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ASTRONOMY
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Disk surface density profile of spiral galaxies and maximal disks

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Abstract. This paper is an attempt to reconcile several contradictory observations in the Galaxy, mainly the microlensing observations, the local dynamical surface density, and the rotation curve. We use a new, original method of inversion of the rotation curve to derive the disk density profile under various assumptions. We show that *a flat rotation curve is compatible with an exponential disk.* However, the maximal disk model is excluded, unless the vertical scale height is larger than 3 kpc. A larger dark halo core radius, of about the size of the Galactic stellar disk, is required to reconcile the observations, although only at a $1-\sigma$ confidence level. This implies in particular a disk surface density of 85 $M_\odot$ pc^{-2}, higher than most dynamical estimates (Bienaymé et al., 1987; Kuijken and Gilmore, 1991; Bahcall et al., 1992; Flynn and Fuchs, 1994), but not irrealistic with regard to the fact that the Sun lies between two spiral arms. Moreover, this result is consistent with the recent discovery of nearby field brown dwarfs (Ruiz et al., 1997; Basri et al., 1997).

Key words: Galaxy: kinematics and dynamics – Galaxy: structure – Galaxy: halo – Galaxies: spiral

1. Introduction

The last data concerning microlensing towards the center of the Galaxy imply a non negligible amount of hidden mass in the galactic disk (Alcock et al., 1997a; Udalski et al., 1994). While the observed surface density follows an exponential law (see for instance Binney and Tremaine 1987, hereafter BT), which leads to a nearly Keplerian rotation curve, the observed rotation curve remains flat to distances larger than the disc size. This problem is one of the reasons for introducing a spherical halo.

The exact repartition of dark matter between a (more or less) spherical halo and the two dimensional disk is still unknown. There has been suggested (Pfenniger and Combes, 1994) that all dark matter may be in the disk, whereas in standard Galactic models (Bahcall and Soneira, 1980; Gilmore et al., 1989, Bienaymé et al., 1987) the disk contains little or no dark matter at all.

The global analysis of microlensing results, together with more traditional observations (star counts, velocity dispersions, rotation curve ...), puts new constraints on possible Galactic models. Such an analysis is presented by Méra et al. (1997b).

In this paper, we use a new method to calculate the contribution of the disk surface density $\Sigma(R)$ needed to reproduce the observed rotation curve of our Galaxy, with respect to some assumed halo distribution. Recall that the method using Bessel functions (see e.g. Binney & Tremaine, 1987, hereafter BT; Mestel, 1963) is inapplicable for two reasons: 1) it needs the radial derivative of the observed rotational velocity, which is not known accurately enough, and 2) it needs the knowledge of the rotation curve from 0 to infinity. In the following section, we explain our method, and we verify its reliability in section 3 with analytic models. Section 4 reports the results of the simulation for our own Galaxy, with different halo parameters. Section 5 summarizes our results and conclusions.

2. The method

Let us suppose that the Galaxy is a bidimensionnal disk with revolution symmetry. We propose to compute the surface density of this disk $\Sigma(R)$ which gives rise to a specific rotation curve.

Our method is based on a numerical resolution of the Poisson equation. For a maximal disk, given the surface density $\Sigma(R)$, the force acting on a given point of the disk reads:

$$\boldsymbol{F} = \int \frac{\Sigma(R)}{R^3} \boldsymbol{R} d\boldsymbol{R}. \quad (1)$$

Such an integral is mathematically divergent, but there are in fact symmetric compensations near the singular point, so that the integral is well defined. Recall that all numerical methods to calculate integrals introduce various discretisations of the integration volume. We will evaluate this integral, and use the discretised equation to derive $\Sigma(R)$ from $\boldsymbol{F}$.

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We model a spiral galaxy by a disk of radius R_g which consists of n massive bodies of mass m_i and position $\boldsymbol{x_i}$ distributed within an axial symmetry. The central point is denoted m_0[1]. For each point its distance to the center is denoted d_i, its velocity v_i and its distance to other stars $d_{ij} = ||\boldsymbol{x_j} - \boldsymbol{x_i}||$.

