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TOWARDS THE PROOF OF ERDÖS-STRAUS CONJECTURE

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Abstract :

In this paper we present new hints for a proof of the Erdös-Straus conjecture of 1948 using the 1 and 2-twin Pythagorean triples.

Keywords: Erdös-Straus conjecture; twin Pythagorean triples

MSC: 11A41

Introduction:

In the previous paper, we give some facts about the life of Paul Erdős (March 26, 1913 in Budapest, Hungary; September 20, 1996 in Warsaw, Poland). The only possession that mattered to him was his little book [2]. He was a prolific researcher in any discipline, with more than 1,500 research articles published. Let us recall one of the favorite maxims of Erdős : "Sometimes you have to complicate a problem to simplify the solution" [1]. The following study of the d-twin primitive Pythagorean triples and their introduction in the context of the Erdös-Straus conjecture is perhaps an example.

The Erdős-Straus conjecture implies that any rational number 4 / n, with n an integer greater than or equal to two, can be written as a sum of three unit fractions, that is to say that there are three natural numbers not zero x, y and z such that: $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$ We propose a road for the proof of this conjecture using the 1 and 2-twin Pythagorean triples.

The definition of d-twin Pythagorean triples :

We introduced the notion of d-twin irreducible Pythagorean triples who are of the form: $F=[n, \sqrt{d(2n+d)}, n+d]$;

for F to be irreducible it is necessary that d(2 n + d) is a square and d is the double of a square or a square of an odd number.

The first possibilities for d are: $\{1, 2, 8, 9, 18, 25, 32, 49, 50, 72, 81, 98, 121, 128\}$.

If d=1 we have: $F1 := [2n(n+1), 1+2n, 2n^2+2n+1]$.

Examples: $[4, 3, 5], [12, 5, 13], [24, 7, 25], [40, 9, 41], [60, 11, 61]$.

We see easily that each 1-twin Pythagorean triple can be transformed to a

$2n^2$- twin Pythagorean triple.

If d=2 we have: $F2 := [4n^2-1, 4n, 4n^2+1]$.

Examples: $[3, 4, 5], [15, 8, 17], [35, 12, 37], [63, 16, 65], [99, 20, 101]$.

We see easily that each 2-twin Pythagorean triple can be transformed to a

$(2n-1)^2$- twin Pythagorean triple.

If d=8 we have: $F8 := [16n^2+8n-3, 4+16n, 16n^2+8n+5]$.

Example:
$[21, 20, 29], [77, 36, 85], [165, 52, 173], [285, 68, 293], [437, 84, 445], [621, 100, 629]$.

But for d=9, 18, 25, … we have, in general, several lists of d-twin irreducible triples which are not too difficult to find.

About the Pythagorean triples:

Theorem 1:

It is said that three numbers a, b and c integers form a Pythagorean triple if they satisfy the relation: $a^2 + b^2 = c^2$. [6,7]

Theorem 2:

(a, b, c) is a Pythagorean triple if and only if for any nonzero integer n, (na; nb; nc) is also a Pythagorean triple. [6,7]

Theorem 3:

If two of the three numbers of a Pythagorean triple have a common divisor d, then d also divides the third number.

So any Pythagorean triple can be reduced to an irreducible Pythagorean triple, where a, b and c are coprime two by two. [6,7]

Theorem 4:

Let a, b, c be integers. (a, b, c) is a Pythagorean triple irreducible if and only if there are two numbers u and v (u> v), of different parities and coprime, such as a $= u^2 - v^2$; $b = 2uv$ $c = u^2 + v^2$. [6,7]

Conjecture 1: Conjecture of Erdös-Straus

Every rational number 4 / n, with n an integer greater than or equal to two, can be written as a sum of three unit fractions (also called Egyptian fractions) , that is to say that there are three nonzero integers x, y and z such that:

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

For a recent survey about this conjecture, the Professor Michel MIZONY gave a conference [3,8].

Theorem 5:

For any decomposition of the fraction 4/p as a sum of three Egyptian fractions, corresponds an irreducible Pythagorean triple.

