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Hypersurfaces with Constant Mean Curvature in Hyperbolic Space Form*

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Abstract. In this article, we prove the following theorem: A complete hypersurface of the hyperbolic space form, which has constant mean curvature and non-negative Ricci curvature Q, has non-negative sectional curvature. Moreover, if it is compact, it is a geodesic distance sphere; if its soul is not reduced to a point, it is a geodesic hypercylinder; if its soul is reduced to a point p, its curvature satisfies $\|\nabla Q\| < \infty$, and the geodesic spheres centered at p are convex, then it is a horosphere.

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1. Introduction

In 1899, H. Liebmann proved that spheres are the only surfaces with constant Gaussian curvature. In 1900, he also proved that spheres are the only ovaloids with constant mean curvature in Euclidean space (see [Hop83] for example). Generalizations of these classical rigidity results were made by many authors (D. Hilbert, H. Hopf, S. S. Chern, among others). K. Nomizu and B. Smyth [NS69] proved in 1969 that a non-negatively curved compact hypersurface with constant mean curvature of Euclidean space or a sphere, is a standard sphere, or a product of two spheres. In 1975, S. T. Yau improved this result when the hypersurface of Euclidean space has non-negative Ricci curvature, showing that, in this case, the hypersurface is a sphere [Yau74]. Recently, R. Walter [Wal85] gave the classification of non-negatively curved compact hypersurfaces in space form, with constant r-mean curvature. In this paper, we shall deal with the same problem in the hyperbolic space, under the weaker assumptions that the hypersurface is only complete, and its Ricci curvature is non-negative. Our conjecture is as follows (we denote by H^{n+1} the simply connected space form of constant sectional curvature-1):

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CONJECTURE. *Let M be a complete hypersurface of the hyperbolic space form H^{n+1}, with non-negative Ricci curvature, and constant mean curvature. Then, M is a geodesic distance sphere, a horosphere, or a geodesic hypercylinder*

(By a geodesic distance sphere in H^{n+1}, we mean the submanifold of points which are at a fixed distance of a fixed point. Such a hypersurface is totally umbilical. A hypercylinder is isometric with $R \times S^{n-2}$, a product of a line with a sphere, and embedded in H^{n+1} as the normal sphere bundle of a geodesic. Finally, to define a horosphere, we begin to remember that H^{n+1} has a standard embedding in Minkowsky space E^{n+2}. A horosphere of H^{n+1} is a flat (umbilical) hypersurface of H^{n+1} obtained as the intersection of a hyperplane of E^{n+2} with H^{n+1}.)

We cannot prove the conjecture in general. Our main observation is that the assumptions imply that the hypersurface M has non-negative curvature and we deal with its soul. If the soul is not a point, we can conclude that its dimension is 1 or $(n-1)$, and we show that M is isoparametric. When its soul is reduced to a point p we cannot conclude. In this case, to prove that M is isoparametric, we must add the following geometric conditions:

(a) $\|\nabla Q\| < \infty$,

(b) *the geodesic spheres centered at p are convex.*

Then, we use the classical classification of isoparametric hypersurfaces of space forms (see [CR85] for instance). The way to prove that M is isoparametric consists in applying the Hopf lemma and its generalization to complete manifolds (due to S. T. Yau [Yau75b]), to suitable functions. We have no problem when this function is a square of the norm of the second fundamental form: it is a smooth function, and we can apply standard methods to compute its Laplacian and apply the Hopf lemma. However, we need to compute the Laplacian of the first eigenvalue of the second fundamental form, which is not smooth in general. That is why we consider a sequence of smooth functions, which approach locally the first principal curvature function, and work with the Laplacian of this sequence. The method used here may be regarded as an attempt to solve the classifying problems under completeness conditions.

2. Local Study of Hypersurfaces of the Hyperbolic Space Form

First of all, we derive from Gauss equation the local behavior of hypersurface of H^{n+1}, with non-negative Ricci curvature. we can summarize the results in the following

PROPOSITION 1. *Let M be a hypersurface with non-negative Ricci curvature, of a space of constant sectional curvature -1. Then, at p.*

1. *The second fundamental form h of M is positive semi-definite, of negative semi-definite.*
2. *$H \geq 1$, and if H is equal to 1 at p, then p is an umbilical and flat point.*
3. *Let $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$ be the principal curvatures at p, and $(e_1, \ldots, e_n)$ be a corresponding local frame of principal vectors, that we extend on a neighborhood of p. The sectional curvature tensor $K_{ij} = K(e_i, e_j)$ satisfies at p, in this frame:*