Using this discretisation to calculate the integral (1), the force $\boldsymbol{F}_i$ acting at the point $\boldsymbol{x_i}$ reads:

$$\boldsymbol{F}_i = \sum_{j \neq i} G \frac{m_i m_j}{d_{ij}^3} \boldsymbol{d_{ij}}.$$

Now if we suppose that these forces give rise to the rotation curve with Newtonian gravitation, we have

$$\boldsymbol{F}_i = m_i \frac{v_i^2}{d_i} \frac{\boldsymbol{x}_i}{d_i}.$$

The two former equations yields:

$$\frac{v_i^2}{d_i} \frac{\boldsymbol{x}_i}{d_i} = \sum_{j \neq i} G \frac{m_j}{d_{ij}^3} \boldsymbol{d_{ij}}. \tag{2}$$

Because of the symmetry of the problem, the first equation for the center ($i = 0$) reduces to $\mathbf{0} = \mathbf{0}$, and the other equations can be projected on the radial axis. Introducing the angle θ_{ij} between the vectors $\boldsymbol{x_i}$ and $\boldsymbol{x_j}$, then $d_{ij}^2 = d_i^2 + d_j^2 - 2 d_i d_j \cos(\theta_{ij})$ and the set of n linear equations (2) reduces to :

$$\sum_{j \neq i} m_j F_{ij} = v_i^2 / d_i, \tag{3}$$

where $F_{ij} = G(d_i - d_j \cos(\theta_{ij}))/d_{ij}^3$.

In order to solve such a linear system of n equations with $n + 1$ unknowns m_i, we have to fix an arbitrary parameter. For each value of this parameter, there is *in general* a unique solution to the whole system. The total mass of the galaxy seems to be the natural parameter. Using dimensionless quantities, we choose the normalized parameter $\omega = \frac{1}{M_g}$, where $M_g = \sum_i m_i$ is the mass of the galaxy.

The physical significance of the existence of a free parameter lies in the fact that only a portion of the rotation curve is known. The range allowed for this parameter corresponds to all possible extensions of the rotation curve from R_g to infinity.

Denoting $\mu_i = m_i / M_g$, we thus have to solve the following $n + 1$ equations with $n + 1$ variables μ_i for each ω:

$$\sum_{j=0, j \neq i}^{j=n} \mu_j F_{ij} = \omega \, v_i^2 / d_i, \tag{4}$$

$$\sum_{j=0}^{n} \mu_j = 1; \tag{5}$$

for $i = 1, ..., n$ with the constraint:

$$\mu_i \geq 0, \tag{6}$$

The constraint (6) is very interesting because it restricts the range of possible values for the parameter ω and yields the following striking result: there exists a *maximal* ω_{max} and a *minimal* ω_{min} possible values for ω such that all masses are positive. Moreover the difference between ω_{max} and ω_{min} is so small (about 10^{-2} or less) that this method provides a *natural evaluation of the mass of a galaxy*, to which must be added the dark halo contribution. As a consequence, we have shown that only a partial knowledge of the rotation curve allows a precise determination of the mass of the Galaxy, despite the theoretical requirement that the whole rotation curve should be known (see Eq. A3). The mathematical proof is given in the appendix.

We recall that for a spherically symmetric distribution of matter, the Gauss theorem implies that only forces due to the matter *inside* the sphere have to be taken into account. This property is no longer valid for a two dimensional distribution, and the integration domain of (1) must include not only the matter inside, but also *outside* a given ring. Not doing this yields erroneous results. This error appears in some astronomy books.

The approach of the thin ellipsoids is properly examined in Mestel's seminal paper (1963), where he explains rigorously the reasons for the apparent difference between the ellipsoid and the thin disk approaches. In particular, the so-called Mestel's disk is the only case for which, by chance, the two methods give the same results without the need of a careful handling of some correcting terms.

The generalization of our numerical method to a galaxy with a known halo contribution is straightforward. The halo contribution to the rotation curve has just to be subtracted to the observations, and the method to be applied to the resulting velocity distribution.

3. Test of the method

In order to test the accuracy of our method, we first apply it to the following three well-known types of velocity curves: an exponential disk, a constant rotation curve (Mestel's disk), and a Keplerian rotation curve.

We used a numerical discretisation of 250 000 points. The points are displayed along 500 rings (500 points on each ring). The ring radii r_i are proportional to i^2, i.e. if the first one has a radius r_1, the second one a radius $r_2 = 4r_1$, $r_3 = 9r_1$ for the third ring and so on (see Fig. 1). The constraint (6) determines ω_{min} and ω_{max}.