In a previous paper we showed this theorem; moreover we obtain the existence: for every number p (prime or not), different from 1 modulo 24, then there exists a primitive Pythagorean triple associated with each decomposition of Erdös-Straus 4/p. We also saw that if p is not 1 mod 24, there exists a decomposition such that the triple associated is a twin Pythagorean triple of the first type, now the 1-twin Pythagorean triple .

Recall the basic identity that will serve as yet systematically:

Whether (α, β, γ) a primitive Pythagorean triple, and a positive integer, then for any integer p we have the identity:

$$\frac{4}{p}=\frac{2\alpha}{(\beta-\alpha+\gamma)a}+\frac{2(2a\beta-\alpha p)}{a\beta p}+\frac{2\alpha}{a(\beta+\alpha+\gamma)} \qquad (1).$$

And so we have alpha/beta = (z-x)(4y-p)/(2yp)

$$\frac{\alpha}{\beta}=\frac{(z-x)(4y-p)}{2yp}$$

Thus to prove the conjecture of Erdös-Straus, it is necessary and sufficient to find for any prime p, a primitive triple and an integer a, such that the three fractions of the second member of the basic identity are unitary.

In the sequel we denote the decompositions obtained in brackets i.e.

4/p=1/x+1/y+1/z is noted 4/p=[1/x, 1/y, 1/z]; and the integer a is always assumed to be (z–x).

<u>Lemma 1:</u> For any prime p, different from 1 or 49 modulo 120, then there is a 1-twin Pythagorean triple giving a decomposition of Erdös-Straus 4/p.

Simply to prove it for p of the form 73+120k and p=97+120k, according to the previous result.

Using a computer algebra system we can quickly find the following equalities:

for p=73+120k, with a=88+144k and the triple $\alpha = 12, \beta = 5, \gamma = 13$ we have:

$$\frac{4}{73+120k} = \left[\frac{1}{2(11+18k)}, \frac{1}{5(11+18k)(73+120k)}, \frac{1}{10(11+18k)}\right];$$

for p=97+60k, with a=(40+24k)(97+60k) and the triple $\alpha = 12, \beta = 5, \gamma = 13$ we have:

$$\frac{4}{97+60k} = \left[\frac{1}{2(5+3k)(97+60k)}, \frac{1}{25+15k}, \frac{1}{10(5+3k)(97+60k)}\right].$$

The following lemma was more difficult to obtain, because it required a little more sophisticated algorithms:

<u>Lemma 2:</u> For any prime p, different from 1, 121, 169, 289, 361 or 529 modulo 840, then there is a 1 or 2-twin Pythagorean triple giving a decomposition of Erdös-Straus of 4/p.

For this consider the set E7 of integers q <840, such that q modulo 120 is equal to 1 or 49.

By Lemma 1, the conjecture is verified for all numbers that are not congruent modulo 120 to one of the elements 1 and 49. So E7 has 12 elements (as a consequence of Chinese Lemma applied to the congruences modulo 120 and those nonzero modulo 7).

To prove this lemma (which is optimal and known in the literature with regard to other approaches to the conjecture of Erdös-Straus), it must be found explicitly, from the identity (1) good Pythagorean triples giving a decomposition for the numbers congruent to 241, 409, 481, 601, 649 and 769 modulo 840. Nothing new there apparently, but a completely new approach to this conjecture. Indeed, the difficulty lies in the fact that we ask a 1 or 2-twin Pythagorean triple. Is this possible? Well yes, because we dared to propose it as a lemma.

So here's what happens:

for p=73+168 k, with a=40(1+2 k)(73+168 k) and the 1-twin Pythagorean triple (220, 21, 221) we have :

$$\frac{1}{73+168k}=\left[\frac{1}{2(1+2k)(73+168k)}, \frac{1}{21+42k}, \frac{1}{42(1+2k)(73+168k)}\right]$$ and as 168 divides 840 and as 241 and 409 are equal to 73 modulo 168, we can remove them.