   $K_{ij} = \lambda_i\lambda_j - 1$ *if $i \neq j$. In particular,*
   $K_{ij} \geq K_{(n-1)n}$ *if $i \neq j$.*
4. *The Ricci curvature tensor $Q_i = Q(e_i)$ satisfies in this frame:*

   $$Q_i = -(n-1) + \lambda_i(nH - \lambda_i).$$

   Moreover,

   $$Q_1 \geq \cdots \geq Q_i \geq \cdots \geq Q_n,$$

   and

   $$Q_n = \inf_{\{v, \|v\|=1\}} Q(v)$$
5. *If $Q_n = 0$, then $K_{1n} = \cdots = K_{(n-1)n} = 0$, and $\lambda_1 = \cdots = \lambda_{n-1}, \lambda_n = \lambda_1^{-1}$.*
6. *$\Sigma_{i,j}(\lambda_i - \lambda_j)^2 K_{ij} \geq 0$.*

Remark. In the following, we shall always assume that h is positive semi-definite. (If not, we replace the normal vector field of the hypersurface by its opposite.)

Proof of Proposition 1. Consider the second fundamental form h as a $(1-1)$-tensor. From the Gauss equation, we deduce immediately:

$$nH\langle h(v), v\rangle = Q(v) + (n-1)\langle v, v\rangle + \langle h(v), h(v)\rangle,$$

for every vector v, tangent to the hypersurface. (1)–(5) are consequences of this equality. (6) is a consequence of the following

LEMMA 1. *Let*

$$a_1 \geq \cdots \geq a_n,$$

$$b_1 \geq \cdots \geq b_n,$$

be two sequences of real numbers. Then,

$$\sum_i a_i b_i \geq \sum_i a_i b_{j_i} \geq \sum_i a_i b_{n-i}$$

for any permutation $j_1, \ldots, j_n$.

PROPOSITION 2. *Let M be a hypersurface with non-negative sectional curvature, of a space of constant sectional curvature -1. Let p be a point of M and let $u, v, \in ST_pM$ be two unit vectors such that:*

$$\frac{\pi}{4} \leq \prec(u,v) \leq \frac{3\pi}{4}.$$

Then,

$$\mathrm{Max}(Q(u), Q(v))_p \geq \frac{H_p - 1}{8(n-1)}.$$

Proof. Take any point p of M. At p, we have:

$$\lambda_{n-1}\lambda_n - 1 \geq 0,$$

$$\lambda_{n-1} \geq \lambda_n,$$

hence $\lambda_{n-1} \geq 1$, and moreover

$$\lambda_1 \geq H.$$

Then,

$$Q_{n-1} = \lambda_{n-1}\lambda_1 - 1 + \sum_{i=2}^{n} K(e_{n-1}, e_i) \geq H - 1.$$

Let u and v be two unit tangent vectors at p such that:

$$\frac{\pi}{4} \leq \prec(u,v) \leq \frac{3\pi}{4}.$$

We put:

$$u = \sum_{i=1}^{n} u^i e_i, \quad v = \sum_{i=1}^{n} v^i e_i.$$

We shall prove that:

$$\sum_{i=1}^{n-1} (u^i)^2 \geq \frac{1}{8(n-1)}, \quad \text{or} \quad \sum_{i=1}^{n-1} (v^i)^2 \geq \frac{1}{8(n-1)}.$$

In fact, if

$$\sum_{i=1}^{n-1} (u^i)^2 < \frac{1}{8(n-1)}, \quad \text{and} \quad \sum_{i=1}^{n-1} (v^i)^2 < \frac{1}{8(n-1)},$$

then,

$$(u^n)^2 > 1 - \frac{1}{8(n-1)}, \quad (v^n)^2 > 1 - \frac{1}{8(n-1)}.$$

This would imply:

$$\begin{aligned} |\cos(\sphericalangle(u,v))| &\geq |u^n v^n| - \left|\sum_{i=1}^{n-1} u^i v^i\right| \\ &\geq |u^n||v^n| - \left(\sum_{i=1}^{n-1} (u^i)^2 (v^i)^2\right)^{\frac{1}{2}} \\ &> 1 - \frac{1}{4(n-1)} > \frac{\sqrt{2}}{2}. \end{aligned}$$

This leads to a contradiction. Finally,

$$\begin{aligned} \mathrm{Max}(Q(u), Q(v)) &\geq \mathrm{Max}\left(\sum_{i=1}^{n-1} (u^i)^2 Q_i, \sum_{i=1}^{n-1} (v^i)^2 Q_i\right) \\ &\geq \mathrm{Max}\left(\sum_{i=1}^{n-1} (u^i)^2 Q_{n-1}, \sum_{i=1}^{n-1} (v^i)^2 Q_{n-1}\right) \\ &\geq \frac{H-1}{8(n-1)}. \end{aligned}$$

The following Proposition is well known ([Che73], for example):