3.1. Exponential disk

For the exponential disk of scale length R_d, the rotation curve is given in BT, p. 78:

$$v_c^2(r) = 4\pi G \Sigma_0 R_d y^2 \left[I_0(y) K_0(y) - I_1(y) K_1(y) \right] \tag{7}$$

where $y = \frac{r}{2R_d}$ and I_0, I_1, K_0, K_1 are modified Bessel functions (see the appendix 1.C.7 of BT).

The surface density we obtain with our method is plotted in Fig. 2, along with the exact, exponential profile associated with the rotation curve (7). In fact, two results are shown on the plot, for the two extreme possibilities $\omega = \omega_{min}$ and $\omega = \omega_{max}$. The differences between the curves are indistinguishable.

[1] This point should not be assimilated to the bulge

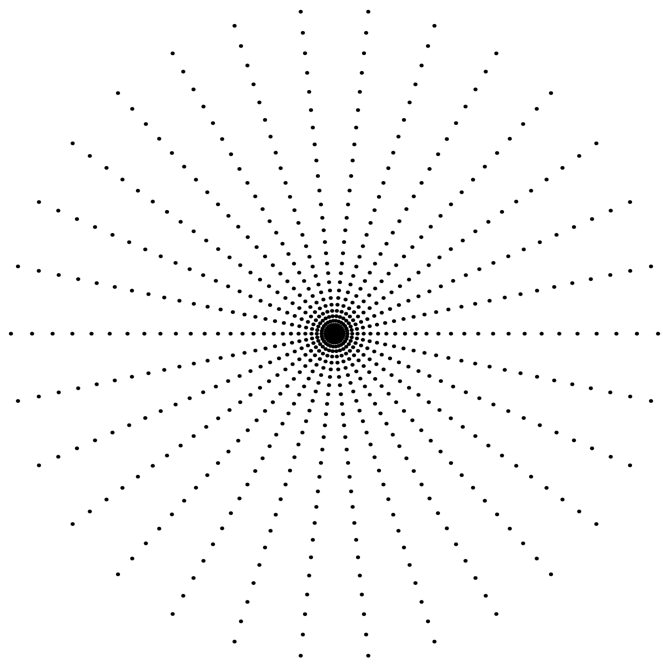

Fig. 1. A schematic representation of the point distribution used in the numerical computation, with only 30×30 points instead of 500×500 for a better visualization. Each ring has a fixed number of points, and its radius is proportional to the square of its order (see text).

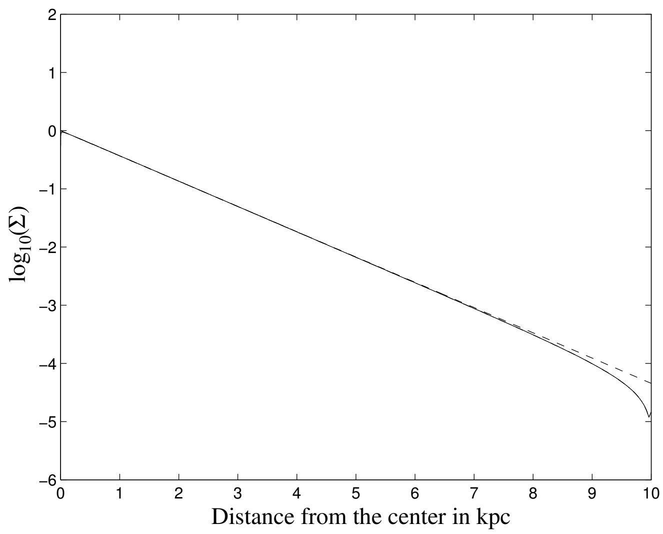

Fig. 2. Surface density for the rotation curve of an exponential disk (solid curve), compared to the exact profile of the *infinite* disc (dashed curve).

The surface density obtained with our method is in remarkable agreement with the exact profile, except near the edge of the disk. Since our method is applicable to a *finite disk* only, the density is lower than the true result near the edge of the disk. A finite exponential yields a larger rotation curve than an infinite one near the edge, so a lower density is required to reproduce the rotation curve (7). The inner part of the galaxy is not affected by this effect because of the exponential decrease of the density.

Another constraint arises from the total mass of the galaxy. As shown in the appendix this constraint is correctly fulfilled by the present method. The mass of an exponential galaxy of central surface density $1M_\odot.pc^{-2}$ is $6.2832 \times 10^6 \mathrm{M}_\odot$: the mass we derive from the allowed values of ω is $6.279 \times 10^6 \mathrm{M}_\odot$, a 0.07% agreement ! The difference arises from 1) the finite size of the disk, and 2) the numerical approximation of the integral by a discrete summation.