The same occurs for p=97+168 k, with a=13(2+3 k)(97+168 k) and the 2-twin Pythagorean triple (195, 28, 197) we have:

$$\frac{1}{97+168k}=\left[\frac{1}{(97+168k)(2+3k)}, \frac{1}{28+42k}, \frac{1}{14(97+168k)(2+3k)}\right]$$ and as 168 divides 840 and as 601 and 769 are equal to 97 modulo 168, we can remove them.

For the remaining two integers to eliminate, 481 and 649, another program gives us :

For p=481 +840 k, a=(136+240 k)(61+105 k) and the 1- twin Pythagorean triple which depends of k (20(17+30 k)(7+12 k), 69+120 k, 20(17+30 k)(7+12 k)+1) the decomposition :

$$\frac{4}{481+840k}=\left[\frac{1}{2(61+105k)}, \frac{1}{3(481+840k)(23+40k)(61+105k)}, \frac{1}{6(23+40k)(61+105k)}\right.$$

for 649 +840 k, a=4(184+240 k)(82+105 k) and the 1-twin Pythagorean triple which depends of k (4(23+30 k)(47+60 k), 93+120 k, 4(23+30 k)(47+60 k)+1) the decomposition :

$$\frac{4}{649+840k}=\left[\frac{1}{2(82+105k)},\frac{1}{3(649+840k)(31+40k)(82+105k)},\frac{1}{6(31+40k)(82+105k)}\right.$$

Thus we find the optimal result of the 6 classes modulo 840.

Obviously, we can continue with a lemma concerning the remaining classes modulo 9240 = 11x840; formulas progressions associated with 1 or 2-twin Pythagorean triples are even more difficult to determine; however, let us give the result: there are 37 classes modulo 9240 ; the maximum found in classical literature is 36 classes, and the optimum of 34 classes by one of authors (forthcoming). Most important is to try to understand why 1 and 2-twin Pythagorean triples should be sufficient to establish the conjecture. Recall that an audit was conducted for all primes less than 10^8 with only 1-twin Pythagorean triple!

Before proceeding to an heuristic about this conjecture, recall different expressions of it: for each prime n, $\frac{4}{n}=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

– 1 the strong conjecture, with two of the three integers x, y, z are multiples of n. An equivalent form is given by: there exist a positive integer m and a divisor d of m^2 such that n+4d=0 modulo 4m-1; so we have x=mn and y=(mn+d)/(4m-1). This form give us a very fast program to obtain a decomposition of n [3].

– 2 the weak conjecture, with at least one of the three integers x, y, z is a multiple of n. An equivalent form is given by : for each prime n=4k+1, there exist a positive integer a and a divisor c of $(k+a)^2$ such that (4k+1)c+k+a=0 modulo gcd(4a-1,c); so we have x=k+a and y=((4k+1)c+k+a)(k+a)/((4a-1)c) [4].

– 3 pentagonal form of the strong conjecture: for each non-pentagonal integer k there exist an integer m and a divisor d of m^2 such that 6k+m+d=0 modulo 4m-1 [8].

– 4 the form of irreducible Pythagorean triples, given by the theorem 5 above, equivalent to the weak form.

– 5 the form based on the 1 and 2-twin Pythagorean triples.

Etc....

Note first that the strong conjecture (- 1 above) has been verified for all p <10^17, so the forms 2, 3, 4 are also, by cons the form 5, stronger than the weak conjecture is not comparable to the strong.

From an heuristic point of view, the strong and weak forms (and those equivalent) collides with the wall, apparently unbridgeable, of arithmetic progressions of solutions (see the Jacobi symbol and the exemplary work of Yamamoto [5] published more than 45 years ago). For example solving the strong conjecture is equivalent to classify and therefore be familiar with the non-pentagonal numbers; little is known about them. By cons the form based on Pythagorean triples has the advantage of relying on the area best known of these irreducible triples, and especially for the last form based on the 1 and 2-twin Pythagorean triples can rely on theorems concerning the generation of all irreducible Pythagorean triples from (4,3,5) and (3,4,5), [9]. We think that the technical lemmas above rested upon this fact.

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