Exercises

July 11, 2019

1. Let P, π be a reversible finite Markov chain. Let $A \subsetneq \Omega$ and let $B = A^c$ with $k = |B|$. Suppose that the sub-stochastic matrix P_B (the restriction of P to B, i.e. $P_B(x,y) = P(x,y)$ for $x, y \in B$) is irreducible, in the sense that for all $x, y \in B$, there exists $n \geq 0$ such that $P_B^n(x,y) > 0$.

(i) Show that P_B has k real eigenvalues

$$1 \geq \gamma_1 > \gamma_2 \geq \ldots \geq \gamma_k.$$

(ii) Show that there exist nonnegative numbers $a_1, \ldots, a_k$ satisfying $\sum_i a_i = 1$ such that for all $t \geq 0$ we have

$$\mathbb{P}_{\pi_B}(\tau_A > t) = \sum_{i=1}^{k} a_i \gamma_i^t$$

(iii) The Perron Frobenius theorem gives that $\gamma_1 > 0$ and $\gamma_1 \geq -\gamma_k$. Using the Courant-Fischer characterisation of eigenvalues or otherwise establish that

$$\gamma_1 \leq 1 - \frac{\pi(A)}{t_{\mathrm{rel}}}.$$

(iv) Deduce that $\mathbb{P}_{\pi_B}(\tau_A > t) \leq \left(1 - \frac{\pi(A)}{t_{\mathrm{rel}}}\right)^t \leq \exp\left(-\frac{t\pi(A)}{t_{\mathrm{rel}}}\right)$.

(v) By the Perron Frobenius theorem the eigenvector v corresponding to $\gamma_1 > 0$ is strictly positive. Let α be the probability distribution given by $\alpha = v / \sum_i v(i)$. Show that when the starting distribution is α, then the law of τ_A is geometric with parameter γ_1.

Prove that for all t and all y

$$\mathbb{P}_\alpha(X_t = y \mid \tau_A > t) = \alpha(y).$$

Finally show that for all $x \notin A$ we have

$$\mathbb{P}_x(X_t = y \mid \tau_A > t) \to \alpha(y) \text{ as } t \to \infty.$$

(The distribution α is called the quasi-stationary distribution.)

2. Let X be a reversible Markov chain on a finite state space E with transition matrix P and invariant distribution π. Prove the Poincaré inequality, i.e. that for all functions $f : E \to \mathbb{R}$

$$\mathrm{Var}_\pi(P^t f) \leq e^{-2t/t_{\mathrm{rel}}} \mathrm{Var}_\pi(f).$$

3. Let X be a reversible Markov chain on a finite state space with transition matrix P and invariant distribution π.

(i) Prove that for all x, y

$$\frac{P^{2t}(x,y)}{\pi(y)} \geq \left(1 - \max_{z,w} \left\|P^t(z,\cdot) - P^t(w,\cdot)\right\|_{\mathrm{TV}}\right)^2.$$

Deduce that

$$P^{2t_{\mathrm{mix}}}(x,y) \geq \frac{1}{4}\pi(y)$$

and that there exists a transition matrix $\widetilde{P}$ such that

$$P^{2t_{\mathrm{mix}}}(x,y) = \frac{1}{4}\pi(y) + \frac{3}{4}\widetilde{P}(x,y)$$

(ii) Let $t_{\mathrm{stop}} = \max_x \min\{\mathbb{E}_x[\Lambda_x] : \Lambda_x \text{ is a stopping time s.t. } \mathbb{P}_x(X_{\Lambda_x} \in \cdot) = \pi(\cdot)\}$. By defining an appropriate stationary time, prove that

$$t_{\mathrm{stop}} \leq 8 t_{\mathrm{mix}}.$$

4. Let X be an irreducible, lazy and reversible Markov chain on a finite state space with transition matrix P and stationary distribution π.

(i) Show that

$$\mathbb{E}_\pi[\tau_\pi] := \sum_{x,y} \pi(x)\pi(y)\mathbb{E}_x[\tau_y] = \sum_{i\geq 2} \frac{1}{1-\lambda_i}.$$

where (λ_i) are all the eigenvalues.

(ii) Show that

$$\sum_{t=k}^{\infty} (P^t(x,x) - \pi(x)) \leq e^{-k/t_{\mathrm{rel}}} \pi(x) \mathbb{E}_\pi[\tau_x].$$

5. For each t let U_t be a uniform random variable on $\{1,\ldots,t\}$ and let Z_t be a geometric random variable of parameter $1/t$. The Cesaro mixing time is defined to be

$$t_{\mathrm{Ces}} = \inf\{t \geq 0 : \max_x \|\mathbb{P}_x(X_{U_t} \in \cdot) - \pi\|_{\mathrm{TV}} \leq 1/4\}.$$

The geometric mixing time is defined to be

$$t_{\mathrm{G}} = \inf\{t \geq 0 : \max_x \|\mathbb{P}_x(X_{Z_t} \in \cdot) - \pi\|_{\mathrm{TV}} \leq 1/4\}.$$

Show that there exist two constants c_1 and c_2 so that for all chains we have

$$c_1 t_{\mathrm{G}} \leq t_{\mathrm{Ces}} \leq c_2 t_{\mathrm{G}}.$$

(You may assume the following form of "sub-multiplicativity": if $d_G(t) \leq \alpha < 1/2$, then there exists a constant c so that $d_G(ct) \leq 1/4$.)

6. This exercise shows that for non-reversible chains t_{stop}, t_{H} and t_{mix} can be of different order.

(i) Let X be an irreducible Markov chain with transition matrix P and invariant distribution π and let

$$\mathbb{E}_a[\tau_\pi] := \sum_x \pi(x)\mathbb{E}_a[\tau_x].$$

Show that $\mathbb{E}_a[\tau_\pi]$ is independent of a. (This is called the random target lemma.)

(ii) Consider the random walk on the greasy ladder, which is a Markov chain with state space $S = \{1, \dots, n\}$ and transition matrix $P(i, i+1) = \frac{1}{2} = 1 - P(i, 1)$ for $i = 1, \dots, n-1$ and $P(n, 1) = 1$. Check that the invariant distribution satisfies

$$\pi(i) = \frac{2^{-i}}{1 - 2^{-n}}$$

Show that the mixing time of the lazy chain is of order 1. Show that $t_{\text{stop}} \asymp n$.

You can use the following that will be proved in the lectures tomorrow.

Definition Let S be a stopping time. A state z is called a **halting** state for the stopping time if $S \le T_z$ a.s. where T_z is the first hitting time of state z.
Theorem [Lovász and Winkler] Let μ be a distribution. Let S be a stopping time such that $\mathbb{P}_\mu(X_S = x) = \pi(x)$ for all x. Then S is mean optimal in the sense that

$$\mathbb{E}_\mu[S] = \min\{\mathbb{E}_\mu[U] : U \text{ is a stopping time s.t. } \mathbb{P}_\mu(X_U \in \cdot) = \pi(\cdot)\}$$

if and only if it has a halting state.

(iii) Let $\mathbb{Z}_n = \{1, 2, \dots, n\}$ denote the n-cycle and let $P(i, i+1) = \frac{2}{3}$ for all $1 \le i < n$ and $P(n, 1) = \frac{2}{3}$. Also $P(i, i-1) = \frac{1}{3}$, for all $1 < i \le n$, and $P(1, n) = \frac{1}{3}$. Find the order of the mixing time of the lazy version of the chain and also of t_{stop} and $\max_{x, A : \pi(A) \ge 1/4} \mathbb{E}_x[\tau_A]$.