Such an error is much smaller than the observational uncertainty in the measurement of the rotation curve. An estimate of this uncertainty can be done on this example. If the precision of the observation is about 10 per cent, then the mass reconstructed from a randomly perturbated rotation curve varies from 5.2 to $7 \times 10^6 \mathrm{M}_\odot$. The error on the mass is hence approximately twice that on the velocity.

3.2. *Point mass*

The finite size effect has no influence at all on this example. It is then a good test of the reliability of the method. A point mass corresponds to a surface density equal to zero everywhere except at the center. The rotation curve has the so-called Keplerian shape $v^2(r) = GM/r$.

For this model, our numerical method reproduces the analytical result within the computer precision. The total mass is recovered exactly, and the density in the rest of the disk is less than 10^{-14} the central density, as expected for a Keplerian profile. It is unusual for a numerical method to give such an accurate result for a singular mass distribution.

3.3. *Mestel's disk*

Mestel (1963) derived the analytical result for the surface density of a finite disk with constant rotation curve (Eq. 56 of his article):

$$\Sigma(R) = \frac{v^2}{2\pi G R}\left(1 - \frac{2}{\pi}\arcsin\frac{R}{R_g}\right) \tag{8}$$

where v is the rotation velocity, and R_g is the disk radius. Of course, the rotation curve of this disk is flat only for $0 < R < R_g$, and it has the so-called Keplerian behavior at infinity. A constant rotation curve from 0 to $+\infty$ corresponds to $R_g \rightarrow +\infty$ and

$$\Sigma(R) = \frac{v^2}{2\pi G R} \tag{9}$$

Our numerical method, applied to a *finite* disk with constant rotation curve on its surface, leads to a density profile which is exactly the theoretical profile given in Eq. (8) (see Fig. 3). Also plotted is the profile of the *infinite* disc (Eq. 9) which appears to be a very poor approximation of Eq. (8), except at the center of the galaxy.

We can see on Fig. 3 that the density of the finite disk with a flat rotation curve is for a large part an exponential of the form

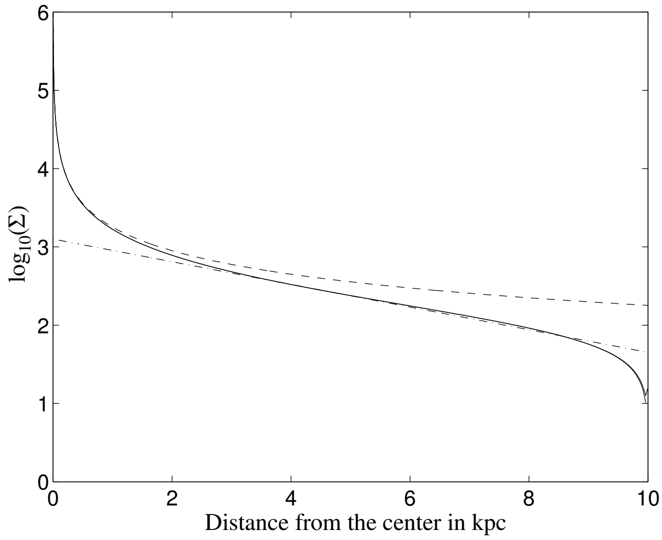

Fig. 3. Surface density for a constant rotation curve, assuming a finite disk (of radius $R_g = 10$ kpc). The exact density of the Mestel's disk (Eq. 8), and the result we get with our method are indistinguishable on the figure (solid line). The dashed curve is the density profile of an *infinite* disc with a flat rotation curve (Eq. 9), which is very different from the *finite* disc density. The dot-dashed curve is an exponential density with a scale length of approximately one third of the galactic radius.

$\Sigma(R) = \Sigma_0 e^{-R/R_d}$ (the plot is logarithmic on the $y-$axis). The scale length R_d derived for this model is approximately $0.3\,R_g$. The density is increasing more rapidly (than the exponential) towards the galactic center, like in a real spiral galaxy with a bulge.

Mestel used this result to determine the mass of the Galaxy and its density at the sun position. However, the value he used for the Galactic radius ($R_g = 10 - 11$ kpc, as recommended by the IAU in 1963) is now known to be incorrect. Moreover the rotation curve of a real spiral galaxy is not exactly constant, especially near the center.

