bar{N}_{i}$ for $\mathbb{E}\left[N_{i}\right]$ show that $\bar{N}_{i}=N(1-1 / N)^{i} \approx N e^{-i / N}$.
(b) We now aim to show that $N_{i}$ stays close to $\bar{N}_{i}$ throughout the process. Specifically, we aim to show that (with high probability)

$$
\bar{N}_{i}-f_{i} \leqslant N_{i} \leqslant \bar{N}_{i}+f_{i} \quad \text { for all } i \geqslant 0
$$

for some sequence $f_{i}$. We define the event $E_{s}^{b}$ to be the event that $N_{i}$ becomes larger than $\bar{N}_{i}+f_{i}$ at step $b+s$ after having increased from $\bar{N}_{i}+\frac{2}{3} f_{i}$ at step $b_{i}$ (and never falling below). More formally, let $E_{s}^{b}$ be the event that
(i) $N_{b-1}<\bar{N}_{b-1}+\frac{2}{3} f_{b-1}$
(ii) $N_{b+i} \geqslant \bar{N}_{b+i}+\frac{2}{3} f_{b+i}$ for all $i=0, \ldots s$, and
(iii) $s$ is minimal such that $N_{b+s}>\bar{N}_{b+s}+f_{b+s}$

Show that the event that $N_{i}>\bar{N}_{i}+f_{i}$ for some $i \leqslant N \log N$ is contained in the union $\bigcup_{b, s: b+s \leqslant N \log N} E_{s}^{b}$.
(c) We now define a sequence of random variables associated with these deviation events. We wish to study

$$
N_{b+i}-\bar{N}_{b+i}-f_{b+i}
$$

as this sequence becomes positive exactly when we break the inequality $N_{i} \leqslant \bar{N}_{i}+f_{i}$. However, it is useful to stop the process if we ever fall too far. Let $\tau$ be the stopping time defined to be the minimum $i$ such that

$$
N_{b+i}<\bar{N}_{b+i}+\frac{2}{3} f_{b+i}
$$

We write $i \wedge \tau$ for the minimum of $i$ and $\tau$ (so that effectively the process is halted at the stopping time). Let

$$
z_{i}^{b}:=N_{b+i \wedge \tau}-\bar{N}_{b+i \wedge \tau}-f_{b+i \wedge \tau}
$$

Show that $E_{s}^{b}$ is contained in the event that

$$
z_{s}^{b}>z_{0}^{b}+\frac{1}{3} f_{b}-\left|f_{b}-f_{b-1}\right|-1
$$

(When using this in the rest of the question feel free to ignore the $\left|f_{b}-f_{b-1}\right|$ and the -1 , they really don't matter).
(d) We shall define the sequence $f_{i}$ as follows:

$$
f_{i}:=N^{1 / 2}(\log N)\left(1-\frac{1}{2 N}\right)^{i} \approx N^{1 / 2}(\log N) e^{-i / 2 N}
$$

Show that (with this choice of $f_{i}$ ) the sequence $Z_{i}^{b}: i \geqslant 0$ is a supermartingale.
(e) Using Freedman's inequality show that $\mathbb{P}\left(E_{s}^{b}\right) \leqslant \exp \left(-c(\log N)^{2}\right)$ for some constant $c>0$ and for all $b+s \leqslant N \log N$.
(f) Show that with high probability

$$
\bar{N}_{i}-f_{i} \leqslant N_{i} \leqslant \bar{N}_{i}+f_{i} \quad \text { for all } i \geqslant 0 .
$$

(g) Would it be possible to replace $f_{i}$ by some sequence that is significantly smaller?
2. This question is a variant of first question. Suppose now there are two opposing armies (not necessarily the British and the French ;). This time one number is selected on each side at each step. When a soldier is activated he shoots someone in the opposing army. Let $M_{i}$ and $N_{i}$ denote the number of soldiers left in each army after $i$ steps, and suppose $M_{0}=N_{0}=N$. We wish to show that both $M_{i}$ and $N_{i}$ remain somewhat close to $\bar{N}_{i}$ for some time. We will use an error function of the form

$$
f_{i}=N^{1 / 2}(\log N) e^{\eta i / N}
$$

for some $\eta \in \mathbb{R}$. For which values of $\eta$ is it possible to prove that with high probability

$$
\bar{N}_{i}-f_{i} \leqslant M_{i}, N_{i} \leqslant \bar{N}_{i}+f_{i} \quad \text { for all } i \geqslant 0 ?
$$

3. Now consider yet another variant. Again we have two opposing armies. However, now an activated soldier shoots someone on the other side with probability $\alpha$ and his own side with probability $1-\alpha$ for some $\alpha \in(0,1)$. Let $\eta(\alpha)$ be the smallest value of $\eta$ we could take in question 2 for this variant of the game. Find $\eta(\alpha)$ for all $\alpha \in(0,1)$.
$\left(^{*}\right)$ 4. How about if there are $k$ equal sized armies and the shooting probabilities are given by some doubly stochastic $k \times k$ matrix $A$ ?
4. This final question will be based on the approach described on Monday. Let us consider the bipartite $C_{4}$-free process. In other words we start with two sets $A, B$ of $n$ vertices and we shall only ever add edges between the two sides. The process runs by adding at each step a uniformly random edge that can be added without creating a $C_{4}$. Let us define $p=p(i)=i / n^{2}$ to be the density after $i$ steps, it may also be useful to use the scaled time $t=i n^{-4 / 3}$ and $q=q(i)=e^{-i^{3} / n^{4}}=e^{-t^{3}}$ which will be approximately the probability that a pair is open ater $i$ steps.
(a) Let $Q(i)$ be the number of open edges in $G_{i}$. Explain why we should expect $Q(i) \approx n^{2} q=n^{2} e^{-t^{3}}$.
(b) Given a pair $u \in A, v \in B$ let $X_{u, v}$ be the number of copies of $C_{4}$ containing $u$ and $v$ such that $G_{i}$ contains one of the other three edges and the remaining two are open. Explain why we should expect $X_{u, v}(i) \approx 3 p n^{2} q^{2}=3 t n^{4 / 3} e^{-2 t^{3}}$.
(c) Given a pair $u \in A, v \in B$ let $Y_{u, v}$ be the number of copies of $C_{4}$ containing $u$ and $v$ such that $G_{i}$ contains two of the other three edges and the remaining edge is open. Explain why we should expect $Y_{u, v}(i) \approx 3 p^{2} n^{2} q=3 t^{2} n^{2 / 3} e-t^{3}$.
(d) Assume ${ }^{1}$ (unrealistically) that the approximate values in (a) and (b) hold precisely, prove that with high probability the approximation in (c) holds for all time, up to an error of the form $f_{i}=n^{3 / 5} e^{C t^{3}}$, for some constant $C$.
(e) Now assume (d), show deterministically that the process runs for at least $c n^{4 / 3}(\log n)^{1 / 3}$ steps for some constant $c>0$.
(f) Conjecture the correct (optimal) value of the constant $c$.