PROPOSITION A. *Let M be a hypersurface with non-negative Ricci curvature, of a space of constant sectional curvature -1. Suppose that M has constant mean curvature. Let De denote the Laplace–Beltrami operator. Then, we have:*

$$\Delta h_{ij} = nH\delta_{ij} - nh_{ij} + nH \sum_l h_{il} h_{lj} - \|h\|^2 h_{ij},$$

$$\Delta \|h\|^2 = 2\|\nabla h\|^2 + \sum_{i,j} (\lambda_i - \lambda_j)^2 K_{ij}.$$

3. Complete Hypersurfaces with Constant Mean Curvature in the Hyperbolic Space Form

3.1. THE COMPACT CASE

In this section, we begin by the simplest case: we assume that the hypersurface is compact, and prove the following

THEOREM 1. *Let M be a compact hypersurface of H^{n+1} with non-negative Ricc curvature, and constant mean curvature. Then M is a (totally umbilic) geodesi distance sphere.*

Proof. The proof of Theorem 1 is standard: Assume that M is compact. Apply ing Proposition 1(6) and the Hopf lemma, we deduce from Proposition A that M has a parallel second fundamental form. Then M is isoparametric ([Law69]). Now since M is compact in H^{n+1}, we know from [Wal85] that M is a geodesic sphere (Remark that this result is related to a special case of the theorem of Alexandro which says that any compact hypersurface embedded in a Euclidean or hyperboli space with constant mean curvature, is a round sphere [Esc89]).

3.2. THE COMPLETE CASE

We continue the discussion and give results on complete hypersurfaces in a hype bolic space form with constant mean curvature, and non-negative Ricci curvature Although we cannot get a general complete theorem of classification, we are abl to solve completely the problem when the soul of the hypersurface is not a poin or when the gradient of the Ricci curvature is finite at infinity. We state our mai theorems:

THEOREM 2. *Let M be a complete (non-compact) hypersurface of H^{n+1} wit non-negative Ricci curvature Q, and constant mean curvature. Then, M has nor negative sectional curvature. Moreover, suppose that one of the two followin conditions hold:*

1. *The soul of M is not reduced to a point.*
2. *The soul of M is reduced to a point p, and*

$$\|\nabla Q\| < \infty,$$

and the geodesic spheres centered at p are convex.

Then, M is a geodesic hypercylinder, or a horosphere.

The proof of this theorem will be done in many steps. It will be a consequence (Theorems 3 and 4 below (Sections 3.7 and 3.8).

3.3. BEHAVIOR AT INFINITY

3.3.1. *Study of the First Principal Curvature of a Complete Hypersurface of H^{n+1}*

On any hypersurface M of any manifold (with oriented normal bundle), we ca define the function λ_1 which associates to each point p of M the largest princip curvature at p. Of course, this function is not of class C^2 in general, but it continuous. We shall say that *the point p is λ_1-regular if λ_1 is C^2 on a neighborhoc* *of* p. It is well known that the measure of the set of points q which are not λ_1-regular is zero in M. We shall say that M is λ_1-regular if λ_1 *is of class* C^2 *in* M.

LEMMA 2. *Let* M *be a hypersurface of* M^{n+1} *with non-negative Ricci curvature, and constant mean curvature. Let* p *be a* λ_1*-regular point of* M. *Then at* p*:*

$$\Delta\lambda_1 \geq \frac{1}{n-1}\sum_i(\lambda_1 - \lambda_i)Q_i \geq 0.$$

Proof. From Proposition A, we have:

$$\begin{aligned}\Delta h_{11} &= nH - n\lambda_1 + nH\lambda_1^2 - \lambda_1\sum_i \lambda_i^2,\\ &= \sum_i(\lambda_1 - \lambda_i)(\lambda_1\lambda_i - 1).\end{aligned}$$

Using Proposition 1(4), and the fact that the Ricci curvature of M is non-negative, we have:

$$Q_i = (n-1)(\lambda_1\lambda_i - 1) \geq 0.$$

From this, we deduce immediately that

$$\Delta\lambda_1 \geq \frac{1}{n-1}\sum_i(\lambda_1 - \lambda_i)Q_i \geq 0.$$

Now, we need the following:

DEFINITION 1. We put

$$\Lambda = \sup_{p\in M} \lambda_1(p),$$

$$S = \sup_{p\in M} \|h\|^2(p).$$

Since H is constant and the principal curvatures are non-negative (see Proposition 1.3), Λ and S are finite. In particular, if M is λ_1-regular, we can apply the generalized maximum principle of S. T. Yau [Yau75a] and immediately get the following

PROPOSITION 3. *Let* M *be a complete (non-compact)* λ_1*-regular hypersurface of* H^{n+1} *with non-negative Ricci curvature and constant mean curvature. Then there exists a sequence* (x_k) *such that:*

1. $\lim_{k\to\infty}\lambda_1(x_k) = \Lambda$,