These examples assess the validity and the accuracy of the method developed in §2. One may argue that the disk is not bi-dimensional. Given a vertical distribution, for instance an exponential density, the method can be adapted to take into account the thickness of the disk, at a higher computational price. With a standard scale height (about 1/10th of the scale length), we have seen that the surface density profile was nearly unaffected by the finite thickness of the disk. Then we will only use the 2D method, accurate enough for our purposes.

4. Application to our Galaxy

Since the accuracy of the present method has been assessed in the previous section by comparison with analytical models, we can apply it with confidence to the case of the Milky Way, first with no halo (i.e. all dark matter is 2D distributed).

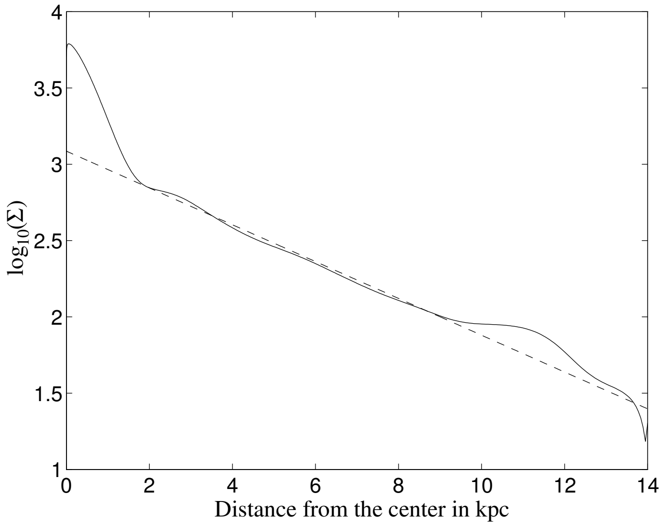

Fig. 4. The surface density of the Milky Way, derived with our method from the rotation curve of Vallée (1994). The straight dashed line is an eye fit by an exponential of scale length 3.6 kpc, and local density $115 M_\odot.pc^{-2}$.

Fich and Tremaine (1991) review observational data on our Galaxy rotation curve. Vallée (1994) takes into account the apparent rise of the rotation curve at the limit of the observations.

The Galactic radius $R_g = 14$ kpc is taken from Robin et al. (1992).

The surface density we obtain is plotted on Figs. 4, along with an exponential profile of scale length 3.6 kpc. A large part of the curve is close to the exponential. Only the central and external parts deviate substantially from the exponential law. The local surface density (i.e. at $R = R_\odot = 8.5$ kpc) is $115 M_\odot.pc^{-2}$, more than twice the visible density $\approx 50 M_\odot.pc^{-2}$.

The local amount of dark matter has been tentatively estimated by several authors in the last ten years from dynamical constraints (see Méra et al. 1997a for a summary of these observations).

The surface density we obtain, with a *maximal disk* is incompatible with the results of Kuijken and Gilmore (1991), and Bienaymé, Robin and Crézé (1987). However, Bahcall, Flynn and Gould (1992) derived 1-sigma limits $50 \leq \Sigma(R_\odot) \leq 115 M_\odot.pc^{-2}$, so that the value we derive, a bit higher than the upper limit, can not be excluded with a high confidence level.

In this model, the rotation curve remains flat only on the disk surface. Beyond 14 kpc, the rotation velocity has a Keplerian decrease, which is not the case for other spiral Galaxies similar to the Milky Way (Casertano and van Gorkom, 1991). This can been solved by taking a larger disk radius, which leads to a larger scale length (roughly proportional to the radius, see section 3.3). Moreover, the local surface density is 190 $M_\odot$ pc^{-2} for a rotation curve flat up to 30 kpc, well over dynamical constraints.

This strongly argues for the presence of a dark halo, which does contribute to the Galaxy rotation curve. The microlensing results towards the galactic bulge (Alcock et al., 1997a; Udalski et al., 1994) and the LMC (Alcock et al., 1997b; Ansari et al., 1996) suggest a light halo and a massive disk.

The standard halo density (see e.g. Alcock et al., 1997b) has the following form:

$$\rho(r) = \frac{v_\infty^2}{4\pi G} \frac{1}{r^2 + a^2}, \tag{10}$$

where r is the distance to the galactic center, a is the core radius, and v_∞ is the asymptotic rotation velocity of this model. The halo size is fixed by dynamical considerations at the level of globular clusters and satellite dwarf galaxies (Kochanek, 1996), or at the scale of the local group (Peebles, 1994). We refer to Méra, Chabrier and Schaeffer (1997a) for a discussion of the Galactic mass. Since we are in this article only interested in the disk, the halo size is not relevant because of the spherical symmetry. The value of v_∞ is also related to large scale dynamics, and is $v_\infty = 220 \pm 20\,\mathrm{km.s^{-1}}$ (Méra et al., 1997a).

We have computed the related disk density profile with our method for several values of the parameters. For each model, the optical depth in the direction of the LMC, assuming a disk scale height of 300 pc, is computed. The bulge optical depth is more dependent on the 3D modelisation of the Galactic center. Moreover, for the bulge, the source stars can not be supposed to lie at the same distance. The distribution of the number of source stars, and of their velocities, is still uncertain because of absorption (Paczyński et al., 1994). Hence, only the LMC optical depth can provide robust constraints on a mass model. The observed values are $\tau_{LMC} = 2.2^{+1.1}_{-0.7} \times 10^{-7}$ for the MACHO collaboration (Alcock et al., 1997b), and $\tau_{LMC} = 0.8 \times 10^{-7}$ for the EROS collaboration (Ansari et al., 1996). The latter estimate relies on only two candidates, and is less robust than the MACHO result (which relies on 6 candidates).

Table 1 shows the relation between the halo core radius a (see Eq. 10) and the disk solar density of the model. We have already mentioned that $50 < \Sigma_\odot < 115 M_\odot.pc^{-2}$, which is verified only for $a > 7$ kpc. The third column of Table 1 gives the model optical depth towards the LMC. The upper limit of MACHO observations leads to the constraint $a > 18$ kpc. However, because the disk has a size of 14 kpc, the resulting rotation curve, which is exactly the observed one (Vallee, 1994) for $R < 14$ kpc, present a dip at $R \approx 15 - 20$ kpc, just before the halo contribution is sufficient to provide a rotation velocity of $\approx v_\infty$, which is the case for $R >> a$. In fact, for that reason, the halo core radius must be *at most* of the order of the disk radius.

The last two columns of Table 1 display respectively the disk mass and scale length. The latter, depending on a, varies between 2.5 and 3.5 kpc, in perfect agreement with observations. The disk mass is not observationally constrained since the amount of dark matter is unknown. The allowed range is $8 - 9 \times 10^{10} M_\odot$.

We have made the same computations for two extreme values of v_∞. In the case $v_\infty = 200\,\mathrm{km.s^{-1}}$, the solar surface density imposes $a > 6$ kpc and the LMC optical depth implies $a > 14$ kpc. For $v_\infty = 240\,\mathrm{km.s^{-1}}$, these limits are respectively $a > 9$ and $a > 20$ kpc. The disk mass and scale length are still compatible with observations.

Table 1. Model parameters for different halo core radius a (see Eq. 10), and for $v_\infty = 220\,\mathrm{km.s^{-1}}$. The second column gives the solar density, the third column display the theoretical optical depth towards the LMC (assuming a disk scale height of 300 pc). The next column corresponds to the disk total mass. The last column is the disk scale length, fitted in the linear part of the density, between 30% and 70% of R_g. The compatibility with observations at the $1 - \sigma$ confidence level imposes $a > 18$ kpc (see text).

a (kpc)	$\Sigma_\odot$ ($M_\odot.pc^{-2}$)	τ_{LMC} ($\times 10^7$)	M_{disk} ($\times 10^{-10} M_\odot$)	H_d (kpc)
2	2.4	6.5	2.9	1.1
3	11.84	6.4	3.957	1.8
4	21.71	6.2	4.82	2.1
5	31.36	5.98	5.53	2.3
6	40.37	5.74	6.14	2.5
7	48.56	5.5	6.65	2.7
8	55.86	5.24	7.08	2.8
9	62.3	5	7.45	2.9
10	67.93	4.76	7.76	3
11	72.85	4.53	8.03	3.1
12	77.14	4.31	8.26	3.2
13	80.87	4.11	8.46	3.2
14	84.13	3.91	8.63	3.3
15	86.98	3.73	8.776	3.3
16	89.48	3.56	8.91	3.4
17	91.68	3.4	9.02	3.4
18	93.61	3.25	9.12	3.5
19	95.33	3.11	9.21	3.5
20	96.85	2.97	9.29	3.5

This study shows that the halo core radius has to be of the order of the disk radius. A larger value leads to a dip in the rotation curve which is not observed in other spiral galaxies. On the contrary, the microlensing towards the LMC suggest a value significantly larger than the standard 5 kpc. The high observed optical depth towards the bulge, although its interpretation is more model-dependent (especially because of the bar structure), is in favor of some non-negligible amount of dark matter in the Galactic disk. For instance, the model with $a = 14$ or 15 kpc is in perfect agreement with microlensing observations, the disk measured scale length, and with the dynamical determination of the solar surface density by Bahcall, Flynn and Gould (1992): this model predicts $\Sigma_\odot = 85 M_\odot.pc^{-2}$ (see table 1).

Moreover, this result shows that the disk does contribute to the flattening of the rotation curve, at least up to its limits.

5. Discussion and conclusions

We have used a new method to calculate the surface density of a disk from the measured rotation curve, assuming a standard halo contribution. This method, whose accuracy has been assessed on several analytic examples, is a useful tool for Galactic modeling. We show that a flat rotation curve for a finite disk *does* correspond to an exponential profile plus a central bulge. The scale length is $\approx 3/10$ of the radius of the galaxy. Therefore, the exponential disk is compatible with a flat rotation curve, and the $1/R$ approximation is valid only in the inner part of the Galaxy.

We have then applied the method to the observed rotation curve of our own Galaxy. The case with no dark halo contribution is excluded. The model of Pfenniger and Combe (1994) implies a solar surface density incompatible with observations, unless the dark matter (whatever its nature) is distributed with a scale height larger than 3 kpc.

This confirms the well-established result that such a maximal disk is inconsistent with various estimates of the mass of the Milky Way (Trimble, 1987; Kochanek, 1996). But the results of microlensing experiments urge to reconsider standard Galactic models. The LMC observed optical depth suggests a halo model with a core radius of the order of the disk size.

The disk surface density in the solar neighborhood is found to be $85 M_{\odot}.pc^{-2}$ in a galactic model with a halo having a large core radius ($a \approx 15$ kpc). This result is within the error bar of the independent determination by Bahcall, Flynn and Gould (1992). The predicted optical depth is in this model $\tau_{LMC} = 3.73 \times 10^{-7}$, compatible with the observations of (Alcock et al., 1997b) at the $1.5 - \sigma$ confidence level.

The determination of a self-consistent galactic model is out of the scope of the present paper, which presents only a new method for this purpose. This ultimate goal is addressed in Méra, Chabrier and Schaeffer (1997a; 1997b), with a consistent analysis of microlensing events, stars counts and viable Galactic models.

Further work implies the application of the method to other galaxies for which we have accurately determined rotation curves, and the study of the stability of the disk. We also plan to include the corrections due to general relativity arising from cosmological considerations. With such a correction, the influence of the local group could also be taken into account.

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Appendix A: mathematical background

Let suppose that the rotation velocity $v(r)$ is known up to infinity. Using Bessel functions J_0 and J_1 we have the following formula (BT):

$$\Sigma(r) = \frac{1}{2\pi G} \int_0^\infty J_0(kr) S(k) k \, dk$$

where

$$S(k) = \int_0^\infty v^2(x) J_1(kx) \, dx.$$

And conversely, using Bessel transformations :

$$S(k) = 2\pi G \int_0^\infty \Sigma(x) J_0(kx) x \, dx$$

$$v^2(r) = r \int_0^\infty S(k) J_1(kr) k \, dk.$$

For the Mestel's disk or for the exponential disk these integrals can be solved analytically.

This method has a theoretical interest: indeed the correspondence $v \leftrightarrow \Sigma(r)$ is a bijection, but since the rotation curve is know from observations up to $r \leq R_g$ only, several surface densities can in principle lead to the same rotation curve $v(r)$ for $r \leq R_g$, but with different $v(r)$ for $r \geq R_g$. Since most of the mass is concentrated in the center of the galaxy, the rotation curves for $r \geq R_g$ must be nearly Keplerian. It is the reason why our method provides a very narrow range $[\omega_{min}, \omega_{max}]$ of solutions.

If we want to compute $\Sigma(r)$ from $v(r)$, we have to consider the following double singular integral:

$$\Sigma(r) = \frac{1}{2\pi G} \int_0^\infty J_0(kr) \left[\int_0^\infty v^2(x) J_1(kx) \, dx \right] k \, dk.$$

In order to carry out the integrations, it is necessary to introduce a principal value integral.

Using properties of hypergeometric series ${}_2F_1$ (Erdelyi, 1953), we write:

$$4\pi G r\Sigma(r) = \lim_{a\to 1}\int_0^a [v^2(r/\sqrt{z})\,\frac{1}{\sqrt{z}}\;{}_2F_1([3/2,1/2],[1],z) - \frac{1}{2}v^2(r\sqrt{z})\;{}_2F_1([3/2,3/2],[2],z)\,]dz. \quad \text{(A1)}$$

Conversely, $v(r)$ can be written as a function of $\Sigma(r)$ in the following form:

$$v^2(r)/r = \pi G\lim_{a\to 1}\int_0^a [\Sigma(r\sqrt{z})\;{}_2F_1([3/2,1/2],[1],z) - \frac{1}{2\sqrt{z}}\Sigma(r/\sqrt{z})\;{}_2F_1([3/2,3/2],[2],z)\,]dz. \quad \text{(A2)}$$

It is also possible to rewrite this latter formula using elliptic integrals. Actually elliptic integrals are hypergeometric functions, so the formula (2-146) of BT leads to the above principal value integral in the plane of the disk.

But the most important consequence is the theoretical formula giving the mass of the galaxy from the rotation curve: If R_g is the radius of the disk then :

$$M = \frac{R_g}{4G}\times \int_0^1 [v^2(R_q\sqrt{z}) + v^2(\frac{R_g}{\sqrt{z}})/\sqrt{z}]\;{}_2F_1([3/2,1/2],[2],z)\,dz.$$

This formula can not be used to compute the mass of a given galaxy because of the unknown term $v^2(R_g/\sqrt{z})$, but we shall use it to prove that our method gives accurate results.

Notice first that if $r \geq R_g$ then the relation (A2) reduces to :

$$v^2(r)/r = \pi G\int_0^1 \Sigma(r\sqrt{z})\;{}_2F_1([3/2,1/2],[1],z)dz,$$

or

$$v^2(r) = \frac{2\pi G}{r}\int_0^{R_g} {}_2F_1([3/2,1/2],[1],u^2/r^2)\,u\,\Sigma(u)\,du. \text{(A3)}$$

Let $v_k^2(r) = \frac{GM}{r} = \frac{2\pi G}{r}\int_0^{R_g} u\,\Sigma(u)\,du$ be the Keplerian rotation curve. Then (A3) can be written:

$$v^2(r) - v_k^2(r) = \frac{2\pi G}{r}\int_0^{R_g} [\,{}_2F_1([3/2,1/2],[1],u^2/r^2) - 1]\,u\,\Sigma(u)\,du.$$

But the function $f(u^2/r^2) = {}_2F_1([3/2,1/2],[1],u^2/r^2) - 1$ has the two following properties :

if $r \geq R_g$ then $\frac{3}{4}\frac{u^2}{r^2} + \frac{2}{3}\frac{u^4}{r^4} \leq f(u^2/r^2)$ and

if $r \geq 2R_g$ then $f(u^2/r^2) \leq \frac{3}{4}\frac{u^2}{r^2} + \frac{u^4}{r^4}$. Denoting $M_n = 2\pi\int_0^{R_g} u^n\Sigma(u)\;du$ the momentum of order n of the surface density, we thus obtain the following estimates:

for $r \geq R_g$,

$$\frac{3}{4}\frac{GM_3}{r^3} + \frac{2}{3}\frac{GM_5}{r^5} \leq v^2(r) - v_k^2(r),$$

and for $r \geq 2R_g$

$$v^2(r) - v_k^2(r) \leq \frac{3}{4}\frac{GM_3}{r^3} + \frac{GM_5}{r^5}.$$

This inequality proves indeed the accuracy of our method: let v_{sup} and v_{inf} be the rotation curves associated with the extremal surface densities. Taking into account the above relations, we get:

$$\frac{M_{sup} - M_{inf}}{M} \leq \frac{4}{\pi}\frac{\delta v_o}{v_o} + \frac{2}{3\pi R_g^2}\frac{M_{sup,3} - M_{inf,3}}{M}$$

where $M_{sup,3}$ and $M_{inf,3}$ are the three momentum of the surface densities associated respectively to the M_{sup} and M_{inf} configurations; and $\delta v_o = \sup_{R_g\leq r\leq 2R_g}(v_+(r) - v_-(r))$. The LHS of this inequality is expected to be small, because the rotation curves are both near the Keplerian curve, and the momentum of order 3 are small.